EXAMPLES OF EXTENDED GEOMETRICALLY FINITE REPRESENTATIONS

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Abstract. This is the second of a pair of papers on extended geometrically finite (EGF) representations, which were originally posted as a single article under the title "An extended definition of Anosov representation for relatively hyperbolic groups." In this paper, we prove that the holonomy representation of a projectively convex cocompact manifold with relatively hyperbolic fundamental group is always an EGF representation. We also prove that EGF representations arise as holonomy representations of convex projective manifolds with generalized cusps and as compositions of projectively convex cocompact representations with symmetric representations of $\mathrm{SL}(d, \mathbb{R})$. We additionally show that any small deformation of a representation of the latter form is still EGF.

Contents

1. Introduction

1.1. **Format of the paper.** This paper constitutes the second part of a preprint originally posted in April 2022 under the title *An extended definition of Anosov representation for relatively hyperbolic groups* [Wei22]. The first part of the original preprint (which introduces EGF representations and proves a relative stability result) can still be found with its original title at the original arXiv posting.

The division was made largely for the sake of decreasing the total length of the paper, so the contents of this article are essentially unchanged from the way they appeared in earlier versions of [Wei22].

1.2. **Overview.** In the last two decades, *Anosov representations* have emerged as important objects in the study of discrete subgroups of semisimple Lie groups. Originally defined for surface groups by Labourie in [Lab06], and extended to arbitrary word-hyperbolic groups by Guichard-Wienhard [GW12], Anosov representations generalize geometric and dynamical properties of convex cocompact representations in rank one: an Anosov representation is always a quasi-isometric embedding $\Gamma \to G$ for a Gromov hyperbolic group Γ and a

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semisimple Lie group G, and comes equipped with an equivariant boundary embedding $\partial\Gamma \to G/P$, where $P \subset G$ is a parabolic subgroup.

Anosov representations have come to be accepted as a suitable higher-rank generalization of convex cocompactness, which raises the question of whether there is also an analogous generalization of geometrical finiteness. Several authors (see [KL18], [Zhu21], [CZZ21], [ZZ22]) have previously defined notions of *relative Anosov* representations, but none of the proposed definitions capture certain natural families of examples of "geometrically finite" behavior in higher rank.

In [Wei22], we introduced a new class of representations of relatively hyperbolic groups into semisimple Lie groups, called *extended geometrically finite (EGF)* representations. EGF representations generalize all existing definitions of relative Anosov representations. In addition, the definition is flexible enough to cover many additional examples of higher-rank "geometrically finite" behavior, and to allow for EGF representations to deform in ways not available to relative Anosov representations. Specifically, EGF representations satisfy the following theorem:

Theorem 1.1 (See [Wei22, Theorem 1.4])**.** *Let* $(\Gamma, \mathcal{H})$ *be a relatively hyperbolic pair, and let* $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ *be an extended geometrically finite representation, with boundary extension* ϕ*. If* $\mathcal{W} \subset \mathrm{Hom}(\Gamma, \mathrm{PGL}(d, \mathbb{R}))$ *is a subspace which is* peripherally stable *with respect to* ρ, ϕ*, then an open subset of* $\mathcal{W}$ *containing* ρ *consists of EGF representations.*

We will review the terminology in Theorem 1.1 in Section 2. The main aim of this article is to explain how various examples fit into the theory of EGF representations, and highlight applications of Theorem 1.1 for these examples. In some cases, the theorem recovers known stability results for various classes of discrete groups, and in other cases it yields new results.

1.3. **Results.** We refer to Section 2 for the definition of an EGF representation. See [Wei22] for further detail.

1.3.1. *Convex cocompact groups in* $\mathrm{PGL}(d, \mathbb{R})$*.* In [DGK17], Danciger-Guéritaud-Kassel introduced a notion of *convex cocompactness* for projective orbifolds, i.e. orbifolds with a real projective structure. Roughly, a group $\Gamma \subset \mathrm{PGL}(d, \mathbb{R})$ is *convex cocompact* if it acts with compact quotient on a certain closed invariant convex subset inside a *properly convex domain* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. We refer to Section 4 for the precise definition.

Convex cocompact groups in $\mathrm{PGL}(d, \mathbb{R})$ are closely related to Anosov subgroups. Let $\{e_1, \ldots, e_d\}$ denote the standard basis for $\mathbb{R}^d$, and let $P_{1,d-1}$ denote the parabolic subgroup stabilizing the flag $(\mathrm{span}\{e_1\}, \mathrm{span}\{e_1, \ldots, e_{d-1}\})$. If Γ is a word-hyperbolic subgroup of $\mathrm{PGL}(d, \mathbb{R})$ which is convex cocompact in the sense of [DGK17], then the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is a $P_{1,d-1}$-Anosov representation. Conversely, any $P_{1,d-1}$-Anosov representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ which preserves some properly convex domain in $\mathbb{P}(\mathbb{R}^d)$ is convex cocompact. Further, for any semisimple Lie group G and any parabolic subgroup $P \subset G$, one can find a representation $\phi : G \to \mathrm{PGL}(d, \mathbb{R})$ so that a representation $\rho : \Gamma \to G$ of a hyperbolic group Γ is P-Anosov if and only if the composition $\phi \circ \rho$ has convex cocompact image (see [GGKW17], [Zim21]). Thus, convex cocompactness in $\mathrm{PGL}(d, \mathbb{R})$ can be used to give a definition of Anosov representation in terms of convex projective geometry.

Unlike several other definitions of Anosov representations, however, projective convex cocompactness immediately generalizes beyond the realm of hyperbolic groups. Indeed, there are a number of different constructions for non-hyperbolic convex cocompact subgroups of $\mathrm{PGL}(d, \mathbb{R})$; see e.g. [Ben06], [BDL15], [CLM20], [DGKLM21], [BV23] (see also [Wei20, Sec.

2.6] for an overview, or the forthcoming work [DGK]). In each of the examples we have mentioned here, the convex cocompact group Γ is (abstractly) a relatively hyperbolic group.

These relatively hyperbolic convex cocompact groups do *not* fit into previously existing theories of relative Anosov representations (see [Wei20, Remark 1.14]). However, we show in this paper that they do always give rise to EGF representations:

Theorem 1.2. *Let Γ be a convex cocompact subgroup of $\mathrm{PGL}(d, \mathbb{R})$, and suppose that Γ is relatively hyperbolic. Then the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is extended geometrically finite.*

Our proof of Theorem 1.2 builds on our earlier study [Wei20] of "boundary maps" from the Bowditch boundary of a relatively hyperbolic convex cocompact group Γ to a quotient of the boundary of a Γ-invariant domain in $\mathbb{P}(\mathbb{R}^d)$, as well as related work of Islam-Zimmer on the same topic [IZ22].

Remark 1.3. Not every convex cocompact group in $\mathrm{PGL}(d, \mathbb{R})$ is abstractly relatively hyperbolic. If Γ is a uniform lattice in $\mathrm{PSL}(n, \mathbb{R})$, then Γ acts on the space $\tilde{X}$ of $n \times n$ positive-definite symmetric matrices, which embeds into the vector space V of symmetric $n \times n$ matrices. The image of $\tilde{X}$ in the projective space $\mathbb{P}(V)$ is a properly convex open set X, and the induced action of Γ on X is properly discontinuous and cocompact, meaning that Γ is convex cocompact in $\mathrm{PGL}(V)$. However, the group Γ is not relatively hyperbolic whenever $n > 2$.

On the other hand, there are no examples known of convex cocompact groups in $\mathrm{PGL}(d, \mathbb{R})$ which are *not* either relatively hyperbolic or isomorphic to a uniform lattice in some semisimple Lie group G.

1.3.2. *Convex projective orbifolds with generalized cusps.* In [CLT18], Cooper-Long-Tillmann studied a different generalization of geometrical finiteness in the context of convex projective geometry. They considered the case of a strictly convex projective $(d-1)$-manifold (possibly with boundary) which decomposes into a compact piece and finitely many *generalized cusps*. In the language of [CLT18], a *generalized cusp* is a strictly convex projective $(d-1)$-manifold homemorphic to $N \times [0, \infty)$, where N is a closed $(d-2)$-manifold with virtually nilpotent fundamental group.

Later, Ballas-Cooper-Leitner [BCL20] classified generalized cusps into d different *cusp types*, in particular showing that a generalized cusp always has virtually abelian fundamental group. A "type 0" generalized cusp is projectively equivalent to a hyperbolic cusp (and has virtually unipotent holonomy), while a "type $(d-1)$" generalized cusp has virtually diagonalizable holonomy. The other cusp types are "interpolations" between type 0 and type $d-1$.

