LIMIT COMPARISON TEST FOR IMPROPER INTEGRALS

UM Math 116
February 13, 2018

The basic question about improper integrals in Math 116 is whether or not they converge.

Example 1. Does $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ converge?

Notice that the function $\frac{1}{\sqrt{1-x^2}}$ has a vertical asymptote at $x = 1$, so this is an improper integral and we will need to consider the appropriate limit. In this case we can use the fundamental theorem of calculus:

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx \\ &= \lim_{b\to 1^-} \arcsin(x)\Big|_0^b \\ &= \lim_{b\to 1^-} \arcsin(b) - \arcsin(0) \\ &= \frac{\pi}{2}.\end{aligned}$$

Therefore the integral converges. □

Often we are asked to determine the convergence of an improper integral which is too complicated for us to compute exactly. When this happens we use an *integral convergence test*.

Example 2. Does $\displaystyle\int_2^\infty \frac{x^2+x+1}{x^3-\sqrt[3]{x}}\,dx$ converge?

Since it looks difficult to find an antiderivative of $\frac{x^2+x+1}{x^3-\sqrt[3]{x}}$, we need a different approach. For large values of x we have

$$\frac{x^2+x+1}{x^3-\sqrt[3]{x}} \approx \frac{x^2}{x^3} = \frac{1}{x},$$

so we should expect this integral to diverge since $\int_2^\infty \frac{1}{x}\,dx$ diverges by the p-test. To show divergence we use the *(direct) comparison test*. (This is referred to in §7.7 of our textbook as "The Comparison Test".) Note that the inequality

$$\frac{x^2+x+1}{x^3-\sqrt[3]{x}} \geq \frac{x^2}{x^3-\sqrt[3]{x}} \geq \frac{x^2}{x^3} = \frac{1}{x}$$

holds for all x in $[2,\infty)$. The integral $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by the p-test with $p = 1$, so $\displaystyle\int_2^\infty \frac{x^2+x+1}{x^3-\sqrt[3]{x}}\,dx$ diverges by direct comparison. □

The direct comparison test works in many cases, but there are situations where the simplest comparison we would like to make is not valid.

Example 3. Does $\displaystyle\int_2^\infty \frac{x^2-x-1}{x^3+\sqrt[3]{x}}\,dx$ converge?

This integral looks almost identical to the one in the previous example, but some of the signs have been flipped. For large values of x we still have

$$\frac{x^2-x-1}{x^3+\sqrt[3]{x}} \approx \frac{x^2}{x^3} = \frac{1}{x},$$

but we now get the inequality

$$\frac{x^2 - x - 1}{x^3 + \sqrt[3]{x}} \leq \frac{1}{x}$$

for all x in $[2, \infty)$. This does not help us show that the original improper integral diverges because of the direction of the inequality.

We can, however, find that $\frac{1}{4}x^2 \leq x^2 - x - 1$ and $2x^3 \geq x^3 + \sqrt[3]{x}$ for all x in $[2, \infty)$. Combining these, we find

$$\frac{1}{8x} \leq \frac{x^2 - x - 1}{x^3 + x^{1/3}}$$

for x in $[2, \infty)$. This lets us use the comparison test, but it took more work to come up with an appropriate function to compare with. $\square$

The *limit comparison test* gives us another strategy for situations like Example 3.

Limit comparison test (LCT) for improper integrals: Suppose $f(x)$ and $g(x)$ are positive, continuous functions defined on $[a, \infty)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$$

where c is a postive number. (In particular, $c \neq 0$ and $c \neq \infty$.)
Then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Let's return to Example 3 to see how the LCT works. We want to show that $\displaystyle\int_2^\infty \frac{x^2 - x - 1}{x^3 + x^{\frac{1}{3}}}\,dx$ diverges. Using the notation from the LCT, we let

$$f(x) = \frac{x^2 - x - 1}{x^3 + x^{1/3}}$$

and choose $g(x) = \frac{1}{x}$, which was the function we originally hoped to compare $f(x)$ with. You can check that both of these functions are positive and continuous on $[2, \infty)$. Next we compute

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x^2-x-1}{x^3+x^{1/3}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^3 - x^2 - x}{x^3 + x^{1/3}} = 1.$$

Since the limit is 1, which is not 0 or ∞, we can apply the LCT. The integral $\int_2^\infty \frac{1}{x}\,dx$ diverges by the p-test with $p = 1$; thus the LCT tells us that $\int_2^\infty f(x)\,dx$ must also diverge.

The limit comparison test is used when we want to determine whether an improper integral $\int_a^\infty f(x)\,dx$ converges, but the function $f(x)$ is too complicated for us to either compute an antiderivative or to easily find a comparison with a simpler function. To choose the function $g(x)$ we look for something that is *simple* and that *grows at a rate that is similar to* $f(x)$. The limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$$

tells us how the growth rates of $f(x)$ and $g(x)$ compare for large values of x.

(1) If $c = 0$, this means that $g(x)$ grows much faster than $f(x)$.
(2) If $c = \infty$, this means that $f(x)$ grows much faster than $g(x)$.
(3) If $c \neq 0$ or ∞, this means that $f(x)$ and $g(x)$ grow at a similar rate.

Note that if this limit does not exist, is 0 or ∞, then the LCT is inconclusive, and cannot be applied.

Steps for using the LCT: Use the LCT when trying to determine whether $\int_a^\infty f(x)\,dx$ converges and the function $f(x)$ is positive and looks complicated.

