Theoretical Basis of the Symmetrization Postulate

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The postulate of symmetrization for identical particles in quantum mechanics does not necessarily follow from the condition of indistinguishability. Greenberg and Messiah have shown that, under suitable conditions, the selection rule implied by the postulate of symmetrization actually follows from the indistinguishability complemented by some broad assumptions about the quantum system. The aim of this paper is to show that the most restrictive of these extra assumptions, namely, *PCT* or time-reversal invariance, is unnecessary.

In a recent paper,[1] Greenberg and Messiah investigated the theoretical and experimental foundation of the symmetrization postulate in the quantum-mechanical treatment of identical particles. They considered systems describable in the framework of a Fock space with a certain evolution operator $U(t)$, either conserving or not conserving the number of particles. They derived in particular a rather strong selection rule for transitions which do not conserve the number of particles, according to which states, which contain no more than one particle in any species for which the symmetrization postulate is still in doubt, cannot perform transitions to states violating the prescription of symmetrization demanded by this postulate. As a consequence, starting from the experimentally very well supported fact that electrons and nucleon are fermions and that photons are bosons, no violation of the symmetrization postulate for the other particles (e.g., π, $K, \Lambda, \cdots$) could be observed in any of the present day experiments, since all of them are collision experiments, in which the initial states never contain more than one particle of possibly questionable species.

This selection rule did not appear, however, as a necessary consequence of the indistinguishability of identical particles. In order to derive it, two additional assumptions had to be made. The first one, expressed as a property of coherence of certain parts of the Hilbert space, put some condition on the correspondence between dynamical states and representative vectors. The second was the invariance of the evolution operator $U(t)$ under a suitable transformation involving the reversal of the time, e.g., time-reversal invariance proper, or *PCT* invariance.

The purpose of this note is to point out that the latter assumption is unnecessary, and that the selection rule holds even if $U(t)$ has none of these symmetry properties.

To show this, we have to prove the following theorem:

If the transition from a state a to a state b stays forbidden in the course of time, i.e., if $\langle b|U(t)|a\rangle=0$ *for all* $t>0$, *the same holds for the reverse transition:*

$$\langle a|U(t)|b\rangle=0 \quad \text{for all} \quad t>0.$$

As can be seen easily by inspection of the argument of Ref. 1, the application of this theorem makes unnecessary any recourse to an invariance property involving the reversal of the time.[2]

The proof of the theorem makes use of the characteristic properties of the evolution operator $U(t)$ of a quantum system. We postulate, as usual, that:

(a) the motion of the system is generated by a Hamiltonian;

(b) its energy spectrum, i.e., the spectrum of this Hamiltonian, has a finite lower bound.

Postulate (a) is equivalent to assuming that the set $\{U(t)\}$ is a continuous unitary representation in the Hilbert space of the group of time translations.[3] According to Stone's theorem,[4] such a set admits the spectral representation

$$U(t)=\int_{-\infty}^{+\infty} e^{-ist}dE_s\,, \tag{1}$$

and the infinitesimal generator H of $U(t)$ can then be defined (in a dense subdomain of the Hilbert space) by

$$H=\int_{-\infty}^{+\infty} s\,dE_s\,; \tag{2}$$

H is the Hamiltonian of the quantum system.

Postulate (b) follows from the requirement that the total energy of a quantum system cannot take negative

[1] A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**, B248 (1964).

[2] The same remark obviously applies to the derivation of the law of conservation of the degree of symmetry types. Thus, this conservation law is a property of $U(t)$ which follows from the indistinguishability of identical particles, without any further assumption.

[3] Otherwise stated, it is a set of operators defined for all real values of the parameter t, with the following properties:

(i) unitarity: $U^\dagger U=UU^\dagger=1$;
(ii) composition law: $U(t_1)U(t_2)=U(t_1+t_2)$;
(iii) weak continuity in t, i.e., continuity in t of every matrix element.

One could, as well, start with a set $\{U(t)\}$ defined in the restricted domain $(0,+\infty)$, i.e., with a semigroup rather than with a group. Taking as a definition of $U(t)$ for negative values of t:

$$U(-t)=U^\dagger(t),$$

one recovers the group defined above.

