Borcherds lifts

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Abstract

In this note, we illustrate the theory of Borcherds lifts. We will follow the presentation of [Bru02].

Let (L, q) be an even lattice of signature $(2, \ell)$ and set $V := L \otimes \mathbb{R}$. Let $O^\circ(V)$ denote the identity component of $O(V)$ and set $\Gamma(L) := O(L) \cap O^\circ(V)$. When L is unimodular, Borcherds constructed a map

$$\left\{\begin{array}{c}\text{weakly holomorphic}\\ \text{modular forms for}\\ \mathrm{SL}_2(\mathbb{Z}) \text{ of weight } 1-\frac{\ell}{2}\end{array}\right\} \to \left\{\begin{array}{c}\text{meromorphic modular forms for}\\ \Gamma(L) \text{ of some weight and}\\ \text{some multiplier system}\end{array}\right\}.$$

More generally, when L is not necessarily unimodular, the Borcherds lift is a map

$$\left\{\begin{array}{c}\mathbb{C}[L'/L]\text{-valued weakly holomorphic}\\ \text{modular forms for}\\ \mathrm{Mp}_2(\mathbb{Z}) \text{ of weight } 1-\frac{\ell}{2}\end{array}\right\} \to \left\{\begin{array}{c}\text{meromorphic modular forms for}\\ \Gamma(L) \text{ of some weight and}\\ \text{some multiplier system}\end{array}\right\},$$

where L' is the dual lattice of L, $\mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic group, and $\mathbb{C}[L'/L]$ is given the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. Here, weak holomorphicity means that the function is holomorphic on $\mathfrak{H}$ but can be meromorphic at the cusps. We will construct the Borcherds lift using the regularized theta lifts of Poincaré series.

Let f be a $\mathbb{C}[L'/L]$-valued weakly holomorphic modular form for $\mathrm{Mp}_2(\mathbb{Z})$ of weight $1 - \ell/2$. We have the Fourier expansion of f at the cusp ∞

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} c(\gamma, n) e^{2\pi i n \tau}[\gamma].$$

Then, the weight of the Borcherds lift Ψ_f of f is given by $c(0,0)/2$. Furthermore, the divisor of Ψ_f is determined by the principal part of f by

$$\mathrm{div}(\Psi_f) = \sum_{\substack{\gamma, n \\ n<0}} c(\gamma, n) H(\gamma, n),$$

where $H(\gamma, n) \subseteq \mathfrak{H}_\ell/\Gamma(L)$ is the Heegner divisor of discriminant (γ, n). This is useful as it allows us to construct many relations among Heegner divisors.

1 The metaplectic group and the Weil representation

Definition 1. The *metaplectic group* is a double cover of $\mathrm{SL}_2(\mathbb{R})$ defined by

$$\mathrm{Mp}_2(\mathbb{R}) := \{(M, \phi) \mid M \in \mathrm{SL}_2(\mathbb{R}) \text{ and } \phi : \mathfrak{H} \to \mathbb{C} \text{ holomorphic such that } \phi(\tau)^2 = c\tau + d\}$$

whose operation is given by $(M_1, \phi_1) \cdot (M_2, \phi_2) := (M_1 M_2, (\phi_1 \circ M_2)\phi_2)$. We also define the subgroup

$$\mathrm{Mp}_2(\mathbb{Z}) := \{(M, \phi) \in \mathrm{Mp}_2(\mathbb{R}) \mid M \in \mathrm{SL}_2(\mathbb{Z})\}.$$

Remark 2. (1) The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ lifts to an action of $\mathrm{Mp}_2(\mathbb{R})$ on $\mathfrak{H}$.
(2) There is a continuous set-theoretic splitting

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Mp}_2(\mathbb{R})$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \widetilde{M} := \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right)$$

using the principal branch of square root.

(3) $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T := \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \quad S := \widetilde{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

subject to the relations $S^8 = 1$, $S^2 = (ST)^3$, $S^2T = TS^2$. The center of $\mathrm{Mp}_2(\mathbb{Z})$ has order 4 and is generated by

$$Z := S^2 = \widetilde{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}.$$

Let (L, q) be an even lattice of signature (b^+, b^-). Here, evenness means that the quadratic form q takes values in $\mathbb{Z}$. Then, $V := L \otimes \mathbb{R}$ has an inner product defined by

$$(v, w) := q(v + w) - q(v) - q(w)$$

that satisfies $(v, w) \in \mathbb{Z}$ and $(v, v) = 2q(v) \in 2\mathbb{Z}$ for all $v, w \in L$.

Consider the $\mathbb{C}$-vector space $\mathbb{C}[L'/L]$ with basis vectors $[\gamma]$ for $\gamma \in L'/L$. We endow $\mathbb{C}[L'/L]$ with the standard Hermitian inner product

$$\left\langle \sum_{\gamma \in L'/L} a_\gamma [\gamma], \sum_{\gamma \in L'/L} b_\gamma [\gamma] \right\rangle := \sum_{\gamma \in L'/L} \overline{a_\gamma} b_\gamma.$$

Definition 3. The *Weil representation* attached to L is a unitary representation $\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \curvearrowright \mathbb{C}[L'/L]$ defined by

$$\rho_L(T)[\gamma] := e^{2\pi i q(\gamma)}[\gamma], \quad \rho_L(S)[\gamma] := \frac{1}{\sqrt{|L'/L|}} \sqrt{i}^{b^- - b^+} \sum_{\delta \in \gamma \in L'/L} e^{-2\pi i (\gamma, \delta)}[\delta].$$

Let ρ_L^* denote the dual representation of ρ_L.

Definition 4. Let $k \in \frac{1}{2}\mathbb{Z}$. The *slash operators* of weight k act on a function $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ by

$$(f|_k(M, \phi))(\tau) := \phi(\tau)^{-2k} \rho_L(M, \phi)^{-1} f(M\tau),$$

$$(f|_k^*(M, \phi))(\tau) := \phi(\tau)^{-2k} \rho_L^*(M, \phi)^{-1} f(M\tau).$$

Suppose that $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ is a meromorphic function that satisfies $f|_k(M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Write $f = \sum_{\gamma \in L'/L} f_\gamma [\gamma]$. Then, $f|_k T = f$ gives $e^{-2\pi i q(\gamma)} f_\gamma(\tau + 1) = f_\gamma(\tau)$, so $e^{-2\pi i q(\gamma)\tau} f_\gamma(\tau)$ is 1-periodic. We obtain a Fourier expansion of f at the cusp ∞

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} c(\gamma, n) e^{2\pi i n \tau} [\gamma].$$

Similarly, if $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ is a holomorphic function that satisfies $f|_\kappa^*(M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, then we obtain a Fourier expansion of f at ∞

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma)} c(\gamma, n) e^{2\pi i n \tau} [\gamma].$$

Definition 5. (1) A $\mathbb{C}[L'/L]$-valued *modular form* for $\mathrm{Mp}_2(\mathbb{Z})$ of weight κ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ with the following properties.

(i) $f|_\kappa^*(M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

(ii) $c(\gamma, n) = 0$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} - q(\gamma)$ with $n < 0$.

It is a *cusp form* if $c(\gamma, n) = 0$ for all γ and $n = 0$ as well. We denote the finite-dimensional $\mathbb{C}$-vector space of modular forms (resp. cusp forms) by $\mathcal{M}_{L,\kappa}$ (resp. $\mathcal{S}_{L,\kappa}$).

(2) A $\mathbb{C}[L'/L]$-valued *weakly modular form* for $\mathrm{Mp}_2(\mathbb{Z})$ of weight k is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ with the following properties.
   (i) $f|_k(M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$,
   (ii) $c(\gamma, n) = 0$ for almost all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$.

   We denote the $\mathbb{C}$-vector space of weakly holomorphic modular forms by $\mathcal{M}^!_{L,k}$.