Now fix a strictly convex projective $(d-1)$-manifold M, and assume that M is a union of a compact manifold and finitely many generalized cusps. The holonomy of M is a relative Anosov representation if and only if all of its cusps have type 0 (see [Zhu21]), and the image of the holonomy is convex cocompact (in the sense mentioned previously) if and only if all of its cusps have type $d-1$. On the other hand, the framework of EGF representations still applies in the presence of all types of cusps:

Theorem 1.4. *For every d and for every $0 \leq t < d-1$, there exists a convex projective $(d-1)$-manifold with a type t generalized cusp whose holonomy is EGF.*

The fact that for every d and every $0 < t < d-1$ there even exists a convex projective $(d-1)$-manifold containing a type-t cusp is a theorem of Bobb [Bob19]; the content of Theorem 1.4 is that Bobb's construction produces examples of EGF representations. Our proof relies on the fact that both EGF representations and holonomy representations of manifolds with generalized cusps are *relatively stable*. Specifically, for a relatively hyperbolic pair $(\Gamma, \mathcal{H})$, we let

$$\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$$

denote the space of *virtual flag* representations of Γ into $\mathrm{PGL}(d, \mathbb{R})$: the space of representations $\rho : \Gamma \to \mathrm{PGL}(d+1, \mathbb{R})$ such that, for each $H \in \mathcal{H}$, there is a finite-index subgroup $H' \subset H$ so that the restriction $\rho|_{H'}$ is discrete faithful and $\rho(H')$ is conjugate to a group of upper-triangular matrices.

Cooper-Long-Tillmann show (see [CLT18, Thm. 0.1]) that holonomy representations of convex projective manifolds with generalized cusps form an open subset of $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$. We prove:

Theorem 1.5. *Let $M = \Omega/\Gamma$ be a finite-volume convex projective $(d-1)$-manifold, and suppose that Ω is strictly convex (so that $\Gamma = \pi_1 M$ is hyperbolic relative to the collection $\mathcal{H}$ of cusp groups, and the holonomy $\rho : \pi_1 M \to \mathrm{PGL}(d, \mathbb{R})$ is 1-EGF with a boundary extension ϕ).*

Then $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ is peripherally stable at (ρ, ϕ). In particular, due to Theorem 1.1, an open subset of $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ consists of EGF representations.

Theorem 1.5 implies Theorem 1.4 because known constructions of convex projective manifolds with generalized cusps (see [Bal21], [BM20], [Bob19]) proceed by starting with a manifold $M = \Omega/\Gamma$ as above, and then performing an (arbitrarily small) deformation of the holonomy of M inside of $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 M, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ to produce a new convex projective structure on M which realizes its ends as generalized cusps.

Remark 1.6. In [CM14], Crampon-Marquis defined several notions of geometrical finiteness for strictly convex projective manifolds, and claimed that their definitions were all equivalent. It appears that this was an error, and some of their definitions are actually stronger than others. None of their definitions allow for the presence of generalized cusps which are not "type 0" in the Ballas-Cooper-Leitner classification.

On the other hand, work of Cooper-Long-Tillmann [CLT15, Thm 11.6] implies that any finite-volume strictly convex projective manifold $M = \Omega/\Gamma$ as in Theorem 1.5 only has cusps of type 0, and is actually geometrically finite in the *strongest* sense defined by Crampon-Marquis. Zhu [Zhu21, Prop 8.7] proved that the holonomy representation of such a geometrically finite manifold is always relative Anosov (relative to the cusp groups), hence EGF by [Wei22, Thm. 1.10]. This justifies the assertion in the first paragraph of Theorem 1.5.

1.3.3. *Compositions with symmetric representations.* The last class of examples we consider in this paper also derive from the convex cocompact representations discussed in Section 1.3.1. Suppose that Γ is a relatively hyperbolic group, relative to a collection $\mathcal{H}$ of virtually abelian subgroups. Let V be the vector space $\mathbb{R}^d$, and let $\bar{\rho} : \Gamma \to \mathrm{PGL}(V)$ be a discrete faithful representation whose image is a convex cocompact group in the sense of Danciger-Guéritaud-Kassel. Possibly after replacing Γ with a finite-index subgroup, we may lift $\bar{\rho}$ to a discrete faithful representation $\rho : \Gamma \to \mathrm{SL}(V)$. Abusing terminology slightly, we will refer to both $\bar{\rho}$ and ρ as *convex cocompact* representations.

If Γ is a hyperbolic group, and ρ is a P-Anosov representation for some parabolic $P \subset \mathrm{SL}(V)$, then the composition of ρ with the symmetric representation

$$\tau_k : \mathrm{SL}(V) \to \mathrm{SL}(\mathrm{Sym}^k(V))$$

is a new representation of Γ which is P'-Anosov, for some parabolic $P' \subset \mathrm{SL}(\mathrm{Sym}^k(V))$ depending only on P. Due to the close connection between $P_{1,d-1}$-Anosov representations and projectively convex cocompact representations, one might hope that even when Γ is *not* hyperbolic, but ρ is convex cocompact, then the composition $\tau_k \circ \rho$ is still convex cocompact.

It appears that this is not the case: we do *not* expect the composition $\tau_k \circ \rho$ to be convex cocompact unless Γ is a hyperbolic group. However, we can still show:

Theorem 1.7. *Let Γ be hyperbolic relative to virtually abelian subgroups, and let $\rho : \Gamma \to \mathrm{SL}(V)$ be projectively convex cocompact. For any $k \geq 1$, the representation $\tau_k \circ \rho$ is EGF, with respect to the parabolic $P' \subset \mathrm{SL}(\mathrm{Sym}^k(V))$ stabilizing a line in a hyperplane.*

In [DGK17], Danciger-Guéritaud-Kassel showed that projectively convex cocompact representations are *absolutely* stable: if $\rho : \Gamma \to \mathrm{SL}(V)$ is projectively convex cocompact, then any sufficiently small deformation of ρ in $\mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is also projectively convex cocompact (in particular, it is discrete with finite kernel). The proof in [DGK17] does *not* apply to the representations in Theorem 1.7. However, we prove the following:

Theorem 1.8. *Let ρ, Γ be as in Theorem 1.7, so that $\tau_k \circ \rho$ is an EGF representation for every $k \geq 1$. Then for some boundary extension ϕ for $\tau_k \circ \rho$, the entire subspace $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^k(V))$ is peripherally stable about $\tau_k \circ \rho$, ϕ.*

In particular, due to Theorem 1.1, an open subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^k(V)))$ containing $\tau_k \circ \rho$ consists of EGF representations.

Thus, taking $k > 1$, Theorem 1.8 provides new examples of discrete subgroups of higher-rank Lie groups which are *absolutely* stable in their representation varieties. (When $k = 1$, the stability of these representations follows from the stability theorem of Danciger-Guéritaud-Kassel.)

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2. Review of EGF representations

In this section we briefly review some of the definitions, terminology, and basic results surrounding extended geometrically finite representations. Although EGF representations can be defined with respect to any (symmetric) parabolic subgroup P of a semisimple Lie group G, in this paper we will only consider the theory in the case where $G = \mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$, and P is the stabilizer of a flag of type $(1, d-1)$ in $\mathbb{R}^d$. We refer to [Wei22] for a more thorough introduction to the theory, including some background regarding relatively hyperbolic groups and semisimple Lie groups.

2.1. Extended convergence group actions.

The definition of an EGF representation is based on a characterization of Anosov representations in terms of topological dynamics—specifically, in terms of *convergence actions*. Recall that if Γ is a group acting by homeomorphisms on a Hausdorff space M, we say that Γ is a *convergence group* and that the action is a *convergence group action* if, for every divergent sequence $\gamma_n \in \Gamma$, after extracting a subsequence, one can find (not necessarily distinct) points $a, b \in M$ so that the restrictions $\gamma_n|_{M - \{a\}}$ converge to the constant map b, uniformly on compacts. The point a can be thought of as an "repelling point" for the divergent sequence γ_n, and the point b can be thought of as an "attracting point;" the complement of the "repelling point" is a "basin of attraction" for the sequence γ_n. When $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, then Γ acts as a convergence group on the Bowditch boundary $\partial(\Gamma, \mathcal{H})$.

In rank one, geometrically finite representations can be characterized using convergence group actions of relatively hyperbolic groups. To define *extended* geometrically finite actions, we broaden the definition in two ways simultaneously. Essentially, we no longer require the "basin of attraction" for a divergent sequence to be the complement of a singleton, and we no longer require the "attracting point" to be a singleton either.

