Extended geometrically finite representations

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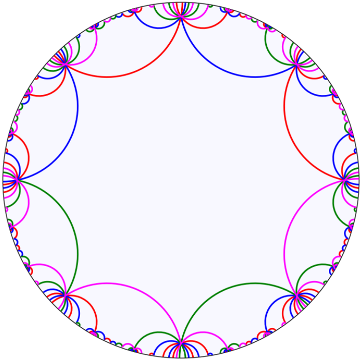
Goal: introduce a notion of *geometrically finite subgroups* of *higher rank* Lie groups (e.g. $G = \mathrm{SL}(d, \mathbb{R})$ for $d > 2$).

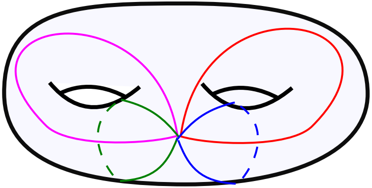
	$\mathrm{Isom}(\mathbb{H}^d)$	Higher rank
hyperbolic	convex cocompact	Anosov representations
relatively hyperbolic	geometrically finite	"relative Anosov" This talk: EGF

Definition

Let Γ be a discrete subgroup of $\mathrm{SO}(d,1)$. We say Γ is *convex cocompact* if Γ acts with compact quotient on a nonempty Γ-invariant convex subset of $\mathbb{H}^d$.

Example: $\Gamma \simeq \pi_1 M$ for M a closed hyperbolic d-manifold.

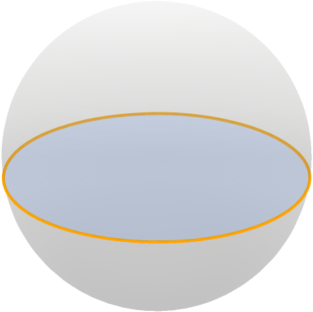




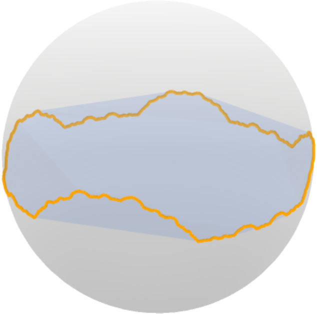
Proposition (Gromov, Coornaert, Bourdon)

A discrete group $\Gamma \subset \mathrm{SO}(d,1)$ *is convex cocompact if and only if* Γ *is (abstractly) word-hyperbolic, and its Gromov boundary* $\partial\Gamma$ *embeds equivariantly into* $\partial\mathbb{H}^d$*.*

S hyperbolic surface,
$\pi_1 S \to \mathrm{SO}(2,1) \hookrightarrow \mathrm{SO}(3,1)$



$\partial\Gamma \simeq \partial\mathbb{H}^2 \hookrightarrow \partial\mathbb{H}^3$



Convex cocompactness in higher rank

Definition (Guéritaud-Guichard-Kassel-Wienhard, Kapovich-Leeb-Porti)

Let $\rho : \Gamma \to \mathrm{SL}(d, R)$ be a representation of a word-hyperbolic group. We say ρ is *P_1-Anosov* if there are ρ-equivariant embeddings

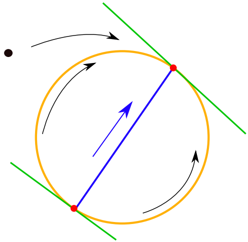
$$\xi : \partial\Gamma \to \mathbb{R}P^{d-1}, \quad \xi^* : \partial\Gamma \to (\mathbb{R}P^{d-1})^*$$

which are *transverse* and *dynamics-preserving*.

S hyperbolic surface
$\rho : \pi_1 S \to \mathrm{SO}(2,1) \hookrightarrow \mathrm{SL}_3(\mathbb{R})$

$\gamma \in \pi_1 S$ acts on $\partial\mathbb{H}^2 \subset \mathbb{R}P^2$

ξ maps attracting fixed points to attracting fixed points



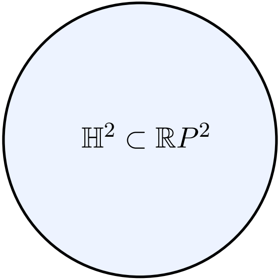
$\mathbb{H}^2 \subset \mathbb{R}P^2$

Theorem (Labourie, Guichard-Wienhard)

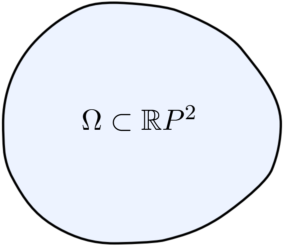
Let $\rho : \Gamma \to \mathrm{SL}_d \mathbb{R}$ be a P_1-Anosov representation. Then an open neighborhood of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SL}_d(\mathbb{R}))$ consists of P_1-Anosov representations.