From now on, we will only consider the case when $b^- - b^+ = \ell - 2$, $\kappa = 1 + \frac{\ell}{2}$, $k = 1 - \frac{\ell}{2}$ where $\ell \geq 3$ is a fixed integer. Most of the following discussion can be generalized to $\ell = 2$ as well (which is necessary in order to apply the theory to Hilbert modular surfaces), but it involves more technicalities.

2 Poincaré series

In this subsection, we introduce Poincaré series and show that they span $\mathcal{S}_\kappa$.

Suppose $b^- - b^+ = \ell - 2$ and $\kappa = 1 + \frac{\ell}{2}$ with $\ell \geq 3$. Fix $\beta \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$ with $m > 0$. We have

$$e^{2\pi i m\tau}[\beta]|^*_\kappa T = e^{2\pi i q(\beta)} e^{2\pi i m(\tau+1)}[\beta] = e^{2\pi i m\tau}[\beta].$$

On the other hand, we have

$$e^{2\pi i m\tau}[\beta]|^*_\kappa Z = i^{-2\kappa} i^{b^- - b^+} e^{2\pi i m\tau}[-\beta] = e^{2\pi i m\tau}[-\beta],$$

hence

$$e^{2\pi i m\tau}[\beta]|^*_\kappa Z^2 = e^{2\pi i m\tau}[\beta].$$

In other words, $e^{2\pi i m\tau}[\beta]$ is invariant under $|^*_\kappa T$ and $|^*_\kappa Z^2$.

Definition 6. The *Poincaré series* of index (β, m) is defined by

$$P^L_{\beta,m}(\tau) := \sum_{(M,\phi) \in \langle T, Z^2 \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} e^{2\pi i m\tau}[\beta]|^*_\kappa (M, \phi).$$

Let us compute the Fourier expansion of $P_{\beta,m}$. Write

$$P_{\beta,m}(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma)} p_{\beta,m}(\gamma, n) e^{2\pi i n\tau}[\gamma].$$

We have

$$\begin{aligned}
p_{\beta,m}(\gamma, n) &= \int_0^1 \langle [\gamma], P_{\beta,m}(\tau) \rangle e^{-2\pi i n\tau} dx \\
&= \sum_{M \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} \int_0^1 \left\langle [\gamma], e^{2\pi i m\tau}[\beta]|^*_\kappa \widetilde{M} \right\rangle e^{-2\pi i n\tau} dx \\
&= \int_0^1 \left\langle [\gamma], e^{2\pi i m\tau}[\beta] \right\rangle e^{-2\pi i n\tau} dx + \int_0^1 \left\langle [\gamma], e^{2\pi i m\tau}[-\beta] \right\rangle e^{-2\pi i n\tau} dx \\
&\quad + \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \atop c \neq 0} \int_0^1 \left\langle [\gamma], e^{2\pi i m\tau}[\beta]|^*_\kappa \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \right\rangle e^{-2\pi i n\tau} dx \\
&= (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) \delta_{m,n} \\
&\quad + \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) / \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \atop c \neq 0} \sum_{r \in \mathbb{Z}} \int_0^1 \left\langle [\gamma], e^{2\pi i m\tau}[\beta]|^*_\kappa \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} T^r \right\rangle e^{-2\pi i n\tau} dx
\end{aligned}$$

$$= (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})\delta_{m,n} + \sum_{c\neq 0}\sum_{d\in(\mathbb{Z}/c)^\times} \int_{-\infty}^{\infty} \left\langle [\gamma], e^{2\pi i m\tau}[\beta]|_\kappa^* \widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \right\rangle e^{-2\pi i n\tau} dx$$

$$= (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})\delta_{m,n} + \sum_{c\neq 0}\sum_{d\in(\mathbb{Z}/c)^\times} \int_{-\infty}^{\infty} \sqrt{c\tau+d}^{-2\kappa} \rho_{\beta,\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \exp\left(2\pi i\left(m\frac{a\tau+b}{c\tau+d} - n\tau\right)\right) dx,$$

where we define $\rho_{\beta,\gamma}(M,\phi) := \langle [\gamma], \rho_L(M,\phi)[\beta]\rangle$. Taking care of the branch cut, the inner sum can be rewritten as

$$\begin{aligned}
&\sum_{d\in(\mathbb{Z}/c)^\times} \int_{-\infty}^{\infty} \sqrt{c\tau+d}^{-2\kappa} \rho_{\beta,\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \exp\left(2\pi i\left(m\tfrac{a\tau+b}{c\tau+d} - n\tau\right)\right) dx \\
&= \operatorname{sgn}(c)^\kappa |c|^{-\kappa} \sum_{d\in(\mathbb{Z}/c)^\times} \rho_{\beta,\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \int_{-\infty}^{\infty} \left(\tau+\tfrac{d}{c}\right)^{-\kappa} \exp\left(2\pi i\left(\tfrac{ma}{c} - \tfrac{m}{c^2(\tau+d/c)} - n\tau\right)\right) dx \\
&= \operatorname{sgn}(c)^\kappa |c|^{-\kappa} \sum_{d\in(\mathbb{Z}/c)^\times} \exp\left(2\pi i\tfrac{ma+nd}{c}\right) \rho_{\beta,\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \int_{-\infty}^{\infty} \tau^{-\kappa} \exp\left(2\pi i\left(\tfrac{m}{c^2\tau} - n\tau\right)\right) dx \\
&= 2\pi|c|^{1-\kappa} \left(i^{-\operatorname{sgn}(c)\kappa}|c|^{-1} \sum_{d\in(\mathbb{Z}/c)^\times} \exp\left(2\pi i\tfrac{ma+nd}{c}\right) \rho_{\beta,\gamma}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} \right) \\
&\qquad \times \left(\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+\infty} z^{-\kappa} \exp\left(2\pi\left(-\tfrac{m}{c^2 z} + nz\right)\right) dz \right) \\
&= 2\pi|c|^{1-\kappa} H^*(\beta,m,\gamma,n;c) \left(\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+\infty} z^{-\kappa} \exp\left(2\pi\left(-\tfrac{m}{c^2 z} + nz\right)\right) dz \right),
\end{aligned}$$

where H^* is the generalized Kloosterman sum. The integral is independent of the constant $\alpha > 0$. If $n \leq 0$, letting $\alpha \to 0$ shows that the integral is 0. If $n > 0$, one can compute the inverse Laplace transform to evaluate

$$\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+\infty} z^{-\kappa} \exp\left(2\pi\left(-\tfrac{m}{c^2 z} + nz\right)\right) dz = \left(\frac{c^2 n}{m}\right)^{(\kappa-1)/2} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where J_ν is the Bessel J-function.

Theorem 7. The Fourier coefficients of $P_{\beta,m}$ are

$$p_{\beta,m}(\gamma,n) = \begin{cases} (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})\delta_{m,n} + 2\pi\left(\frac{n}{m}\right)^{(\kappa-1)/2} \sum_{c\neq 0} H^*(\beta,m,\gamma,n;c) J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{|c|}\right) & \text{if } n > 0, \\ 0 & \text{if } n \leq 0. \end{cases}$$

Corollary 8. $P_{\beta,m} \in \mathcal{S}_\kappa$.

Next, we show that $P_{\beta,m}$ span $\mathcal{S}_\kappa$. We introduce the Petersson inner product on $\mathcal{S}_\kappa$ defined by

$$(f,g) := \int_{\mathcal{F}} \langle f(\tau), g(\tau)\rangle y^\kappa \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$.