Definition 2.1. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, acting on a Hausdorff space M by homeomorphisms. Let $\Lambda \subset M$ be a closed Γ-invariant set. We say that a Γ-equivariant surjective map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ *extends the convergence group action* of Γ on $\partial(\Gamma, \mathcal{H})$ if, for each $z \in \partial(\Gamma, \mathcal{H})$, there exists an open set $C_z \subset M$ satisfying:

(1) For every $z \in \partial(\Gamma, \mathcal{H})$, we have $\Lambda \subset C_z \cup \phi^{-1}(z)$.
(2) For every sequence $\gamma_n \in \Gamma$ such that $\gamma_n \to z_+$ and $\gamma_n^{-1} \to z_-$ for $z_\pm \in \partial(\Gamma, \mathcal{H})$, every compact subset $K \subset C_{z_-}$, and every open set $U \subset M$ containing $\phi^{-1}(z_+)$, we have $\gamma_n K \subset U$ for all sufficiently large n.

2.2. **Linear actions on flags in $\mathbb{R}^d$.** An extended geometrically finite representation into $\mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$ is essentially just an extended convergence action of a relatively hyperbolic group on the space of *flags* in $\mathbb{R}^d$, with an extra condition taking into account some of the additional structure on this flag space.

Notation 2.2. When V is a real vector space, we use $\mathrm{Gr}(k, V)$ to denote the Grassmannian of k-planes in V; recall that $\mathrm{Gr}(1, V)$ is the same as the projective space $\mathbb{P}(V)$. When $\dim(V) = d$, recall that there is also a canonical equivariant identification of $\mathrm{Gr}(d-1, V)$ with the *dual* projective space $\mathbb{P}(V^*)$, since the projectivization of a nonzero linear functional in V^* is determined by its kernel.

We let $\mathcal{F}(V)$ denote the space of *flags* of type $(1, \dim(V) - 1)$ in V, i.e. pairs of subspaces (V_i, V_j) such that $V_i \subset V_j \subset V$ and $\dim(V_i) = \mathrm{codim}(V_j) = 1$. We will write $\mathrm{Gr}(k, d)$ and $\mathcal{F}(d)$ as shorthand for $\mathrm{Gr}(k, \mathbb{R}^d)$ and $\mathcal{F}(\mathbb{R}^d)$, respectively.

Definition 2.3. We say that a pair of flags $F = (V, W)$, $F' = (V', W')$ in $\mathcal{F}(d)$ are *transverse* if $V \oplus W' = \mathbb{R}^d$ and $W \oplus V' = \mathbb{R}^d$.

For any $1 \le k < d$, and any subspace $V_k \subset \mathrm{Gr}(k, d)$, we let $\mathrm{Opp}(\xi) \subset \mathrm{Gr}(d-k, d)$ denote the space of $(d-k)$-planes transverse to V_k. Similarly, for a fixed flag $\xi \in \mathcal{F}(d)$, we let $\mathrm{Opp}(\xi)$ denote the space of flags

$$\{\nu \in \mathcal{F} : \nu \text{ is transverse to } \xi\}.$$

This is always an open dense subset of $\mathcal{F}(d)$. For a subset $X \subset \mathcal{F}$, we also let $\mathrm{Opp}(X)$ denote the set

$$\bigcap_{\xi \in X} \mathrm{Opp}(\xi).$$

Definition 2.4. Let g_n be a sequence in $\mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$, and let $1 \le k < d$.

(1) The sequence g_n is called *k-contracting* if there is a nonempty open subset $U \subset \mathrm{Gr}(k, d)$ such that $g_n U$ converges to a singleton $\{\xi\}$. The point $\xi \in \mathrm{Gr}(k, d)$ is called the *k-limit* of the sequence g_n.
(2) The sequence g_n is called *k-divergent* if every subsequence of g_n has a further subsequence which is k-contracting.
(3) A subgroup Γ in $\mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$ is called *k-divergent* if every sequence of pairwise distinct elements in Γ is k-divergent.

Any divergent sequence in $\mathrm{PGL}(d, \mathbb{R})$ must be k-divergent for at least one k with $1 \le k < d$. When g_n is a k-divergent sequence in $\mathrm{PGL}(d, \mathbb{R})$, then the sequence of inverses g_n^{-1} is $(d-k)$-divergent. More precisely, we have the following fact, which can be verified using the singular value decomposition of elements in $\mathrm{PGL}(d, \mathbb{R})$ (or for a general statement see [KLP17, Lemma 4.19] and [Wei22, Appendix A]):

Fact 2.5. For a sequence $g_n \in \mathrm{PGL}(d, \mathbb{R})$ and points $\xi_- \in \mathrm{Gr}(d-k, d)$ and $\xi_+ \in \mathrm{Gr}(k, d)$, the following are equivalent:

(1) g_n is k-contracting and $g_n|_{\mathrm{Opp}(\xi_-)} \to \xi_+$ uniformly on compacts.
(2) g_n is k-divergent, g_n has unique k-limit point ξ_+, and g_n^{-1} has unique $(d-k)$-limit point ξ_-.

When a sequence g_n in $\mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$ is both 1-contracting and $(d-1)$-contracting (resp. 1-divergent and d-divergent), we will say that it is $(1, d-1)$-*contracting* or $(1, d-1)$-*divergent*. Owing to Fact 2.5, any 1-divergent subgroup of $\mathrm{PGL}(d, \mathbb{R})$ or $\mathrm{SL}(d, \mathbb{R})$ is automatically also $(d-1)$-divergent, and vice-versa, so we will also call such subgroups $(1, d-1)$-divergent.

Remark 2.6. The more usual definition of $(1, d-1)$-divergence (or more generally, P-*divergence* for a parabolic subgroup P in a semisimple Lie group G) is stated in terms of the behavior of the *Cartan projection* of a sequence $g_n \in G$. We refer again to [KLP17, Lemma 4.19] and [Wei22, Appendix A] for the equivalence.

2.3. EGF representations.

Definition 2.7. Let $\Lambda \subset \mathcal{F}(d)$ be a subset, and let $\phi : \Lambda \to Z$ be a surjective map to some space Z. We say that ϕ is *transverse* if for every pair of distinct points $z_1, z_2 \in Z$, every flag in $\phi^{-1}(z_1)$ is transverse to every flag in $\phi^{-1}(z_2)$.

Definition 2.8. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ (or to $\mathrm{SL}(d, \mathbb{R})$) is 1-*extended geometrically finite* or (1-EGF) if there exists a compact ρ-invariant subset $\Lambda \subset \mathcal{F}(d)$ and a surjective transverse map $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ extending the convergence action of Γ on $\partial(\Gamma, \mathcal{H})$.

The map ϕ is called a *boundary extension* of the representation ρ, and the set Λ is called a *boundary set*. Since we are only concerned with 1-EGF representations in this paper, we will almost always just refer to 1-EGF representations as *EGF representations*.

Remark 2.9. In general there may be more than one possible choice for the boundary set and boundary extension associated to a given EGF representation; there might also be more than one possible choice for the open sets $C_z \subset \mathcal{F}(d)$ specified in Definition 2.1. It is also always possible to choose the boundary extension ϕ so that the preimage of any *conical limit point* (see Definition 2.10 below) in $\partial(\Gamma, \mathcal{H})$ is a singleton; see [Wei22, Proposition 4.8].

2.4. An alternative characterization.

In order to use Definition 2.8 to directly verify that a given representation is EGF, we must consider the dynamical behavior of (essentially) arbitrary divergent sequences in Γ, since any divergent sequence γ_n in a relatively hyperbolic group has a subsequence which satisfies $\gamma_n \to z_+$ and $\gamma_n^{-1} \to z_-$ for points $z_\pm \in \partial(\Gamma, \mathcal{H})$. Often, we will only want to consider sequences in Γ which either limit to the boundary of Γ *along geodesics* (i.e. *conical limit sequences*), or sequences which diverge inside of a fixed peripheral subgroup of Γ.

In Proposition 2.11 below, we give an alternative characterization of EGF representations which allows us to restrict our attention to sequences of one of these two forms. We first recall some terminology:

Definition 2.10. Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair. We say that a sequence $\gamma_n \in \Gamma$ *limits conically to* a point $z \in \partial(\Gamma, \mathcal{H})$ if there exist *distinct* points $a, b \in \partial(\Gamma, \mathcal{H})$ such that $\gamma_n^{-1} z \to a$ and $\gamma_n^{-1} y \to b$ for every $y \neq z$. If a sequence $\gamma_n \in \Gamma$ limits conically to z, then we say that z is a *conical limit point*.

Every point in the Bowditch boundary of a relatively hyperbolic pair $(\Gamma, \mathcal{H})$ is either a conical limit point or a *parabolic point*, i.e. the (unique) fixed point of a peripheral subgroup in $\mathcal{H}$. When p is a parabolic point, we let $\Gamma_p \in \mathcal{H}$ denote the peripheral subgroup stabilizing p.