S hyperbolic surface

$\pi_1 S \to \mathrm{SO}(2,1) \hookrightarrow \mathrm{SL}_3(\mathbb{R})$



Deform in $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$:



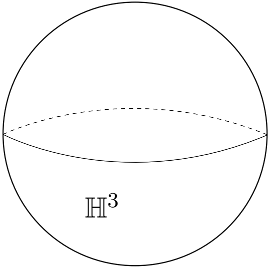
Invariant under deformed action, quotient is *convex projective surface*

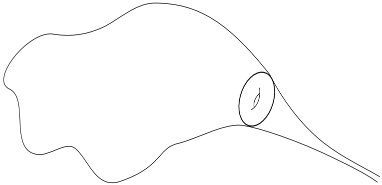
What about geometrically finite groups?

Definition

Let $\Gamma \subset \mathrm{SO}(d,1)$ be a *finitely generated* discrete group. We say Γ is *geometrically finite* if it acts with finite covolume on a convex Γ-invariant subset of $\mathbb{H}^d$ *with nonempty interior*.

Example: M = complete finite-volume noncompact hyperbolic 3-manifold, $\Gamma = \pi_1 M \subset \mathrm{SO}(3,1)$.

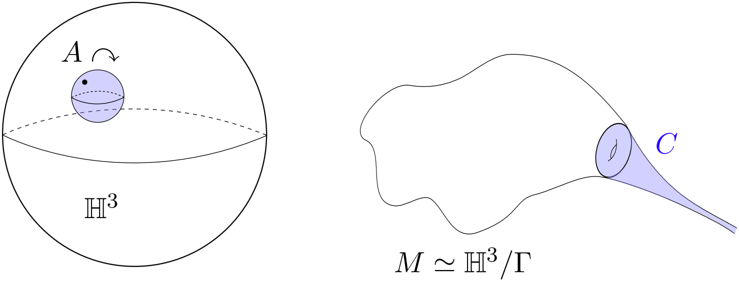




$M \simeq \mathbb{H}^3/\Gamma$

Γ is *not* a word-hyperbolic group.

Any geometrically finite group Γ is *relatively hyperbolic*, relative to its *cusp subgroups* $\mathcal{P} = \{\pi_1 C : C \text{ a cusp of } \mathbb{H}^d/\Gamma\}$.



The *parabolic subgroup* $A \simeq \mathbb{Z}^2$ is the fundamental group of the cusp $C \subset M$.

A is the stabilizer of a point in $\partial\mathbb{H}^3 = \partial(\Gamma, \mathcal{P})$, the *Bowditch boundary* of the pair $(\Gamma, \mathcal{P})$

Relative hyperbolicity in higher rank

Definition (Guéritaud-Guichard-Kassel-Wienhard, Kapovich-Leeb-Porti)

Let $\rho : \Gamma \to \mathrm{SL}(d, R)$ be a representation of a word-hyperbolic group. We say ρ is P_1-*Anosov* if there are ρ-equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{R}P^{d-1}, \quad \xi^* : \partial\Gamma \to (\mathbb{R}P^{d-1})^*$$

which are *transverse* and *dynamics-preserving*.

Definition (Kapovich-Leeb)

Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation of a relatively hyperbolic group. We say ρ is *relatively asymptotically embedded* if there are ρ-equivariant embeddings

$$\xi : \partial(\Gamma, \mathcal{P}) \to \mathbb{R}P^{d-1}, \quad \xi^* : \partial(\Gamma, \mathcal{P}) \to (\mathbb{R}P^{d-1})^*$$

which are *transverse* and *dynamics-preserving*.

Definition (Kapovich-Leeb)

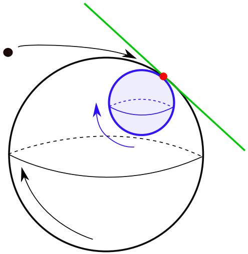
Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation of a relatively hyperbolic group. We say ρ is *relatively asymptotically embedded* if there are ρ-equivariant embeddings

$$\xi : \partial(\Gamma, \mathcal{P}) \to \mathbb{R}P^{d-1}, \quad \xi^* : \partial(\Gamma, \mathcal{P}) \to (\mathbb{R}P^{d-1})^*$$

which are *transverse* and *dynamics-preserving*.