Proposition 9. Let $f \in \mathcal{S}_\kappa$ with Fourier coefficients $c(\gamma,n)$. Then,

$$(P_{\beta,m}, f) = \frac{2\Gamma(\kappa-1)}{(4\pi m)^{\kappa-1}} c(\beta,m).$$

Proof. We have

$$\begin{aligned}
(P_{\beta,m}, f) &= \int_{\mathcal{F}} \langle P_{\beta,m}(\tau), f(\tau)\rangle y^\kappa \frac{dxdy}{y^2} \\
&= \sum_{(M,\phi)\in\langle T,Z^2\rangle\backslash \mathrm{Mp}_2(\mathbb{Z})} \int_{\mathcal{F}} \langle e^{2\pi i m\tau}[\beta]|_\kappa^*(M,\phi), f(\tau)\rangle y^\kappa \frac{dxdy}{y^2} \\
&= \sum_{(M,\phi)} \int_{\mathcal{F}} \langle e^{2\pi i m\tau}[\beta]|_\kappa^*(M,\phi), f(\tau)|_\kappa^*\rangle y^\kappa \frac{dxdy}{y^2} \qquad (\because f \in \mathcal{S}_\kappa)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(M,\phi)} \int_{\mathcal{F}} \langle \rho_L^*(M,\phi)^{-1} e^{2\pi i m M\tau}[\beta], \rho_L^*(M,\phi)^{-1} f(M\tau)\rangle \Im(M\tau)^\kappa \frac{dxdy}{y^2} \\
&= \sum_{(M,\phi)} \int_{\mathcal{F}} \langle e^{2\pi i m M\tau}[\beta], f(M\tau)\rangle \Im(M\tau)^\kappa \frac{dxdy}{y^2} \qquad (\because \rho_L^* \text{ unitary}) \\
&= 2\int_{\mathcal{G}} \langle e^{2\pi i m\tau}[\beta], f(\tau)\rangle y^\kappa \frac{dxdy}{y^2} \\
&= 2\int_0^\infty \left(\int_0^1 \langle [\beta], f(\tau)\rangle e^{-2\pi i m x} dx \right) e^{-2\pi m y} y^{\kappa-2} dy \\
&= 2\int_0^\infty c(\beta,m) e^{-4\pi m y} y^{k-2} dy \\
&= \frac{2\Gamma(\kappa-1)}{(4\pi m)^{\kappa-1}} c(\beta,m).
\end{aligned}
$$

Here, $\mathcal{G}$ means a fundamental domain for $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathfrak{H}$. □

Corollary 10. $\mathcal{S}_\kappa$ is spanned by all $P_{\beta,m}$.

Proof. The dual space $\mathcal{S}_\kappa^*$ is spanned by the linear functionals $\alpha_{\beta,m} \in \mathcal{S}_\kappa^*$ for $\beta \in L'/L$ and $m \in \mathbb{Z} - q(\beta)$ with $m > 0$ that maps $f \in \mathcal{S}_\kappa$ to its Fourier coefficient of degree (β, m). □

3 Eisenstein series

Fix $\beta \in L'/L$ such that $q(\beta) \in \mathbb{Z}$ and set $m = 0$.

Definition 11. The *Eisenstein series* of index β is defined by

$$
E_\beta^L(\tau) := \sum_{(M,\phi) \in \langle T, Z^2 \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} [\beta]|_\kappa^*(M,\phi).
$$

The computation of the Fourier expansion of E_β is almost the same as that of Poincaré series. The only difference is the evaluation of the inverse Laplace transform.

Theorem 12. The Fourier coefficients of E_β are

$$
p_{\beta,m}(\gamma,n) = \begin{cases} \frac{(2\pi)^\kappa n^{\kappa-1}}{\Gamma(\kappa)} \sum_{c\neq 0} |c|^{1-\kappa} H^*(\beta,0,\gamma,n;c) & \text{if } n > 0, \\ \delta_{\beta,\gamma} + \delta_{-\beta,\gamma} & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}
$$

Corollary 13. (1) $E_\beta \in \mathcal{M}_\kappa$ but $E_\beta \notin \mathcal{M}_\kappa$.
(2) Let B denote the orbit space of $\{\pm 1\}\backslash\{\beta \in L'/L \mid q(\beta) \in \mathbb{Z}\}$. Then,

$$
\mathcal{M}_\kappa = \mathcal{S}_\kappa \oplus \mathrm{span}\{E_\beta \mid \beta \in B\}.
$$

4 Non-holomorphic Poincaré series

In this subsection, we introduce non-holomorphic Poincaré series, whose linear combinations can express any weakly holomorphic modular form. These Poincaré series will also be inputs of the regularized theta lifting.

Suppose $b^- - b^+ = \ell - 2$ and $k = 1 - \frac{\ell}{2}$ with $\ell \geq 3$. Fix $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$.

Definition 14. The *hyperbolic Laplacian* of weight k is

$$
\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right).
$$

Denote the Whittaker M-function (resp. W-function) by M_ν (resp. W_ν). For simplicity, we define functions

$$\mathcal{M}_s(y) := y^{-k/2} M_{-k/2,s-1/2}(y) \text{ for } y > 0,$$

$$\mathcal{W}_s(y) := |y|^{-k/2} W_{-k/2,s-1/2}(|y|) \text{ for } y \neq 0.$$

Then, $\mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]$ is an eigenfunction of Δ_k of eigenvalue $s(1-s) - \frac{k}{2}\left(1-\frac{k}{2}\right)$. We also have

$$\mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]|_\kappa^* T = e^{-2\pi i q(\beta)} \mathcal{M}_s(4\pi|m|y)e^{2\pi im(x+1)}[\beta] = \mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]$$

while

$$\mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]|_\kappa^* Z = i^{-2k} i^{b^+ - b^-} \mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[-\beta] = \mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[-\beta],$$

so that

$$\mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]|_\kappa^* Z^2 = \mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta].$$

Definition 15. The non-holomorphic *Poincaré series* of index (β, m) is defined for $\Re(s) > 1$ by

$$F_{\beta,m}^L(\tau) := \frac{1}{\Gamma(2s)} \sum_{(M,\phi) \in \langle T, Z^2 \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathcal{M}_s(4\pi|m|y)e^{2\pi imx}[\beta]|_\kappa^*(M,\phi).$$

Let us compute the Fourier-Whittaker expansion of $F_{\beta,m}$. Write

$$F_{\beta,m}(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} a_{\beta,m}(\gamma,n,s,y) e^{2\pi inx}[\gamma].$$

We have

$$\begin{aligned}
a_{\beta,m}(\gamma,n,s,y) &= \frac{1}{\Gamma(2s)} \int_0^1 \langle [\gamma], F_{\beta,m}(\tau)\rangle e^{-2\pi inx} dx \\
&= \frac{1}{\Gamma(2s)} (\delta_{\beta,\gamma} + \delta_{-\beta,\gamma}) \delta_{m,n} \mathcal{M}_s(4\pi|m|y) \\
&\quad + \frac{1}{\Gamma(2s)} \sum_{c \neq 0} \sum_{d \in (\mathbb{Z}/c)^\times} \int_{-\infty}^{\infty} \sqrt{c\tau+d}^{-2k} \overline{\rho_{\gamma,\beta}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \\
&\qquad\qquad \times \mathcal{M}_s\left(4\pi|m|\Im\left(\frac{a\tau+b}{c\tau+d}\right)\right) \exp\left(2\pi i\left(m\Re\left(\frac{a\tau+b}{c\tau+d}\right) - nx\right)\right) dx.
\end{aligned}$$