We can now state our alternative characterization of EGF representations:

Proposition 2.11 (See [Wei22, Proposition 4.6])**.** *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a representation of a relatively hyperbolic group, and let $\Lambda \subset \mathcal{F}(d)$ be a closed $\rho(\Gamma)$-invariant set. Suppose that $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ is a continuous surjective ρ-equivariant transverse map.*

Then ρ is an EGF representation with EGF boundary extension ϕ if and only if both of the following conditions hold:

1. *For any sequence $\gamma_n \in \Gamma$ limiting conically to some point in $\partial(\Gamma, \mathcal{H})$, the sequences $\rho(\gamma_n^{\pm 1})$ are $(1, d-1)$-divergent, and every $(1, d-1)$-limit point of $\rho(\gamma_n^{\pm 1})$ lies in Λ.*
2. *For every parabolic point $p \in \partial(\Gamma, \mathcal{H})$, there exists an open set $C_p \subset \mathcal{F}$, with $\Lambda \subset C_p \cup \phi^{-1}(p)$, such that for any compact $K \subset C_p$ and any open set U containing $\phi^{-1}(p)$, for all but finitely many $\gamma \in \Gamma_p$, we have $\rho(\gamma) \cdot K \subset U$.*

2.5. **Peripheral stability.** The central result of [Wei22] is that EGF representations are relatively stable: any sufficiently small deformation which satisfies a dynamical condition on peripheral subgroups is still EGF.

For an EGF representation ρ with boundary extension ϕ, if p is a parabolic point in $\partial(\Gamma, \mathcal{H})$, then the set $\phi^{-1}(p)$ can be thought of as *both* an "attracting set" and a "repelling set" for the action of Γ_p on $\mathcal{F}(d)$. More precisely, if K is any compact set in C_p, and U is any neighborhood of $\phi^{-1}(p)$, there is always some finite set F such that $\rho(\Gamma_p - F)K \subset U$.

We want to consider deformations of the representation ρ which satisfy the property that some set close to $\phi^{-1}(p)$ is still an "attracting set" for Γ_p with "basin of attraction" C_p. We also want to ask for the "strength" of the attraction to not decrease too much, which is quantified by the compact set $K \subset C_p$, the open set $U \supset \phi^{-1}(p)$, and the finite set F mentioned previously.

Definition 2.12 (Peripheral stability)**.** Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be an EGF representation with boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$. We say that a subspace $\mathcal{W} \subset \mathrm{Hom}(\Gamma, \mathrm{PGL}(d, \mathbb{R}))$ is *peripherally stable* (with respect to the data (ρ, ϕ)) if, for every parabolic point $p \in \partial(\Gamma, \mathcal{H})$, every open subset $U \subset \mathcal{F}(d)$ containing $\phi^{-1}(p)$, every compact set $K \subset C_p$, and every finite subset $F \subset \Gamma_p$ such that

$$\rho(\Gamma_p - F)K \subset U,$$

there is an open neighborhood $\mathcal{W}' \subset \mathcal{W}$ containing ρ, such that every $\rho' \in \mathcal{W}$ satisfies

$$\rho'(\Gamma_p - F)K \subset U.$$

Theorem 1.1 asserts that small deformations of EGF representations inside of peripherally stable subspaces remain EGF. The point of the peripheral stability condition is that it can be verified by only considering how deformations of some representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ behave when they are restricted to peripheral subgroups of Γ. If the peripheral subgroups are not too complicated (for instance, if they are virtually nilpotent, or even virtually abelian), then one can hope to get a direct understanding of how their actions on $\mathcal{F}(d)$ deform in certain subspaces of the representation variety $\mathrm{Hom}(\Gamma, \mathrm{PGL}(d, \mathbb{R}))$; then Theorem 1.1 makes it possible to upgrade this understanding to the deformed action of the *entire* group Γ on the same subspace.

2.6. **EGF representations and relative Anosov representations.** As stated in the introduction, EGF representations generalize previous definitions of relative Anosov representations provided by Kapovich-Leeb [KL18], Zhu [Zhu21], and Zhu-Zimmer [ZZ22].

By a “relative Anosov representation,” in this paper we mean the following:

Definition 2.13. Let Γ be a subgroup of $\mathrm{PGL}(d, \mathbb{R})$, and suppose that $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair. We say that Γ is *relatively* 1*-Anosov* if Γ is 1-divergent, and there is a Γ-equivariant transverse embedding $\partial(\Gamma, \mathcal{H}) \to \mathcal{F}(d)$ whose image is the set of $(1, d-1)$-limit points of Γ.

In the work of Kapovich-Leeb [KL18], groups as in Definition 2.13 are called *relatively asymptotically embedded* subgroups of $\mathrm{PGL}(d, \mathbb{R})$; in [ZZ22], it is shown that the definiton essentially agrees with Zhu's notion [Zhu21] of a *relatively dominated* representation.

The relation to EGF representations is given by the following:

Theorem 2.14 (See [Wei22, Theorem 1.10])**.** *Let Γ be a subgroup of $\mathrm{PGL}(d, \mathbb{R})$, and suppose that $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair. Then the following are equivalent:*

(1) The group Γ is relatively 1*-Anosov.*
(2) The inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is a 1*-EGF representation, and there is an associated boundary extension $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$ which is injective.*

Moreover, in this case the associated boundary set $\Lambda \subset \mathcal{F}(d)$ is precisely the set of $(1, d-1)$-limit points of Γ.

3. Convex projective geometry: notation and background

In this section we briefly discuss some background material related to convex projective geometry, as all of the examples of EGF representations in this paper derive from convex projective structures on manifolds. We also establish a few routine results that we will need in later sections.

Notation 3.1. We fix the rest of the following conventions for the rest of the paper.

- If V is a real vector space, then $\mathbb{P}(V)$ denotes the projective space over V, i.e. the space $\mathrm{Gr}(1, V)$ of lines in V.
- For any $x \in V - \{0\}$, we let $[x]$ denote the image of x under the quotient map $V - \{0\} \to \mathbb{P}(V)$.
- If W is a subset of V, then we let $[W]$ denote the image of $W - \{0\}$ in $\mathbb{P}(V)$. If $W \subseteq V$ is a vector subspace, we will identify $\mathbb{P}(W)$ with $[W] \subset \mathbb{P}(V)$.
- When $B \subset \mathbb{P}(V)$, then the *span* of B, denoted $\mathrm{span}(B)$, is the subspace of V spanned by any lift of B in V. The *projective span* of B is the subset $[\mathrm{span}(B)] \subset \mathbb{P}(V)$.
- If w is any element of the *dual* projective space $\mathbb{P}(V^*)$, we let $[\ker w]$ denote the subset of $\mathbb{P}(V)$ given by the image of $\ker \tilde{w}$, where $\tilde{w}$ is any lift of w in V^*.
- If $w \in \mathbb{P}(V^*)$ and $v \in \mathbb{P}(V)$ satisfy $v \notin [\ker w]$, then we say w and v are *transverse* and write $w \perp v$.

3.1. **Convex projective structures.** Let V be a real d-dimensional vector space. Recall that a subset $\Omega \subset \mathbb{P}(V)$ is *properly convex* if $\overline{\Omega}$ is a convex subset of an affine chart in $\mathbb{P}(V)$. A manifold M (possibly with boundary) has a *convex projective structure* if M can be realized as a quotient Ω/Γ for a discrete subgroup $\Gamma \subseteq \mathrm{Aut}(\Omega)$, where

$$\mathrm{Aut}(\Omega) = \{\gamma \in \mathrm{PGL}(V) : \gamma \cdot \Omega = \Omega\}.$$

There are several different possible meanings for the "boundary" of a properly convex set in $\mathbb{P}(V)$, so below we make things explicit:

Definition 3.2. Let $\Omega \subset \mathbb{P}(V)$ be a properly convex set with nonempty interior. In general, Ω might not be either open or closed.

- The *frontier* of Ω is $\mathrm{Fr}(\Omega) = \overline{\Omega} - \mathrm{int}(\Omega)$.
- The *nonideal boundary* of Ω is $\partial_n\Omega = \mathrm{Fr}(\Omega) \cap \Omega$.
- The *ideal boundary* of Ω is $\partial_i\Omega = \mathrm{Fr}(\Omega) - \partial_n\Omega$.

When Ω is a properly convex *open* set, we will use the notation $\partial\Omega$ to mean the *ideal boundary* $\partial_i(\Omega)$, which coincides with the frontier $\mathrm{Fr}(\Omega)$ in this case. Note that this conflicts with the convention in e.g. [CLT18; BCL20].

When Ω is open, we say it is a *properly convex domain*. Then any manifold $M = \Omega/\Gamma$ has empty boundary. If $M = \Omega/\Gamma$ is a convex projective manifold with boundary, then its boundary ∂M is identified with $\partial_n\Omega/\Gamma$. When $\partial_n\Omega$ contains no nontrivial projective segments, then we say that the manifold M has *strictly convex boundary*.