(1) Find a function $g(x)$ which grows at a similar rate as $f(x)$ and is simpler.
(2) Compute
$$c = \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$
Make sure the limit exists and is not 0 or ∞.
(3) Check by some other means whether $\int_a^\infty g(x)\,dx$ converges or diverges.
(4) Cite the LCT to conclude whether $\int_a^\infty f(x)\,dx$ converges or diverges.

Following are some additional examples in which the LCT is used.

Example 4. Determine whether $\displaystyle\int_1^\infty \frac{7\sqrt{x}+\sqrt[3]{x}}{\sqrt{x^5}+\cos(x)}\,dx$ converges.

The function $f(x) = \frac{7\sqrt{x}+\sqrt[3]{x}}{\sqrt{x^5}+\cos(x)}$ looks pretty complicated. In this situation we turn immediately to the limit comparison test. What function do we choose for $g(x)$? Looking at the most significant terms in the numerator and denominator we see that for large values of x,

$$\frac{7\sqrt{x}+\sqrt[3]{x}}{\sqrt{x^5}+\cos(x)} \approx \frac{\sqrt{x}}{\sqrt{x^5}} = \frac{1}{x^2}.$$

Let $g(x) = \frac{1}{x^2}$. Next compute the necessary limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{7\sqrt{x}+\sqrt[3]{x}}{\sqrt{x^5}+\cos(x)}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{7x^{5/2}+x^{7/3}}{x^{5/2}+\cos(x)}.$$

In the numerator, $7x^{5/2}$ dominates $x^{7/3}$, that is, when x is very large, $7x^{5/2}$ becomes so much larger than $x^{7/3}$ that the latter term becomes negligible. In the denominator, $\cos(x)$ is between -1 and 1, and so is dominated by $x^{5/2}$ when x is very large. This tells us that

$$\lim_{x\to\infty} \frac{7x^{5/2}+x^{7/3}}{x^{5/2}+\cos(x)} = 7.$$

Since the limit is 7 and not 0 or ∞, the LCT applies. The integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges by the p-test with $p = 2$. Thus, the LCT tells us that

$$\int_1^\infty \frac{7\sqrt{x}+\sqrt[3]{x}}{\sqrt{x^5}+\cos(x)}\,dx$$

must also converge. □

Example 5. Does $\displaystyle\int_0^\infty \frac{e^x+x^3}{e^{3x}-x^3}\,dx$ converge?

For large x,

$$\frac{e^x+x^3}{e^{3x}-x^3} \approx \frac{e^x}{e^{3x}} = e^{-2x}.$$

Let $g(x) = e^{-2x}$. Then we compute

$$\lim_{x\to\infty} \frac{\frac{e^x+x^3}{e^{3x}-x^3}}{e^{-2x}} = \lim_{x\to\infty} \frac{e^{2x}(e^x+x^3)}{e^{3x}-x^3}.$$

In the numerator, e^x dominates x^3 when x is large, which means numerator overall is dominated by e^{3x}. In the denominator, e^{3x} dominates x^3 when x is large. Thus this limit is 1.

Since the limit converged to 1 and not 0 or ∞, the LCT applies. The integral $\int_0^\infty e^{-2x}\,dx$ converges by the exponential decay test, and so it follows that

$$\int_0^\infty \frac{e^x + x^3}{e^{3x} - x^3}\,dx$$

converges by the LCT. $\square$

Our next example highlights one pitfall of the LCT.

Example 6. Determine whether $\displaystyle\int_1^\infty \frac{x(3+\cos(x))}{x^3 - e^{-x}}\,dx$ converges.

A rough estimate tells us that for large x,

$$\frac{x(3+\cos(x))}{x^3 - e^{-x}} \approx \frac{x}{x^3} = \frac{1}{x^2}.$$

But we run into trouble when we try to compute the required limit for the LCT:

$$\lim_{x\to\infty} \frac{\frac{x(3+\cos(x))}{x^3-e^{-x}}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^3(3+\cos(x))}{x^3 - e^{-x}} = \lim_{x\to\infty} \frac{3+\cos(x)}{1 - \frac{e^{-x}}{x^3}}.$$

We get to the last equality by dividing the numerator and denominator by x^3. Here we can see that the denominator will approach 1 as x approaches infinity, but the numerator will never approach a single number, as $\cos(x)$ never stops oscillating through values in $[-1, 1]$. To deal with this we use the inequality

$$3 + \cos(x) \leq 4$$

to see that

$$\frac{x(3+\cos(x))}{x^3 - e^{-x}} \leq \frac{4x}{x^3 - e^{-x}}.$$

Now we wish to apply the LCT with our new function $f(x) = \dfrac{4x}{x^3 - e^{-x}}$ and $g(x) = \frac{1}{x^2}$. We find

$$\lim_{x\to\infty} \frac{\frac{4x)}{x^3-e^{-x}}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{4x^3}{x^3 - e^{-x}} = \lim_{x\to\infty} \frac{4}{1 - \frac{e^{-x}}{x^3}} = 4.$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}$ converges by the p-test with $p = 2$, we apply the LCT as we had hoped to show that

$$\int_1^\infty \frac{4x}{x^3 - e^{-x}}\,dx$$

converges, and then use the inequality

$$\frac{x(3+\cos(x))}{x^3 - e^{-x}} \leq \frac{4x}{x^3 - e^{-x}}$$

to conclude that our original integral converges by the direct comparison test. $\square$