The inner sum is

$$\begin{aligned}
&\sum_{d \in (\mathbb{Z}/c)^\times} \int_{-\infty}^{\infty} \sqrt{c\tau+d}^{-2k} \overline{\rho_{\gamma,\beta}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \mathcal{M}_s\left(4\pi|m|\Im\left(\tfrac{a\tau+b}{c\tau+d}\right)\right) \exp\left(2\pi i\left(m\Re\left(\tfrac{a\tau+b}{c\tau+d}\right) - nx\right)\right) dx \\
&= \mathrm{sgn}(c)^k |c|^{-k} \sum_{d \in (\mathbb{Z}/c)^\times} \overline{\rho_{\gamma,\beta}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \int_{-\infty}^{\infty} \left(\tau + \tfrac{d}{c}\right)^{-k} \mathcal{M}_s\left(\tfrac{4\pi|m|y}{|c\tau+d|^2}\right) \exp\left(2\pi i\left(\tfrac{ma}{c} - \tfrac{m(x+d/c)}{c^2|\tau+d/c|^2} - nx\right)\right) dx \\
&= i^k |c|^{1-k} \left(i^{-\mathrm{sgn}(c)k} |c|^{-1} \sum_{d \in (\mathbb{Z}/c)^\times} \exp\left(2\pi i \tfrac{ma+nd}{c}\right) \overline{\rho_{\gamma,\beta}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \right) \\
&\qquad \times \left(\int_{-\infty}^{\infty} \tau^{-k} \left(\tfrac{4\pi|m|y}{c^2|\tau|^2}\right)^{-k/2} M_{-k/2,s-1/2}\left(\tfrac{4\pi|m|y}{c^2|\tau|^2}\right) \exp\left(2\pi i\left(\tfrac{ma}{c} - \tfrac{mx}{c^2|\tau|^2} - nx\right)\right) dx \right) \\
&= i^k (4\pi|m|y)^{-k/2} |c| H(\beta,m,\gamma,n;c) \\
&\qquad \times \left(\int_{-\infty}^{\infty} \left(-\tfrac{\tau}{\bar\tau}\right)^{-k/2} M_{-k/2,s-1/2}\left(\tfrac{4\pi|m|y}{c^2|\tau|^2}\right) \exp\left(2\pi i\left(\tfrac{ma}{c} - \tfrac{mx}{c^2|\tau|^2} - nx\right)\right) dx \right),
\end{aligned}$$

where H is the generalized Kloosterman sum. Using Hejhal's evaluation of the integral, we obtain

$$a_{\beta,m}(\gamma,n,s,y)=\begin{cases}\frac{2\pi|n/m|^{(k-1)/2}}{\Gamma(s-k/2)}\sum_{c\neq 0}H(\beta,m,\gamma,n;c)I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{|c|}\right)\mathcal{W}_s(4\pi ny) & \text{if } n>0,\\ \frac{4^{1-k/2}\pi^{1+s-k/2}|m|^{s-k/2}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}\sum_{c\neq 0}|c|^{1-2s}H(\beta,m,\gamma,0;c)y^{1-s-k/2} & \text{if } n=0,\\ \frac{1}{\Gamma(2s)}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma})\delta_{m,n}\mathcal{M}_s(4\pi|m|y) & \\ +\frac{2\pi|n/m|^{(k-1)/2}}{\Gamma(s+k/2)}\sum_{c\neq 0}H(\beta,m,\gamma,n;c)J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{|c|}\right)\mathcal{W}_s(4\pi ny) & \text{if } n<0,\end{cases}$$

where J_ν (resp. I_ν) is the Bessel J-function (resp. I-function).

Theorem 16. The Fourier-Whittaker expansion of $F_{\beta,m}$ is

$$\begin{aligned}F_{\beta,m}(\tau,s)=&\frac{1}{\Gamma(2s)}\mathcal{M}_s(4\pi|m|y)e^{2\pi imx}([\beta]+[-\beta])\\&+\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\neq 0}}\widetilde{b}_{\beta,m}(\gamma,n,s)\mathcal{W}_s(4\pi ny)e^{2\pi inx}[\gamma]\\&+\sum_{\substack{\gamma\in L'/L\\ q(\gamma)\in\mathbb{Z}}}\widetilde{b}_{\beta,m}(\gamma,0,s)y^{1-s-k/2}[\gamma],\end{aligned}$$

where

$$\widetilde{b}_{\beta,m}(\gamma,n,s):=\begin{cases}\frac{2\pi|n/m|^{(k-1)/2}}{\Gamma(s-k/2)}\sum_{c\neq 0}H(\beta,m,\gamma,n;c)I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{|c|}\right) & \text{if } n>0,\\ \frac{4^{1-k/2}\pi^{1+s-k/2}|m|^{s-k/2}}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}\sum_{c\neq 0}|c|^{1-2s}H(\beta,m,\gamma,0;c) & \text{if } n=0,\\ \frac{2\pi|n/m|^{(k-1)/2}}{\Gamma(s+k/2)}\sum_{c\neq 0}H(\beta,m,\gamma,n;c)J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{|c|}\right) & \text{if } n<0.\end{cases}$$

Using the fact that

$$\mathcal{M}_s(|y|)=\frac{\Gamma(1+k/2-s)}{\Gamma(1-2s)}\mathcal{W}_s(y)-\frac{\Gamma(1+k/2-s)\Gamma(2s-1)}{\Gamma(1-2s)\Gamma(s+k/2)}\mathcal{M}_{1-s}(|y|),$$

we can slightly reformulate the theorem.

Corollary 17.

$$\begin{aligned}F_{\beta,m}(\tau,s)=&\frac{\Gamma(1+k/2-s)}{\Gamma(2-2s)\Gamma(s+k/2)}\mathcal{M}_{1-s}(4\pi|m|y)e^{2\pi imx}([\beta]+[-\beta])\\&+\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\neq 0}}b_{\beta,m}(\gamma,n,s)\mathcal{W}_s(4\pi ny)e^{2\pi inx}[\gamma]\\&+\sum_{\substack{\gamma\in L'/L\\ q(\gamma)\in\mathbb{Z}}}b_{\beta,m}(\gamma,0,s)y^{1-s-k/2}[\gamma],\end{aligned}$$

where

$$b_{\beta,m}(\gamma,n,s):=\begin{cases}\widetilde{b}_{\beta,m}(\gamma,n,s) & \text{if } n\geq 0,\\ \frac{\Gamma(1+k/2-s)}{\Gamma(2s)\Gamma(1-2s)}(\delta_{\beta,\gamma}+\delta_{-\beta,\gamma})\delta_{m,n}+\widetilde{b}_{\beta,m}(\gamma,n,s) & \text{if } n<0.\end{cases}$$

In the special case of $s=1-\frac{k}{2}$ when $F_{\beta,m}$ is harmonic, we obtain the following expansion.

Corollary 18.

$$\begin{aligned}F_{\beta,m}(\tau,1-\tfrac{k}{2})=&e^{2\pi im\tau}([\beta]+[-\beta])+\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\geq 0}}b_{\beta,m}(\gamma,n)e^{2\pi in\tau}[\gamma]\\&+\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}}b_{\beta,m}(\gamma,n)\mathcal{W}_{1-k/2}(4\pi ny)e^{2\pi inx}[\gamma],\end{aligned}$$

where

$$b_{\beta,m}(\gamma,n):=b_{\beta,m}(\gamma,n,1-\frac{k}{2})$$

$$= \begin{cases} 2\pi \left|\frac{n}{m}\right|^{(k-1)/2} \sum_{c\neq 0} H(\beta, m, \gamma, n; c) I_{1-k}\left(\frac{4\pi\sqrt{mn}}{|c|}\right) & \text{if } n > 0, \\ \frac{(2\pi)^{2-k}|m|^{1-k}}{\Gamma(2-k)} \sum_{c\neq 0} |c|^{1-2s} H(\beta, m, \gamma, 0; c) & \text{if } n = 0, \\ \frac{-1}{\Gamma(1-k)}(\delta_{\beta,\gamma} + \delta_{-\beta,\gamma})\delta_{m,n} & \\ \frac{2\pi}{\Gamma(1-k)} \left|\frac{n}{m}\right|^{(k-1)/2} \sum_{c\neq 0} H(\beta, m, \gamma, n; c) J_{1-k}\left(\frac{4\pi\sqrt{mn}}{|c|}\right) & \text{if } n < 0. \end{cases}$$

The first two terms in the expansion are holomorphic while the last term is anti-holomorphic.