3.1.1. *Faces in convex domains.* If $\Omega \subset \mathbb{P}(V)$ is a properly convex domain, a *face* of Ω is an equivalence class in $\partial\Omega$ under the relation which identifies distinct points $x, y \in \partial\Omega$ if there is an open projective line segment in $\partial\Omega$ containing both x and y.

If $x \in \partial\Omega$, we let $F_\Omega(x)$ denote the unique face of Ω containing x. The *support* of a face F, denoted $\mathrm{supp}(F)$, is the projective span $[\mathrm{span}(F)] \subset \mathbb{P}(V)$. The *boundary* of a face ∂F is the boundary of F when F is viewed as a convex open subset of $\mathrm{supp}(F)$.

3.1.2. *The Hilbert metric.* Whenever Ω is a properly convex domain, one can define the *Hilbert metric* on Ω as follows:

Definition 3.3. For a pair of distinct points $x, y \in \Omega$, we let

$$d_\Omega(x, y) = \frac{1}{2}\log[u, v; x, y],$$

where u, v are the two points in $\partial\Omega$ such that u, x, y, v lie on a projective line in that order, and $[a, b; c, d]$ is the *cross-ratio*

$$[a, b; c, d] = \frac{(d-a)(c-b)}{(c-a)(d-b)}.$$

Here the differences can be measured under any affine identification of the projective line spanned by x, y with $\mathbb{R} \cup \{\infty\}$.

It turns out that (Ω, d_Ω) is a proper geodesic metric space, on which $\mathrm{Aut}(\Omega)$ acts by isometries. This means that $\mathrm{Aut}(\Omega)$ acts properly on Ω—in particular, discrete subgroups of $\mathrm{Aut}(\Omega)$ act properly discontinuously.

The Hilbert metric gives us another perspective on the faces of Ω. Each face F of Ω is a properly convex subset of $\mathbb{P}(V)$, open in its own projective span. This allows us to define a restricted Hilbert metric d_F on F.

The proposition below is a standard result in the theory of convex projective structures, and is a direct consequence of the definition of the Hilbert metric.

Proposition 3.4. *Let Ω be a properly convex domain, let F be a face of Ω, and let x_n be a sequence in Ω converging to some $x_\infty \in F$.*

For any $D > 0$, if $y_n \in \Omega$ is a sequence satisfying

$$d_\Omega(x_n, y_n) \leq D,$$

then any accumulation point y_∞ of y_n lies in F, and

$$d_F(x_\infty, y_\infty) \leq D.$$

3.1.3. *Dynamics of* $\mathrm{Aut}(\Omega)$. Fix a divergent sequence $g_n \in \mathrm{PGL}(d, \mathbb{R})$. This sequence is k-divergent for some k with $1 \leq k < d$, so fix such a k, and choose a subsequence of g_n which is k-contracting and whose sequence of inverses is $(d-k)$-contracting. There are then uniquely determined projective subspaces $E_+, E_- \subset \mathbb{R}^d$ with complementary dimension such that for any $x \in \mathbb{P}(\mathbb{R}^d) - E_-$, the sequence $g_n x$ accumulates on E_+, uniformly on compact subsets of $\mathbb{P}(\mathbb{R}^d) - E_-$. We refer to the subspaces E_+, E_- as *attracting* and *repelling* subspaces, respectively. We emphasize that E_+ and E_- are *not* necessarily transverse.

The faces of Ω are related to attracting and repelling subspaces of divergent sequences in $\mathrm{Aut}(\Omega)$:

Proposition 3.5. *Let γ_n be a divergent sequence in* $\mathrm{Aut}(\Omega)$ *for a properly convex domain Ω, and suppose that for some $x \in \Omega$, the sequence $\gamma_n x$ accumulates on a face F_+ of Ω. Then, after extracting a subsequence, there is an attracting subspace E_+ of γ_n such that $[E_+] \subseteq \mathrm{supp}(F_+)$.*

Proof. Let B be the ball of radius 1 about x with respect to the Hilbert metric on Ω, and let $x_\infty \in F_+$ be an accumulation point of the point of the sequence $\gamma_n x$. Using a diagonal argument, we can replace γ_n with a subsequence so that for every y in a countable dense subset of B, the sequence $\gamma_n \cdot y$ has a well-defined limit in the compact space $\mathbb{P}(V)$. Proposition 3.4 then implies that for every point $y \in B$, the sequence $\gamma_n y$ converges to a unique point in F_+.

Let B_∞ be the set of accumulation points of $\gamma_n \cdot B$, and let W_∞ be the subspace $\mathrm{span}(B_\infty) \subset V$.

Proposition 3.4 implies that B is a subset of the face $F = F_\Omega(x_\infty)$, so $[W_\infty]$ is a projective subspace of $\mathrm{supp}(F)$. Let $k = \dim(W_\infty)$. We claim that there is an open subset U of the Grassmannian $\mathrm{Gr}(k, V)$ so that

$$\gamma_n U \to \{W_\infty\}.$$

This implies the desired result by e.g. [Wei22, Prop. 3.6].

To see the claim, fix k points $z_1, \ldots, z_k \in B_\Omega(x, 1)$ so that the limits of the sequences $\gamma_n z_1, \ldots \gamma_n z_k$ span the projective subspace $[W_\infty]$. Proposition 3.4 implies that for some fixed $\varepsilon > 0$, if z_i' lies in the ball of radius ε about z_i, then the limits of the sequences

$$\gamma_n z_1', \ldots, \gamma_n z_k'$$

are in general position, and therefore also span $[W_\infty]$.

For each $1 \leq i \leq k$, let B_i denote the ball of radius ε about z_i. Then, if U is the open set

$$\{W \in \mathrm{Gr}(k, V) : W = u_1 \oplus \ldots \oplus u_k, [u_i] \in B_i\},$$

we have that $\gamma_n U \to \{W_\infty\}$, as required. ∎

4. Convex cocompactness in projective space

Fix a real vector space V of dimension d. Our goal in this section is to prove Theorem 1.2, which says that *convex cocompact* representations of relatively hyperbolic groups in $\mathrm{PGL}(V)$ give examples of EGF representations. We begin by recalling the precise definition of a (projectively) convex cocompact group in $\mathbb{P}(V)$.

Definition 4.1 ([DGK17], Definitions 1.10 and 1.11)**.** Let Ω be a properly convex domain in $\mathbb{P}(V)$, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$ be discrete.

(1) The *full orbital limit set* $\Lambda_\Omega(\Gamma)$ of Γ is the union (over all $x \in \Omega$) of the set of accumulation points in $\partial\Omega$ of $\Gamma \cdot x$.
(2) The group Γ *acts convex cocompactly on* Ω if the convex hull of $\Lambda_\Omega(\Gamma)$ is nonempty, and Γ acts cocompactly on the convex hull of $\Lambda_\Omega(\Gamma)$ in Ω.
(3) Γ is *convex cocompact in* $\mathbb{P}(V)$ if it acts convex cocompactly on some properly convex domain $\Omega \subset \mathbb{P}(V)$.

Danciger-Guéritaud-Kassel show that if Γ is a hyperbolic group acting convex cocompactly in $\mathbb{P}(V)$, then the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ is a 1-Anosov representation.

In [Wei20], we showed that any group acting convex cocompactly on some domain $\Omega \subset \mathbb{P}(V)$ acts with "Anosov-like" expansion dynamics on the faces of $\partial\Omega$. We further showed that, when $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, Γ acts convex cocompactly on Ω, and each $H \in \mathcal{H}$ also acts convex cocompactly on Ω, then there is an equivariant embedding of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ into a *quotient* of $\partial\Omega$.

In subsequent work [IZ22], Islam-Zimmer showed that one does not need to assume that each $H \in \mathcal{H}$ acts convex cocompactly on Ω: it turns out that this follows automatically from the fact that Γ acts convex cocompactly. Together with our earlier work, this gives the following:

Theorem 4.2 (See [Wei20], Theorem 1.16 and [IZ22], Theorem 1.6)**.** *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and suppose $\rho(\Gamma)$ acts convex cocompactly on a properly convex domain Ω. Then there is an equivariant homeomorphism*

$$\psi : \partial(\Gamma, \mathcal{H}) \to \Lambda_\Omega(\Gamma)/\sim,$$

where $x \sim y$ if $x, y \in \Lambda_\Omega(H)$ for some $H \in \mathcal{H}$.

Remark 4.3. Islam-Zimmer also proved a version of Theorem 4.2 in the more general context of *naive* convex cocompact group actions, which we do not discuss here.

Before we can proceed with the proof of Theorem 1.2, we will need a slightly different perspective on the full orbital limit set of a discrete group $\Gamma \subseteq \mathrm{Aut}(\Omega)$.