M finite-vol. hyp. 3-manifold
$\pi_1 M \to \mathrm{SO}(3,1) \hookrightarrow \mathrm{SL}_4(\mathbb{R})$

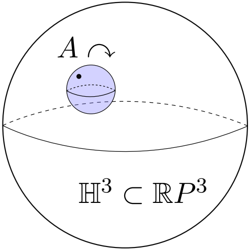
Cusp group $A \subset \pi_1 M$ acts on $\partial(\Gamma, \mathcal{P}) = \partial \mathbb{H}^3 \subset \mathbb{R}P^3$



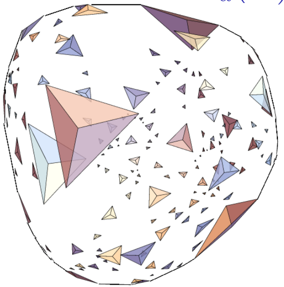
$\mathbb{H}^3 \subset \mathbb{R}P^3$

Deforming relative Anosov representations in $\mathrm{SL}_d(\mathbb{R})$

$\pi_1 M \to \mathrm{SO}(3,1) \hookrightarrow \mathrm{SL}_4(\mathbb{R})$

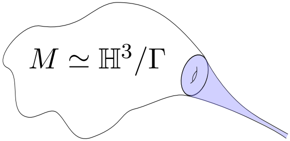


deform
in
$\mathrm{SL}_4(\mathbb{R})$
$\to$

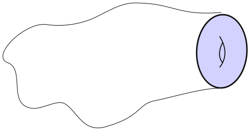


(image from Ballas-Danciger-Lee)

$A \simeq \mathbb{Z}^2 \subset \{\text{upper triangular}\}$



$A' \subset \{\text{diagonalizable}\}$



Get *convex projective 3-manifd.*

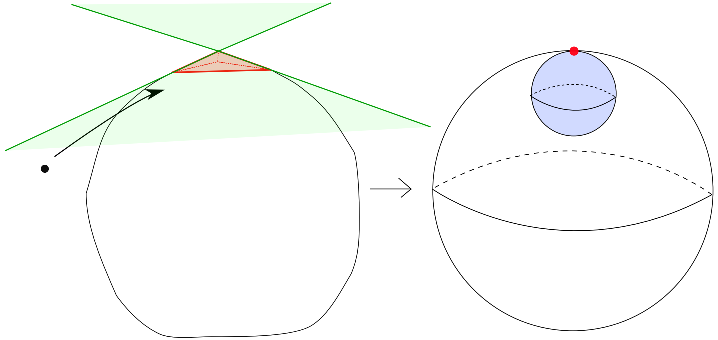
Bowditch boundary $\partial\mathbb{H}^3$ is *not* equivariantly embedded into $\mathbb{R}P^3$!

Definition (W.)

Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation of a relatively hyperbolic group. We say that ρ is *extended geometrically finite* if there are Γ-invariant subsets $\Lambda \subset \mathbb{R}P^{d-1}$, $\Lambda^* \subset (\mathbb{R}P^{d-1})^*$ and surjective transverse maps

$$\phi : \Lambda \to \partial(\Gamma, \mathcal{H}), \quad \phi^* : \Lambda^* \to \partial(\Gamma, \mathcal{H})$$

which *extend convergence dynamics*.



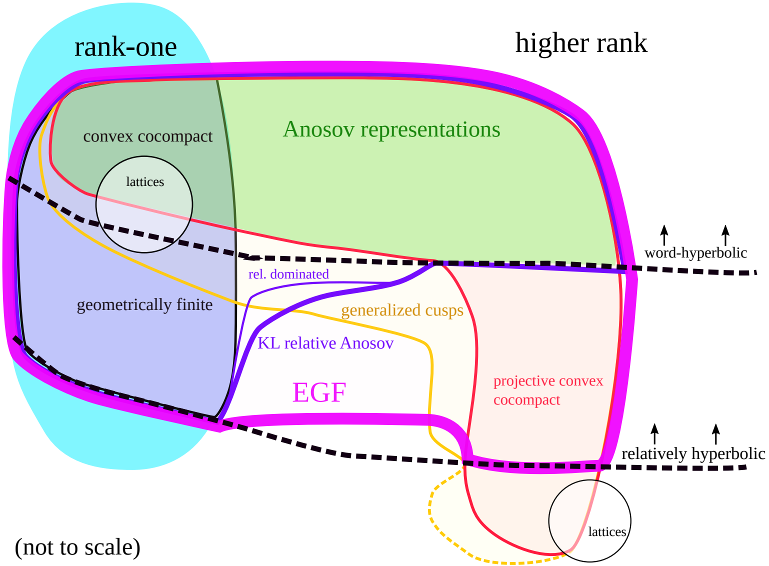
Extended geometrically finite representations are *relatively stable.*

Theorem (W.)

Let $\rho : \Gamma \to G$ *be EGF, and let* $W \subseteq \mathrm{Hom}(\Gamma, G)$ *be a peripherally stable subspace at* ρ*. Then an open subset of* W *containing* ρ *consists of EGF representations.*

In particular, the deformation of $\pi_1 M \to \mathrm{SO}(3,1) \hookrightarrow \mathrm{SL}_4 \mathbb{R}$ shown previously is peripherally stable.

This works for any relatively hyperbolic group Γ and semisimple Lie group G.


rank-one
higher rank
convex cocompact
Anosov representations
lattices
word-hyperbolic
rel. dominated
geometrically finite
generalized cusps
KL relative Anosov
projective convex
cocompact
EGF
relatively hyperbolic
lattices
(not to scale)