[4] See, for instance, F. Riesz and B. Sz. Nagy, *Leçons d'analyse fonctionnelle* (Gauthier-Villars, Paris, 1965), 4th ed., Chap. X.

values (spectral condition). It means that the support of the spectral measure dE_s has a finite lower bound. Accordingly, the evolution operator $U(t)$ admits a spectral decomposition of the following form:

$$U(t)=\int_m^{+\infty} e^{-ist}dE_s. \tag{3}$$

Equation (3) summarizes all the properties of $U(t)$ that are needed in our proof.

The proof goes according to the following standard argument.

From (3), we can express the transition amplitudes under investigation as Lebesgue-Stieltjes integrals. Using the notation

$$\mu(s)\equiv\langle b|E(s)|a\rangle,$$

we have

$$\langle b|U(t)|a\rangle=f(t)\equiv\int_m^{+\infty} e^{-ist}d\mu(s),$$

and, since

$$U^\dagger(t)=U(-t),$$

$$\langle a|U(t)|b\rangle=f^*(-t).$$

Thus, we must prove that, if $f(t)=0$ for all $t>0$, it identically vanishes on the whole domain $(-\infty,+\infty)$.

We may assume $m>0$ without any loss of generality, since an overall shift Δ in the energy scale merely results in multiplying $f(t)$ by the phase factor $e^{i\Delta t}$ and does not change the final conclusion. Let us then introduce the following function of the complex variable τ:

$$F(\tau)\equiv\int_m^{+\infty} e^{-is\tau}d\mu(s),\quad \mathrm{Im}\tau<0. \tag{4}$$

Since the integral converges uniformly in τ when $\mathrm{Im}\tau<0$, $F(\tau)$ is analytic in the lower half-plane. Furthermore, its limit on the real axis is $f(t)$:

$$f(t)=\lim_{\epsilon\to 0^+}F(t-i\epsilon). \tag{5}$$

We have now to take into account the condition

$$f(t)=0\quad\text{for all}\quad t>0. \tag{6}$$

It means that $F(\tau)$ vanishes on a certain segment of the real axis. Therefore, it can be continued through the operation of Schwarz's principle of reflection beyond this portion of the boundary of its domain of analyticity. In the continuation process, this portion of the boundary becomes a line inside the domain of analyticity. Since the function vanishes on this line, it necessarily vanishes on its whole domain and its boundary value vanishes on the whole boundary. Thus, $f(t)=0$ on the whole domain $(-\infty,+\infty)$. Q.E.D.

Clearly, postulate (b) played an important role in our argument. In fact, the above theorem does not follow from postulate (a) alone, as is shown by the following counterexample.

Let $\varphi(x)$ be an infinitely differentiable function of the real variable x, with support contained in the interval $[-1,+1]$. For concreteness, let us take

$$\varphi(x)\equiv e^{-1/(1-x^2)}\theta(1-x^2)$$

with

$$\theta(y)=1\quad\text{when}\quad y>0$$
$$=0\quad\text{when}\quad y<0.$$

In the Hilbert space of square integrable functions of x, we consider the two vectors $|a\rangle$ and $|b\rangle$, which are defined, respectively, by the functions

$$a(x)=\varphi(x),\quad b(x)=\varphi(x+2).$$

We then consider the continuous group of unitary operators $\{U(t)\}$ defined by

$$(U(t)f)(x)=f(x-t).$$

$U(t)$ performs the translation of each state by an amount t along the x axis. It is generated by a Hamiltonian whose spectrum is continuous and extends from $-\infty$ to $+\infty$. Thus, postulate (a) is fulfilled, but postulate (b) is not. Now, we obviously have $\langle b|U(t)|a\rangle=0$ for all $t>0$, but $\langle a|U(t)|b\rangle\neq 0$ when $0<t<4$.

Thus, in the cases when H is neither bounded from above nor from below, we cannot expect the theorem to hold for every pair of states a and b. It still holds, however, for certain pairs of states. In particular, it holds when either a or b is an analytic vector.[5] Indeed, under this condition $\langle b|U(t)|a\rangle$ is analytic in t over the whole real axis; therefore, if it vanishes when $t>0$, it necessarily vanishes everywhere. The above counterexample reflects the lack of analyticity of both $a(x)$ and $b(x)$.

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[5] Cf. E. Nelson, Ann. Math. **70**, 572 (1959).