Definition 4.4. Let Ω be a convex projective domain. The *dual domain* $\Omega^* \subset \mathbb{P}(V^*)$ is the set

$$\Omega^* = \{w \in \mathbb{P}(V^*) : [\ker w] \cap \overline{\Omega} = \emptyset\}.$$

If $\Gamma \subset \mathrm{PGL}(V)$ is a subgroup of $\mathrm{Aut}(\Omega)$, then its dual $\Gamma^* \subset \mathrm{PGL}(V^*)$ is a subgroup of $\mathrm{Aut}(\Omega^*)$. If Γ is convex cocompact in $\mathbb{P}(V)$, then [DGK17], Proposition 5.6 implies that there is some Γ-invariant domain Ω so that Γ acts convex cocompactly on Ω, and Γ^* acts convex cocompactly on Ω^*.

Definition 4.5. Let $\Gamma \subseteq \mathrm{Aut}(\Omega)$.

(1) The *dual full orbital limit set* $\Lambda^*_\Omega(\Gamma)$ is the full orbital limit set of Γ^* in $\partial\Omega^*$.

(2) The *flag-valued full orbital limit set* $\hat{\Lambda}_\Omega(\Gamma)$ is the set
$$\hat{\Lambda}_\Omega(\Gamma) := \{(x, w) \in \mathcal{F}(V) : x \in \Lambda_\Omega(\Gamma), w \in \Lambda^*_\Omega(\Gamma)\}.$$
(3) The *maximal domain* $\Omega_{\max}(\Gamma)$ is the unique connected component of
$$\mathbb{P}(V) - \bigcup_{w \in \Lambda^*_\Omega(\Gamma)} [\ker w]$$
containing Ω. Equivalently, $\Omega_{\max}(\Gamma)$ is the dual of the convex hull of $\Lambda^*_\Omega(\Gamma)$ in Ω^*.

We emphasize that $\Omega_{\max}(\Gamma)$ is *not* necessarily a properly convex set, which means that we cannot always define a Hilbert metric on it (so we do not have a guarantee that Γ acts properly discontinuously in general). However, when Γ acts convex cocompactly on Ω, we do get a properly discontinuous action, thanks to the following argument suggested by Jeff Danciger and Fanny Kassel:

Proposition 4.6. *Let Γ act convex cocompactly on a properly convex comain Ω. For any sequence $\gamma_n \in \Gamma$ and any $x \in \Omega_{\max}(\Gamma)$, the sequence $\gamma_n \cdot x$ accumulates in $\Lambda_\Omega(\Gamma)$, uniformly on compacts. In particular, Γ acts properly discontinuously on $\Omega_{\max}(\Gamma)$.*

Proof. When $\Omega_{\max}(\Gamma)$ is a *properly* convex domain, this follows immediately from Proposition 4.18 in [DGK17], which says that whenever Γ acts convex cocompactly on some domain Ω, and Ω' is any Γ-invariant properly convex domain containing Ω, then Γ acts convex cocompactly on Ω' and $\Lambda_\Omega(\Gamma) = \Lambda_{\Omega'}(\Gamma)$.

So, we consider the case where $\Omega_{\max}(\Gamma)$ is *not* properly convex. We may assume our domain Ω is chosen so that Γ acts convex cocompactly on both Ω and $\Omega^* \subset \mathbb{P}(V^*)$. Since $\Omega_{\max}(\Gamma)$ is not properly convex, its dual $\Omega_{\max}(\Gamma)^*$ (given by the convex hull of $\Lambda^*_\Omega(\Gamma)$ in Ω^*) has empty interior (i.e. it spans a proper projective subspace of $\mathbb{P}(V^*)$).

Given any $\varepsilon > 0$, we let Ω^*_ε be the uniform ε-neighborhood of $\Omega_{\max}(\Gamma)^*$, with respect to the Hilbert metric on Ω^*. We let $\Omega_\varepsilon \subset \mathbb{P}(V)$ denote the dual of Ω^*_ε. Note that Ω_ε is a Γ-invariant properly convex subset of $\Omega_{\max}(\Gamma)$, containing Ω.

Since duality reverses inclusions, and the intersection
$$\bigcap_{\varepsilon \to 0} \Omega^*_\varepsilon$$
is exactly the set $\Omega_{\max}(\Gamma)^*$, the union
$$\bigcup_{\varepsilon \to 0} \Omega_\varepsilon$$
is the set $\Omega_{\max}(\Gamma)$. So, if we fix a compact set $K \subset \Omega_{\max}(\Gamma)$, for some $\varepsilon > 0$ we have $K \subset \Omega_\varepsilon$. Then we apply Proposition 4.18 in [DGK17] to the *properly* convex domain Ω_ε to see that for any sequence $\gamma_n \in \Gamma$, $\gamma_n \cdot K$ accumulates in $\Lambda_\Omega(\Gamma)$. ☐

We need a few more results before we can prove Theorem 1.2. We quote the following observation from [Wei20]:

Proposition 4.7 ([Wei20], Proposition 8.1)**.** *Let $\Gamma \subset \mathrm{PGL}(V)$ act convex cocompactly on a properly convex domain Ω. If Γ is hyperbolic relative to a collection of subgroups $\mathcal{H}$ also acting convex cocompactly on Ω, then every nontrivial projective segment in $\Lambda_\Omega(\Gamma)$ is contained in $\Lambda_\Omega(H)$ for some $H \in \mathcal{H}$.*

Using a theorem of Danciger-Guéritaud-Kassel ([DGK17], Theorem 1.18), one can strengthen this result:

Corollary 4.8. *In the context of Proposition 4.7, it is possible to choose the convex domain Ω so that every nontrivial projective segment in $\partial\Omega$ is contained in $\Lambda_\Omega(H)$ for some $H \in \mathcal{H}$.*

We also observe:

Proposition 4.9. *Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, let $\rho : \Gamma \to \mathrm{PGL}(V)$ be representation such that $\rho(\Gamma)$ acts convex cocompactly on a domain Ω, and let $\psi : \partial(\Gamma, \mathcal{H}) \to \Lambda_\Omega(\Gamma)/\sim$ be the map coming from Theorem 4.2.*

If $z \in \partial(\Gamma, \mathcal{H})$ is a conical limit point, and γ_n is a sequence limiting to z in $\overline{\Gamma} = \Gamma \sqcup \partial(\Gamma, \mathcal{H})$, then $\psi(z) \in \mathbb{P}(V)$ is the unique one-dimensional attracting subspace for $\rho(\gamma_n)$ in $\mathbb{P}(V)$.

Proof. It suffices to show that any subsequence of γ_n has a further subsequence which has $\psi(z)$ as a one-dimensional attracting subspace. So, using the convergence group property, we can take a subsequence and assume that there is some point $y \in \partial(\Gamma, \mathcal{H})$ so that γ_n converges to z on every point in the set $\partial(\Gamma, \mathcal{H}) - \{y\}$. We can further assume that the pair $(\Gamma, \mathcal{H})$ is not elementary, so $\partial(\Gamma, \mathcal{H})$ contains infinitely many points.

So, by Corollary 4.8, we can find distinct points $u, v \in \Lambda_\Omega(\Gamma)$ so that the projective line segment (u, v) lies in Ω, and $\rho(\gamma_n)u$, $\rho(\gamma_n)v$ both converge to $\psi(z)$. Corollary 4.10 in [DGK17] implies that $F_\Omega(\psi(z)) \subset \Lambda_\Omega(\Gamma)$, and then Corollary 4.8 implies that $F_\Omega(\psi(z)) = \{\psi(z)\}$.

Then for any $x \in (u, v)$, $\rho(\gamma_n)x$ converges to $\psi(z)$, and we are done by Proposition 3.5. ☐

Proof of Theorem 1.2. Fix a d-dimensional real vector space V, let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair, and let $\rho : \Gamma \to \mathrm{PGL}(V)$ be a representation such that $\rho(\Gamma)$ acts convex cocompactly on a properly convex domain $\Omega \subset \mathbb{P}(V)$. This implies (see [IZ22]) that each $H \in \mathcal{H}$ also acts convex cocompactly on Ω.

The first step in the proof is to define a $\rho(\Gamma)$-invariant subset $\hat{\Lambda} \subset \mathcal{F}(V)$ and a boundary extension $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{H})$. We use the map ψ coming from Theorem 4.2 to define an equivariant surjection $\phi : \Lambda \to \partial(\Gamma, \mathcal{H})$, where the preimage of each parabolic point p in $\partial(\Gamma, \mathcal{H})$ is exactly $\Lambda_\Omega(H)$ for $H = \mathrm{Stab}_\Gamma(p)$.