While $F_{\beta,m}(-, 1-\frac{k}{2})$ itself is not holomorphic, any $f \in \mathcal{M}_k^!$ can be expressed as a linear comination of $F_{\beta,m}(-, 1-\frac{k}{2})$.

Proposition 19. Let $f \in \mathcal{M}_k^!$ with Fourier coefficients $c(\gamma, n)$. Then,

$$f(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) F_{\beta,m}(\tau, 1-\frac{k}{2}).$$

Proof. Let $g(\tau)$ denote the difference between the left-hand side and the right-hand side.

- $\Delta_k g = 0$.
- $g|_k(M, \phi) = g$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.
- $f|_k Z = f$ gives $c(\gamma, n) = c(-\gamma, n)$. Then by Corollary 3.17, $g(\tau) \to 0$ as $y \to \infty$.

These conditions force g to become identically zero. □

Theorem 20. There exists $f \in \mathcal{M}_k^!$ with prescribed principal part

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) e^{2\pi i m\tau}[\beta]$$

if and only if

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) \alpha_{\beta,-m}$$

in the dual space $\mathcal{S}_\kappa^*$, where $\alpha_{\beta,-m} \in \mathcal{S}_\kappa^*$ is the linear functional that maps $g \in \mathcal{S}_\kappa$ to its Fourier coefficient of degree $(\beta, -m)$.

Proof. ($\Rightarrow$) Suppose such $f \in \mathcal{M}_k^!$ exists. By 19, we have

$$f(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) F_{\beta,m}(\tau, 1-\frac{k}{2}).$$

Applying $\frac{\partial}{\partial \overline{\tau}}$ to both sides, the holomorphic terms in Corollary 3.17 disappear and we obtain

$$\sum_{\beta,m} c(\beta, m) b_{\beta,m}(\gamma, n) = 0$$

for each $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Now, observe that the two types of generalized Kloosterman sums are related by $H(\beta, m, \gamma, n) = -H^*(\gamma, -n, \beta, -m; -c)$. This implies $b_{\beta,m}(\gamma, n) = p_{\gamma,-n}(\beta, -m)$, hence

$$\sum_{\beta,m} c(\beta, m) p_{\gamma,-n}(\beta, -m) = 0.$$

We can rewrite this as

$$\sum_{\beta,m} c(\beta, m) \alpha_{\beta,-m}(P_{\gamma,-n}) = 0.$$

Since $P_{\gamma,-n}$ span $\mathcal{S}_\kappa$ by Corollary 3.7, we are done.

($\Leftarrow$) Reverse the argument. □

5 Regularized theta lifts

Definition 21. The *positive Grassmannian* is the smooth manifold

$$\mathrm{Gr}(L) := \{v \text{ subspace of } V \text{ of dimension } b^+ \mid q \text{ positive definite on } v\}.$$

For $v \in \mathrm{Gr}(L)$ and $x \in V$, we denote the orthogonal projections of x to v (resp. $v^\perp$) by x_v (resp. $x_{v^\perp}$). Note that $q(x) = q(x_v) + q(x_{v^\perp})$.

Definition 22. (1) The *Siegel theta function* $\theta_L : \mathfrak{H} \times \mathrm{Gr}(L) \to \mathbb{C}$ is defined by

$$\theta_L(\tau, v) := \sum_{\lambda \in L} \exp(2\pi i(\tau q(\lambda_v) + \overline{\tau} q(\lambda_{v^\perp}))).$$

(2) More generally, given $\gamma \in L'/L$ and $r, t \in V$, the *Borcherds theta function* $\theta_{L,\gamma}(-,-;r,t) : \mathfrak{H} \times \mathrm{Gr}(L) \to \mathbb{C}$ is defined by

$$\theta_{L,\gamma}(\tau, v; r, t) := \sum_{\lambda \in \gamma + L} \exp(2\pi i(\tau q((\lambda + t)_v) + \overline{\tau} q((\lambda + t)_{v^\perp}) - (\lambda + t/2, r))).$$

We denote $\theta_{L,\gamma}(\tau, v; 0, 0)$ by $\theta_{L,\gamma}(\tau, v)$.

(3) Given $r, t \in V$, we define a function $\Theta_{L,\gamma}(-,-;r,t) : \mathfrak{H} \times \mathrm{Gr}(L) \to \mathbb{C}[L'/L]$ by

$$\Theta_L(\tau, v; r, t) := \sum_{\gamma \in L'/L} \theta_{L,\gamma}(\tau, v; r, t)[\gamma].$$

We denote $\Theta_L(\tau, v; 0, 0)$ by $\Theta_L(\tau, v)$.

Using the Poisson summation formula, one can prove the transformation law of Θ_L.

Theorem 23 (Borcherds)**.** For $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$\Theta(M\tau, v; ar + bt, cr + dt) = \phi(\tau)^{b^+} \overline{\phi(\tau)}^{b^-} \rho_L(M, \phi)\Theta(\tau, v; r, t).$$

In particular, this implies that $\langle \Theta_L(\tau, v), F_{\beta,m}(\tau, s)\rangle y^{b^+/2}$ is $\mathrm{Mp}_2(\mathbb{Z})$-invariant. We want to study the theta integral

$$\int_{\mathcal{F}} \langle \Theta_L(\tau, v), F_{\beta,m}(\tau, s)\rangle y^{b^+/2} \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. However, $F_{\beta,m}$ grows exponentially as $y \to \infty$, so we need to regularize the integral accordingly to make it converge.

Definition 24. Let $m \in \mathbb{Z} - q(\beta)$ with $m < 0$. The *regularized theta integral* is defined by

$$\Phi^L_{\beta,m}(v, s) := \lim_{u \to \infty} \int_{y \le u} \left(\int_{\{x \mid (x,y) \in \mathcal{F}\}} \langle \Theta_L(\tau, v), F_{\beta,m}(\tau, s)\rangle dx \right) y^{b^+/2-2} dy.$$

Definition 25. Let $m \in \mathbb{Z} - q(\beta)$ with $m < 0$. If $b^+ \ge 1$, we set

$$H(\beta, m) := \bigcup_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} \{v \in \mathrm{Gr}(L) \mid q(\lambda_v) = 0\} \subseteq \mathrm{Gr}(L).$$

If $b^+ = 0$, we set $H(\beta, m) := \varnothing \subseteq \mathrm{Gr}(L)$.

Theorem 26. (1) $\Phi_{\beta,m}(v, s)$ converges when $v \in \mathrm{Gr}(L) - H(\beta, m)$ and $\Re(s) > \sigma_0 := \max(\frac{b^+ - k}{2}, 1)$.

(2) For a fixed $v \in \mathrm{Gr}(L)$, $\Phi_{\beta,m}$ admits a meromorphic continuation to $\Re(s) > 1$ with a simple pole of residue $b_{\beta,m}(0, 0, \frac{b^+ - k}{2})$ at $s = \frac{b^+ - k}{2}$.