If we let Λ^* be the full orbital limit set of the Γ-action on Ω^*, we can similarly find an equivariant surjection $\phi^* : \Lambda^* \to \partial(\Gamma, \mathcal{H})$, where ${\phi^*}^{-1}(z)$ is a single hyperplane if z is a conical limit point, and the dual full orbital limit set of $\mathrm{Stab}_\Gamma(z)$ if z is a parabolic point.

We consider the set

$$\hat{\Lambda} = \{(x, w) \in \mathcal{F} : x \in \Lambda, w \in \Lambda^*\}.$$

Each element of Λ^* is a supporting hyperplane of the domain Ω. Corollary 4.8 implies that for every point (x, w) in $\hat{\Lambda}$, either:

(1) $x = \phi^{-1}(z)$ and $w = {\phi^*}^{-1}(z)$ for a conical limit point $z \in \partial(\Gamma, \mathcal{H})$, or
(2) $x \in \Lambda_H$ and $w \in \Lambda^*_H$ for a peripheral subgroup $H \in \mathcal{H}$.

This allows us to combine ϕ and ϕ^* to get a well-defined equivariant surjection $\hat{\phi} : \hat{\Lambda} \to \partial(\Gamma, \mathcal{H})$.

The next step is to define the open subsets $C_z \subset \mathcal{F}$ for each $z \in \partial(\Gamma, \mathcal{H})$. If $z \in \partial(\Gamma, \mathcal{H})$ is a conical limit point, we define the set C_z by

$$C_z = \{\nu \in \mathcal{F} : \nu \text{ is opposite to } \phi^{-1}(z)\}.$$

Otherwise, if z is a parabolic point, we consider the maximal domain $\Omega_{\max}(H) \subset \mathbb{P}(V)$ for $H = \mathrm{Stab}_\Gamma(z)$. Dually, we can define $\Omega^*_{\max}(H) \subset \mathbb{P}(V^*)$, viewing $\Lambda_\Omega(H)$ as the dual full orbital limit set of H^* acting on Ω^*.

Then, we define

$$C_z = \{(x, w) \in \mathcal{F}(V) : x \in \Omega_{\max}(H), w \in \Omega^*_{\max}(H)\}.$$

For every $z \in \partial(\Gamma, \mathcal{H})$, C_z is open, and Corollary 4.8 implies that C_z contains $\hat{\phi}^{-1}(z')$ for every $z' \neq z$ in $\partial\Gamma$.

The last step is to check that the map $\hat{\phi}$ actually extends the convergence group action of Γ on $\partial(\Gamma, \mathcal{H})$, using the sets C_z. To do so, we appeal to Proposition 2.11.

First, let γ_n be a sequence in Γ limiting conically to $z \in \partial(\Gamma, \mathcal{H})$. Proposition 4.9 implies that $\phi^{-1}(z)$ is the unique one-dimensional attracting subspace for $\rho(\gamma_n)$ in $\mathbb{P}(V)$. Dually, $(\phi^*)^{-1}(z)$ is the unique one-dimensional attracting subspace for $\rho(\gamma_n)$ in $\mathbb{P}(V^*)$. So the sequence $\rho(\gamma_n)$ is $(1, d-1)$-divergent, and so is the sequence $\rho(\gamma_n^{-1})$.

Further, observe that if γ_n is any sequence in Γ such that $\rho(\gamma_n)$ is 1-divergent, then any 1-limit point of $\rho(\gamma_n)$ must lie $\Lambda_\Omega(\Gamma)$. The same holds for $(d-1)$-divergent sequences and $\Lambda^*_\Omega(\Gamma)$. Together this implies that the first condition of Proposition 2.11 is satisfied.

On the other hand, if z is a parabolic point and γ_n is an infinite sequence in $\mathrm{Stab}_\Gamma(z)$, Proposition 4.6 implies that for any compact $K \subset C_z$, the set $\rho(\gamma_n) \cdot K$ eventually lies in any given neighborhood of $\phi^{-1}(z)$, which fulfills the second condition of Proposition 2.11. ☐

5. Generalized cusps

In this section, we wish to consider deformations of the holonomy of a finite-volume convex projective manifold $M = \Omega/\Gamma$, where Ω is a strictly convex subset of projective space with nonempty interior. (Note that, in contrast to the previous section, in this section we assume that Ω is strictly convex, but we do *not* assume that it is an open subset of projective space).

As mentioned in the introduction to this paper, whenever M is such a manifold, work of Cooper-Long-Tillman [CLT15], Crampon-Marquis [CM14], and Zhu [Zhu21] implies that the holonomy representation $\rho : \pi_1 M \to \mathrm{PGL}(d, \mathbb{R})$ is relatively 1-Anosov in the sense of Definition 2.13, and therefore an EGF representation by Theorem 2.14.

Our goal in this section is to prove Theorem 1.5, which says that certain deformations of ρ are peripherally stable, and hence give rise to EGF representations. These peripherally stable deformation spaces can contain representations ρ' preserving a properly convex domain Ω' which is *not* strictly convex. The quotient $\Omega'/\rho'(\pi_1 M)$ is in general an *infinite*-volume convex projective manifold homeomorphic to M: it is the union of a compact piece and finitely many *generalized cusps*. We provide the definition below:

Definition 5.1 (See [CLT18], [BCL20])**.** Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex set with nonempty interior, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$ be discrete. A manifold $C = \Omega/\Gamma$ is a *generalized cusp* if C has compact and strictly convex boundary, $\Gamma \simeq \pi_1 C$ is virtually abelian, and C is homeomorphic to $\partial C \times [0, \infty)$.

5.1. Flag stability for generalized cusps.

In [CLT18], Cooper-Long-Tillmann prove that if one deforms the holonomy representation of a generalized cusp in certain controlled way, then (for a small enough deformation), the resulting representation is still the holonomy of a generalized cusp. To state their result, for any virtually abelian group H, let $\mathrm{Hom}_{\mathrm{VF}}(H, \mathrm{PGL}(d, \mathbb{R})$ denote the space of *virtual flag* representations $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$, consisting of representations whose image is virtually conjugate to a group of upper-triangular matrices.

Theorem 5.2 ([CLT18], Theorem 6.25)**.** *Let C be a generalized cusp, and let $\mathcal{U}$ be the set of holonomies of convex projective structures on C with strictly convex boundary. Then $\mathcal{U}$ is an open subset of* $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 C)$.

Using a gluing procedure, Cooper-Long-Tillmann can then show that whenever M is a finite-volume convex projective manifold, certain restricted deformations of its holonomy give rise to new convex projective structures on M. (In fact, their result holds in greater generality, but in this paper we consider only the finite-volume case.) For this result, recall from the introduction that when $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ denotes the space of representations $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ such that the restriction of ρ to each $H \in \mathcal{H}$ lies in $\mathrm{Hom}_{\mathrm{VF}}(H, \mathrm{PGL}(d, \mathbb{R}))$.

Theorem 5.3 (See [CLT18], Theorem 0.1)**.** *Let $M = \Omega/\Gamma$ be a finite-volume convex projective manifold with holonomy $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$. Let $\mathcal{H}$ be the collection of conjugates of cusp groups of M. Then an open neighborhood of ρ in $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ consists of holonomy representations of convex projective manifolds homeomorphic to M, each of which is the union of a compact piece and finitely many generalized cusps.*

We can think of the relative stability theorem for EGF representations as a kind of dynamical analog of this "geometric gluing" result, since Theorem 1.1 gives us a way to understand the *topological dynamics* of the deformation of some representation, provided we have control over the topological dynamics of the restrictions of the representation to its cusp groups. Theorem 1.5 says that small deformations inside of $\mathrm{Hom}_{\mathrm{VF}}(\Gamma, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ are controlled in precisely the sense we need for this to apply.

Proving Theorem 1.5 therefore boils down to analyzing the dynamics implicit in Theorem 5.2: using the geometry of generalized cusps, we show that if $\rho : \pi_1 C \to \mathrm{PGL}(d, \mathbb{R})$ is the holonomy of a hyperbolic cusp, then the large-scale dynamics of any nearby representation $\rho' : \pi_1 C \to \mathrm{PGL}(d, \mathbb{R})$ in $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 C, \mathrm{PGL}(d, \mathbb{R}))$ are "close" to the dynamics of ρ (in the sense made precise by the definition of peripheral stability).

Remark 5.4. We expect that a version of Theorem 1.5 also holds with weaker assumptions on M, which are more in line with the original assumptions in the Cooper-Long-Tillmann stability result. For instance, we conjecture that whenever M is a convex projective manifold with strictly convex boundary, each end of M is a generalized cusp, and $\pi_1 M$ is relatively hyperbolic (relative to a collection of subgroups $\mathcal{H}$), then the holonomy of M is an EGF representation, and $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 M, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ is a peripherally stable subspace.