Proof. By 16, we have

$$\int_1^\infty \left(\int_0^1 \langle \Theta(\tau, v), F_{\beta,m}(\tau, s)\rangle dx \right) y^{b^+/2-2} dy$$

$$
\begin{aligned}
&= \sum_{\lambda\in L'}\int_1^\infty \sum_{\gamma\in L'/L}\left(\int_0^1 \langle[\gamma], F_{\beta,m}(\tau,s)\rangle e^{-2\pi i q(\lambda)x}dx\right)\exp(-2\pi q(\lambda_v)y+2\pi q(\lambda_{v^\perp})y)y^{b^+/2-2}dy\\
&= \int_1^\infty \sum_{\lambda\in L'} a_{\beta,m}(\gamma,q(\gamma),y,s)\exp(-2\pi q(\lambda_v)y+2\pi q(\lambda_{v^\perp})y)y^{b^+/2-2}dy\\
&= b_{\beta,m}(0,0,s)\int_1^\infty y^{b^+/2-k/2-s-1}dy\\
&\quad+\frac{2}{\Gamma(2s)}\int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)=m}} \exp(-4\pi q(\lambda_v)y+2\pi my)\mathcal{M}_s(4\pi|m|y)y^{b^+/2-2}dy\\
&\quad+\int_1^\infty \sum_{\substack{0\neq\lambda\in L'\\ q(\lambda)\neq 0}} \widetilde{b}_{\beta,m}(\lambda,q(\lambda),s)\exp(-2\pi q(\lambda_v)y+2\pi q(\lambda_{v^\perp})y)\mathcal{W}_s(4\pi q(\lambda)y)y^{b^+/2-2}dy\\
&\quad+\int_1^\infty \sum_{\substack{0\neq\lambda\in L'\\ q(\lambda)=0}} b_{\beta,m}(\lambda,0,s)\exp(-4\pi q(\lambda_v))y^{b^+/2-k/2-s-1}dy.
\end{aligned}
$$

The first term converges when $\Re(s)>\frac{b^+-k}{2}$ but can be continued meromorphically to all s with a simple pole at $s=\frac{b^+-k}{2}$ of residue $b_{\beta,m}(0,0,\frac{b^+-k}{2})$. The remaining terms converge absolutely uniformly on compacta when $v\in \mathrm{Gr}(L)-H(\beta,m)$ and $\Re(s)>1$. □

Definition 27. For $v\in \mathrm{Gr}(L)-H(\beta,m)$, we define $\Phi^L_{\beta,m}(v)$ to be the constant term of the Laurent expansion of $\Phi^L_{\beta,m}(v,s)$ at $s=1-\frac{k}{2}$. (Note that there is a pole here only when $b^+=2$.)

Theorem 28. $\Phi_{\beta,m}$ is real analytic on $\mathrm{Gr}(L)-H(\beta,m)$. Furthermore, when $b^+=1$ or 2, $\Phi_{\beta,m}$ has a singularity along $H(\beta,m)$ locally on U of type

$$
\begin{cases}
-4\sqrt{2}\pi\sum_{\lambda\in S(\beta,m,U)}(\lambda_v,\lambda_v)^{1/2} & \text{if } b^+=1,\\
-2\sum_{\lambda\in S(\beta,m,U)}\log(\lambda_v,\lambda_v) & \text{if } b^+=2,
\end{cases}
$$

where $U\subseteq \mathrm{Gr}(L)$ is an open set with compact clousre and $S(\beta,m,U):=\{\lambda\in\beta+L\mid q(\lambda)=m \text{ and } \exists v\in U\ q(\lambda_v)=0\}$ is a finite set.

Proof. Proceed as in the proof of 26 but use 17 instead of 16.

$$
\begin{aligned}
\Phi_{\beta,m}(v) &= 2\int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)=m}} \exp(-4\pi q(\lambda_v)y+2\pi my)\mathcal{M}_{k/2}(4\pi|m|y)y^{b^+/2-2}dy\\
&\quad+\int_1^\infty \sum_{\substack{0\neq\lambda\in L'\\ q(\lambda)\neq 0}} b_{\beta,m}(\lambda,q(\lambda),s)\exp(-2\pi q(\lambda_v)y+2\pi q(\lambda_{v^\perp})y)\mathcal{W}_{1-k/2}(4\pi q(\lambda)y)y^{b^+/2-2}dy\\
&\quad+\int_1^\infty \sum_{\substack{0\neq\lambda\in L'\\ q(\lambda)=0}} b_{\beta,m}(\lambda,0,s)\exp(-4\pi q(\lambda_v))y^{b^+/2-2}dy+(\text{real analytic function})
\end{aligned}
$$

The singularity is caused by the first term. Since $\mathcal{M}_{k/2}(y)=e^{y/2}$, we have

$$
\begin{aligned}
&2\int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)=m}} \exp(-4\pi q(\lambda_v)y+2\pi my)\mathcal{M}_{k/2}(4\pi|m|y)y^{b^+/2-2}dy\\
&= 2\int_1^\infty \sum_{\substack{\lambda\in L'\\ q(\lambda)=m}} \exp(-4\pi q(\lambda_v)y)y^{b^+/2-2}dy\\
&= 2\sum_{\lambda\in S(\beta,m,U)}(4\pi q(\lambda-v))^{1-b^+/2}\Gamma(b^+/2-1,4\pi q(\lambda_v))+(\text{real analytic function})
\end{aligned}
$$

on U, where $\Gamma(a,x) := \int_x^\infty e^{-t}t^a\frac{dt}{t}$ is the incomplete Gamma function. The singularity type follows from the asymptotics of $\Gamma(-1/2,x)$ (resp. $\Gamma(0,x)$) near $x=0$. □

We introduce notations $x^2 := (x,x)$ and $|x| := |(x,x)|^{1/2}$ for $x \in V$.

We now derive expressions for $\Phi_{\beta,m}$ when $b^+ = 1$ or 2. Fix a primitive isotropic vector $z \in L$ (i.e. $q(z)=0$ and $\mathbb{Q}z \cap L = \mathbb{Z}z$). Fix $z' \in L'$ such that $(z,z') = 1$ and set $K := L \cap z^\perp \cap z'^\perp$, which is a lattice of signature (b^+-1, b^-1). For $x \in V$, denote the orthogonal projection of x to $K\otimes\mathbb{R}$ by x_K. Write $(z,L) = N\mathbb{Z}$ and fix $\zeta \in L$ such that $(z,\zeta) = N$. One can check that $L = K \oplus \mathbb{Z}z \oplus \mathbb{Z}\zeta$, and that there is a surjection

$$\begin{aligned} p : L_0' := \{\lambda \in L' \mid (z,\lambda) \in N\mathbb{Z}\} &\to K' \\ \lambda &\mapsto \left(\lambda - \frac{(z,\lambda)}{N}\zeta\right)_K \end{aligned}$$

that maps L onto K. Applying the Poission summation formula only to the ζ-direction, Borcherds reduced theta functions for L to theta functions for K.

Lemma 29 (Borcherds)**.** We have

$$\theta_{L,\gamma}(\tau,v) = \frac{1}{\sqrt{2yz_v^2}} \sum_{\substack{c,d\in\mathbb{Z} \\ c\equiv(\gamma,z)\ (N)}} \exp\left(2\pi i\left(-\frac{|c\tau+d|^2}{4iyz_v^2} - d(\gamma,z') + cdq(z')\right)\right)\theta_{K,p(\gamma-cz')}(\tau,w;d\mu,-c\mu),$$

where w (resp. $w^\perp$) is the orthogonal complement of z_v in v (resp. $z_{v^\perp}$ in $v^\perp$) and

$$\mu := -z' + \frac{z_v}{2z_v^2} + \frac{z_{v^\perp}}{2z_{v^\perp}^2}.$$

Substituting this into the definition of $\Phi^L_{\beta,m}(v,s)$, we obtain the following theorem.

Theorem 30. [Borcherds] If $z_v^2 < \frac{1}{4|m|}$, then

$$\begin{aligned} \Phi^L_{\beta,m}(v,s) = {} & \frac{1}{\sqrt{2}|z_v|}\Phi^K_{\beta,m}(w,s) \\ & + \frac{2}{\sqrt{\pi}}\left(\frac{2z_v^2}{\pi}\right)^{s-b^+/4-b^-/4}\Gamma\left(s+\frac{1}{2}-\frac{b^+}{4}-\frac{b^-}{4}\right) \\ & \quad \times \sum_{r\in\mathbb{Z}/N} b_{\beta,m}\left(\frac{rz}{N},0,s\right)\sum_{n\geq 1} e^{2\pi irn/N}n^{b^+/2+b^-/2-2s-1} \\ & + \frac{\sqrt{2}}{|z_v|}\sum_{0\neq\lambda\in K'}\sum_{n\geq 1} e^{2\pi i(\lambda,\mu)n}\sum_{\substack{\delta\in L_0'/L \\ p(\delta)=\lambda+K}} e^{2\pi i(\delta,z')n} \\ & \quad \times \int_0^\infty a_{\beta,m}(\delta,q(\lambda),y,s)\exp\left(-\frac{\pi n^2}{2yz_v^2} - 2\pi q(\lambda_w)y + 2\pi q(\lambda_{w^\perp})y\right)y^{b^+/2-5/2}dy, \end{aligned}$$

where

$$\Phi^K_{\beta,m}(w,s) := \begin{cases} \Phi^K_{p(\beta),m} & \text{if } (\beta,z)\equiv 0\ (N), \\ 0 & \text{if } (\beta,z)\not\equiv 0\ (N). \end{cases}$$

First, consider the case $(b^+,b^-) = (1,\ell-1)$.

Definition 31. (1) We identify $\mathrm{Gr}(L)$ with the *hyperboloid model*

$$\begin{aligned} V_1 := \{v_1 \in V \mid (v_1,v_1) = 1, (z,v_1) = 0\} &\to \mathrm{Gr}(L) \\ v_1 &\mapsto \mathbb{R}v_1. \end{aligned}$$

(2) A *Weyl chamber* of index (β,m) is a connected component of $V_1 - H(\beta,m)$.

Definition 32. (1) On V_1, we define a real analytic function

$$\xi^L_{\beta,m}(v_1) := \frac{\Phi^K_{\beta,m}}{\sqrt{2}}\left(\frac{1}{(z,v_1)} - 2(z',v_1)\right) + \frac{4\sqrt{2}\pi}{(z,v_1)} \sum_{\substack{\lambda\in p(\beta)+K\\ q(\lambda)=m}} (\lambda,v_1)^2$$
$$+ 4\sqrt{2}\left(\frac{\pi}{(z,v_1)}\right)^{-k} \sum_{0\neq\lambda\in K'} \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e^{2\pi i(\delta,z')n} b_{\beta,m}(\delta,q(\lambda))|\lambda|^{1-k}$$
$$\times \sum_{n\geq 1} n^{-k-1}\exp\left(2\pi i\left(\frac{(\lambda,v_1)}{(z,v_1)} + (\delta,z')\right)n\right) K_{1-k}\left(\frac{2\pi n|\lambda|}{(z,v_1)}\right),$$

where K_ν is the Bessel K-function.

(2) On $V_1 - H(\beta,m)$, we define a real analytic function $\psi^L_{\beta,m}(v_1) := \Phi^L_{\beta,m}(v_1) - \xi^L_{\beta,m}(v_1)$.

Using 30, one can verify that $\psi^L_{\beta,m}$ on $\{(z,v_1)^2 < \frac{1}{4|m|}\}$ is locally given by a linear combination of (z,v_1), (z',v_1), (λ,v_1). By analyticity, $\xi^L_{\beta,m}$ is linear on each Weyl chamber that intersects $\{(z,v_1)^2 < \frac{1}{4|m|}\}$. By 28, this linearity result propagates, and $\xi^L_{\beta,m}$ is linear on every Weyl chamber.

Definition 33. The *Weyl vector* for a Weyl chamber W is the unique vector $\rho_{\beta,m}(W) \in V$ that satisfies $\psi^L_{\beta,m}(v_1) = 8\sqrt{2}\pi(\rho^L_{\beta,m}(W), v_1)$ on W.

Theorem 34. Consider a linear combination $\xi(v_1) = \sum_{\beta,m} c(\beta,m)\xi^L_{\beta,m}$. If ξ is a rational function in v_1, then ξ is identically zero.

Proof. (idea) Express $\Phi^L_{\beta,m}$ with respect to two different choices of a primitive isotropic vector $z \in L$ and compare the formulas to deduce that $\Phi^{\beta,m}_L$ itself must be piecewise linear. □

Next, consider the case $(b^+,b^-) = (2,\ell)$.

Definition 35. Let $\mathfrak{H}_\ell$ be one of the two connected components of $\{Z = X + iY \in K\otimes\mathbb{C} \mid q(Y) > 0\}$. We identify $\mathrm{Gr}(L)$ with the *upper half space model*

$$\mathfrak{H}_\ell \to \mathrm{Gr}(L)$$
$$Z = X + iY \mapsto \mathbb{R}X_L \oplus \mathbb{R}Y_L,$$

where $Z_L = X_L + iY_L := Z - (q(Z)+q(z'))z + z'$. In particular, we can view $\mathrm{Gr}(L)$ as a complex manifold.

Definition 36. (1) On $\mathfrak{H}_\ell$, we define a real analytic function

$$\xi^L_{\beta,m}(Z) = \frac{|Y|}{\sqrt{2}}\xi^K_{\beta,m}\left(\frac{Y}{|Y|}\right) - b_{\beta,m}(0,0)\log Y^2$$
$$+ \frac{2}{\sqrt{\pi}} \sum_{\substack{\lambda\in K'\\ q(\lambda)<0}} \sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} e^{2\pi i(\delta,z')n} b_{\beta,m}(\delta,q(\lambda)) \sum_{n\geq 1} n^{-1} e^{2\pi i(\lambda,X)n} \mathcal{V}_{2-k}(\pi n|\lambda||Y|, \pi n(\lambda,Y)),$$

where

$$\xi^K_{\beta,m}(v_1,s) := \begin{cases} \xi^K_{p(\beta),m}(v_1,s) & \text{if } (\beta,m) \equiv 0 \ (N),\\ 0 & \text{if } (\beta,m) \not\equiv 0 \ (N). \end{cases}$$

and $\mathcal{V}_\kappa$ is a function defined by

$$\mathcal{V}_\kappa(A,B) := \int_0^\infty \Gamma(\kappa-1, A^2y)e^{-B^2y-1/y}y^{-3/2}dy.$$

(2) On $\mathfrak{H}_\ell - H(\beta,m)$, we define a real analytic function $\psi^L_{\beta,m}(v_1) := \Phi^L_{\beta,m}(v_1) - \xi^L_{\beta,m}(v_1)$.

(3) For a Weyl chamber W and $Z \in \mathfrak{H}$ with $q(Y) > |m|$, we define

$$\Psi_{\beta,m}^{L,W}(Z) := \exp(2\pi i(\rho_{\beta,m}(W), Y)) \prod_{\substack{\lambda \in \pm p(\beta)+K \\ (\lambda,W)>0 \\ q(\lambda)=m}} (1 - \exp(2\pi i(\pm(\beta, z') + (\lambda, Z))))$$
$$\times \prod_{\substack{\lambda \in K' \\ (\lambda,W)>0 \\ q(\lambda)\geq 0}} \prod_{\substack{\delta \in L_0'/L \\ p(\delta)=\lambda+K}} (1 - \exp(2\pi i((\delta, z') + (\lambda, Z))))^{b_{\beta,m}(\delta,q(\lambda))}.$$

Using 28, 30, and pluriharmonicity of $\xi_{\beta,m}^L$, one can show that $\Psi_{\beta,m}^{L,W}$ admits an analytic continuation to $\mathfrak{H}_\ell$ such that

(i) $\Psi_{\beta,m}(Z)\prod_{\lambda \in S(\beta,m,U)}(\lambda, Z_L)^{-1}$ is nonzero on an open set $U \subseteq \mathfrak{H}_\ell$ with compact closure, and
(ii) $\log|\Psi_{\beta,m}^{L,W}(Z)| = -\frac{1}{4}(\psi_{\beta,m}^L(Z) - C_{\beta,m})$ on $\mathfrak{H}_\ell - H(\beta, m)$ for some constant $C_{\beta,m}$.

In particular, $\Psi_{\beta,m}^{L,W}$ is independent of W up to multiplication by a constant of absolute value 1.

6 Borcherds lifts

Let L be an even lattice of signature $(2, \ell)$. So far, we only know that $|\Psi_{\beta,m}(Z)|e^{-\xi_{\beta,m}(Z)/4}$ is $\Gamma(L)$-invariant. In this subsection, we take a weakly holomorphic modular form for $\mathrm{Mp}_2(\mathbb{Z})$ and construct a finite product in which $e^{-\xi_{\beta,m}(Z)/4}$ cancel out, and therefore produce a meromorphic modular form for $\Gamma(L)$.

Definition 37. (1) For $g \in O^\circ(V)$ and $Z \in \mathfrak{H}_\ell$, we define the *automorphy factor* $j(g, Z) := (z, gZ_L)$.
(2) Fix a branch of $\log j(g, Z)$. For $r \in \mathbb{Q}$, we define $j(g, Z)^r := e^{r \log j(g,z)}$. This induces a set-theoretic function $\omega_r : O^\circ(V) \times O^\circ(V) \to S^1$ that satisfies $j(gh, Z)^r = \omega_r(g, h)j(g, hZ)^r j(h, Z)^r$.
(3) Let $\Gamma \leq O^\circ(V)$ be a subgroup. A *multiplier system* for Γ of weight r is a function $\chi : \Gamma \to S^1$ that satisfies $\chi(gh) = \omega_r(g, h)\chi(g)\chi(h)$.

Definition 38. A *meromorphic modular form* for $\Gamma(L)$ of weight r and multiplier system χ is a meromorphic function F on $\mathfrak{H}_\ell$ that satisfies $F(gZ) = \chi(g)j(g, Z)^r F(Z)$ for all $g \in \Gamma(L)$.

Lemma 39. Let F be a meromorphic function on $\mathfrak{H}_\ell$ such that $|\Psi(Z)|q(Y)^{r/2}$ is $\Gamma(L)$-invariant. Then, F is a meromorphic modular form for $\Gamma(L)$ of weight r and some multiplier system.

Proof. Since $q(\Im(gZ)) = \frac{q(Y)}{|j(g,Z)|^2}$, the hypothesis implies that $\frac{|\Psi(gZ)|}{|\Psi(Z)|}|j(g, Z)|^{-r} = 1$ is constant. By the maximum modulus principle, $\chi(g) := \frac{\Psi(gZ)}{\Psi(Z)}j(g, Z)^{-r}$ is independent of Z. Then, F is a meromorphic modular form of weight r and multiplier system χ. □

Definition 40. Let $f \in \mathcal{M}_k^!$ with principal part

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m)e^{2\pi i m\tau}[\beta].$$

(1) A *Weyl chamber* for f is a connected component of $\mathrm{Gr}(K) - \bigcup_{c(\beta,m)\neq 0} H(\beta, m)$. A Weyl chamber W can be uniquely written as $\bigcap_{c(\beta,m)\neq 0} W_{\beta,m}$ where $W_{\beta,m}$ is a Weyl chamber of index $(p(\beta), m)$.
(2) The *Weyl vector* for a Weyl chamber W is defined by

$$\rho_f(W) := \frac{1}{2} \sum_{c(\beta,m)\neq 0} c(\beta, m)\rho_{p(\beta),m}(W_{\beta,m}).$$

Theorem 41 (Borcherds)**.** We define the *Borcherds lift* of $f \in \mathcal{M}_k^!$ by

$$\Psi_f(Z) := \prod_{\beta,m} \Psi_{\beta,m}(Z)^{c(\beta,m)/2}.$$

(1) Ψ_f is a meromorphic modular form for $\Gamma(L)$ of weight $c(0,0)/2$ and some multiplier system.

(2) The divisor of Ψ_f is given by

$$\operatorname{div}(\Psi_f) = \frac{1}{2}\sum_{\beta,m} c(\beta,m)H(\beta,m).$$

(3) Ψ_f admits a product expansion

$$\Psi_f(Z) = C\exp(2\pi i(\rho_f(W),Z)) \prod_{\substack{\lambda\in K'\\(\lambda,W)>0}} \prod_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} (1-\exp(2\pi i((\delta,z')+(\lambda,Z))))^{c(\delta,q(\lambda))}$$

called the *Borcherds product*, where C is a constant of absolute value 1.

Proof. Consider

$$\Phi_f(Z) := \frac{1}{2}\sum_{\beta,m} c(\beta,m)\Phi_{\beta,m}^L(Z)$$

and

$$\begin{aligned}
\xi_f(Z) &:= \frac{1}{2}\sum_{\beta,m} c(\beta,m)\xi_{\beta,m}^L(Z)\\
&= \frac{1}{2}\sum_{\beta,m} c(\beta,m)\left(\frac{|Y|}{\sqrt{2}}\xi_{\beta,m}^K\left(\frac{Y}{|Y|}\right) - b_{\beta,m}(0,0)\log Y^2\right)\\
&\quad + \frac{1}{\sqrt{\pi}}\sum_{\beta,m}\sum_{\substack{\lambda\in K'\\ q(\lambda)<0}}\sum_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda+K}} c(\beta,m)b_{\beta,m}(\delta,q(\lambda))\\
&\qquad \times \sum_{n\geq 1}(\delta,q(\lambda))n^{-1}\exp(2\pi i((\delta,z')+(\lambda,X))n)\mathcal{V}_{2-k}(\pi n|\lambda||Y|,\pi n(\lambda,Y)).
\end{aligned}$$

As in the proof of 20, we have

$$\sum_{\beta,m} c(\beta,m)b_{\beta,m}(\delta,q(\lambda)) = 0,$$

hence

$$\xi(Z) = \frac{|Y|}{2\sqrt{2}}\sum_{\beta,m} c(\beta,m)\xi_{\beta,m}^K\left(\frac{Y}{|Y|}\right) - \frac{1}{2}\log Y^2\sum_{\beta,m} c(\beta,m)b_{\beta,m}(0,0).$$

The same argument shows that

$$\sum_{\beta,m} c(\beta,m)\xi_{\beta,m}^K(v_1))$$

is rational in v_1, so is identically zero by 34. On the other hand, by 19, we have

$$\frac{1}{2}\sum_{\beta,m} c(\beta,m)b_{\beta,m}(0,0) = c(0,0).$$

We conclude that

$$\xi_f(Z) = -c(0,0)\log Y^2.$$

In particular, $|\Psi_f(Z)|q(Y)^{c(0,0)/4} = (\text{const})e^{-\Phi_f(Z)/4}$ is $\Gamma(L)$-invariant. Therefore, by 39, Ψ_f is a meromorphic modular form for $\Gamma(L)$ of weight $c(0,0)/2$ and some multiplier system. The rest of the theorem is clear. □

Remark 42. (1) The theory or Borcherds lifts can be generalized to congruence subgroups.

(2) The theory of Borcherds lifts can be generalized to lattices of signature $(2,2)$ using Hecke's trick, although there are more technicalities involved. One has to consider the Eisenstein series of weight

0 in order to obtain all weakly holomorphic modular forms, and the Eisenstein series of weight 2 are no longer holomorphic.

(3) Equip $H := \mathbb{Z}^2$ with the quadratic form $q((a,b)) = ab$ and consider the even unimodular lattice $L := H \oplus H$. In this case, modular forms for $\Gamma(L)$ can be identified with modular forms on $\mathfrak{H} \times \mathfrak{H}$ for $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. The Heegner divisors $H(m) = T(|m|)$ are the Hecke correspondences. In particular, $H(1)$ consists of the translates of the diagonal. The Borcherds lift of $j - 744 \in \mathcal{M}^!_{0,L}$ yields a product expansion

$$j(\tau_1) - j(\tau_2) = \exp(-2\pi i \tau_1) \prod_{m \geq 1} \prod_{n \in \mathbb{Z}} (1 - \exp(2\pi i (m\tau_1 + n\tau_2)))^{c(mn)}.$$

References

[Bru02] Jan H. Bruinier. *Borcherds Products on O(2, l) and Chern Classes of Heegner Divisors*. Springer-Verlag Berlin Heidelberg, 2002.