5.2. **Generalized horospheres.** In [CLT18], Cooper-Long-Tillmann show that if C is a generalized cusp with holonomy $\rho : \pi_1 C \to \mathrm{PGL}(d, \mathbb{R})$, there is a finite-index subgroup $\Gamma_1 \subseteq \pi_1 C$ (depending only on $\pi_1 C$ and d) so that $\rho(\Gamma_1)$ is a lattice in a *syndetic hull* of $\rho(\Gamma_1)$: a uniquely determined connected Lie group $T(\rho) \subset \mathrm{PGL}(d, \mathbb{R})$, conjugate into the group of upper triangular matrices. This group is called the *translation group* of the cusp.

We may assume that Γ_1 is free abelian, so it is a lattice in $\Gamma_1 \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{d-2}$. The restriction

$$\rho|_{\Gamma_1} : \Gamma_1 \to \mathrm{PGL}(d, \mathbb{R})$$

extends uniquely to an embedding of Lie groups

$$\iota_\rho : \mathbb{R}^{d-2} \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$$

with image $T(\rho)$.

We observe the following:

Proposition 5.5. *Let C be a generalized cusp. The embedding*

$$\iota_\rho : \mathbb{R}^{d-2} \to \mathrm{PGL}(d, \mathbb{R})$$

varies continuously with $\rho \in \mathrm{Hom}_{\mathrm{VF}}(\pi_1 C, \mathrm{PGL}(d, \mathbb{R}))$ in the compact-open topology on maps from $\mathbb{R}^{d-2}$ into $\mathrm{PGL}(d, \mathbb{R})$.

Proof. The Lie algebra of $\mathrm{PGL}(d,\mathbb{R})$ is identified with $\mathfrak{sl}(d,\mathbb{R})$. Fix a finite set S of generators for Γ_1. The subspace $\mathfrak{s}_\rho \subset \mathfrak{sl}(d,\mathbb{R})$ spanned by $\log(\rho(S))$ varies smoothly with ρ, and the induced map

$$\mathbb{R}^{d-2} \to \mathfrak{sl}(d,\mathbb{R})$$

with image $\mathfrak{s}_\rho$ varies continuously in the compact-open topology. The embedding ι_ρ is given by composition with the exponential map. □

When $C = \Omega/\Gamma$ is projectively equivalent to a cusp in some hyperbolic manifold, we can assume that Ω is a closed horoball in $\mathbb{H}^d$. The nonideal boundary $\partial_n\Omega$ is a horosphere preserved by Γ, which carries a Γ-invariant Euclidean metric. In this case the translation group is the group of Euclidean translations on $\partial_n\Omega$.

When C is a generalized cusp, we can always find some orbit of the translation group $T(\rho)$ in $\mathbb{P}(\mathbb{R}^d)$ which is a strictly convex hypersurface (see Proposition 6.22 in [CLT18]). This hypersurface is called a *generalized horosphere*. Its convex hull in $\mathbb{P}(\mathbb{R}^d)$ is a *generalized horoball*. We can always choose this horoball so that its quotient by Γ is contained in the generalized cusp C.

Now consider a convex projective manifold $M = \Omega/\Gamma$ with strictly convex boundary which can be written as a union of a compact piece and finitely many generalized cusps. It is always possible to choose the cusps so that each cusp $C \subset M$ is a quotient $C = \Omega_C/\Gamma_C$, where $\Omega_C \subset \Omega$ is a convex subset whose nonideal boundary is a generalized horosphere, and $\Gamma_C = \pi_1 C$ is the cusp group. The boundary $\partial_n\Omega_C$ is homogeneous, in the sense that the translation group $T(\Gamma_C)$ acts simply transitively on it.

5.3. **The ideal boundary.** The Ballas-Cooper-Leitner classification of generalized cusps allows us to get a more explicit description of the ideal boundary of Ω. Given a generalized cusp $C = \Omega/\Gamma$, we let Ω_C denote the "standard" Γ-invariant domain with homogeneous nonideal boundary, alluded to above.

Proposition 5.6 (Lemmas 1.24 and 1.25 in [BCL20])**.** *Let $C = \Omega_C/\Gamma$ be a generalized cusp. The ideal boundary of Ω_C is a projective k-simplex Δ_C. There is a unique supporting hyperplane H_C of Ω_C containing Δ_C, and the affine chart*

$$\mathbb{A}_C = \mathbb{P}(\mathbb{R}^d) - H_C$$

is the unique affine chart containing Ω_C as a closed subset.

The vertices of Δ_C must be preserved by Γ, and in fact they are all eigenvectors for the translation group $T(\Gamma)$.

Each generalized horosphere S_C for C is a strictly convex hypersurface contained in the affine chart $\mathbb{A}_C$. The closure of this hypersurface in $\mathbb{P}(\mathbb{R}^d)$ is either $S_C \cup \partial\Delta_C$ (if C is a "type $d-1$" cusp) or $S_C \cup \Delta_C$ (if C is any other type of cusp).

5.4. **Deformations of convex hypersurfaces.** The main ingredient in the proof of Theorem 1.5 is the following:

Lemma 5.7. *Let C be a hyperbolic cusp with holonomy ρ. Let p_C be the cusp point, and let H_C be the unique supporting hyperplane of Ω_C at p_C.*

Let K be a compact subset in $\mathbb{A}_C = \mathbb{P}(\mathbb{R}^d) - H_C$, let $U \subset \mathbb{P}(\mathbb{R}^d)$ be an open subset containing p_C, and let $F \subset \pi_1 C$ be a cofinite subset such that $\rho(F)K \subset U$.

Then there exists a neighborhood $\mathcal{W}$ of ρ in $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 C, \mathrm{PGL}(d,\mathbb{R}))$ so that for any $\rho' \in \mathcal{W}$, we have

$$\rho'(F)K \subset U.$$

Proof. [CLT18], Theorem 6.25 implies that we can choose a neighborhood $\mathcal{W}$ of ρ in $\mathrm{Hom}_{\mathrm{VF}}(\pi_1 C, \mathrm{PGL}(d, \mathbb{R}))$ consisting of holonomies of generalized cusps. For any $\rho' \in \mathcal{W}$, we let Ω' denote a "standard" properly convex set preserved by $\rho'(\pi_1 C)$, whose non-ideal boundary is a generalized horosphere.

Since p_C and H_C are respectively the unique eigenvector and fixed hyperplane of $\rho(\pi_1 C)$, we can choose our neighborhood $\mathcal{W}$ so that for any $\rho' \in \mathcal{W}$, any eigenvectors and fixed hyperplanes of $\rho'(\pi_1 C)$ are close to p_C, H_C.

In particular, we can choose $\mathcal{W}$ so that the ideal boundary of Ω' is a k-simplex Δ' contained in U. And, by applying a small conjugation in $\mathrm{PGL}(d+1, \mathbb{R})$, we can assume that H_C is the unique supporting hyperplane of Ω' containing Δ'.

Let $T(\rho)$ be the translation group of ρ. For any $x \in K$, the orbit $T(\rho) \cdot x$ is a paraboloid in $\mathbb{A}_C$, and as x varies in K we obtain a family of paraboloids foliating a region of $\mathbb{A}_C$. We can write $\mathbb{A}_C = \mathbb{R}^{d-1} \times \mathbb{R}$, and then view each paraboloid as the graph of a function

$$f_\rho : \mathbb{R}^{d-2} \to \mathbb{R}.$$

The function f_ρ is determined by the condition

$$(u, f_\rho(u)) = \iota_\rho(u) \cdot x.$$

Here $\iota_\rho : \mathbb{R}^{d-2} \to \mathrm{PGL}(d, \mathbb{R})$ restricts to ρ on a finite-index subgroup $\Gamma_1 \subset \pi_1 C$, with Γ_1 identified with $\mathbb{Z}^{d-2} \subset \mathbb{R}^{d-2}$.

If $T(\rho')$ is the translation group of ρ', then Lemma 6.24 in [CLT18] implies that for each $x \in K$, the orbit $T(\rho') \cdot x$ is a strictly convex hypersurface $S' \subset \mathbb{A}_C$. The hypersurface S' is the graph of a map $f_{\rho'} : \mathbb{R}^{d-2} \to \mathbb{R}$, satisfying

$$(u, f_{\rho'}(u)) = \iota_{\rho'}(u) \cdot x.$$

Proposition 5.5 implies that $f_{\rho'}$ varies continuously (in the compact-open topology on continuous maps $\mathbb{R}^{d-2} \to \mathbb{R}$) as ρ' varies in $\mathcal{W}$ and as x varies in K.

We fix a norm $||\cdot||$ on $\mathbb{R}^{d-2}$. There is a constant D so that for any $(u, v) \in \mathbb{A}_C$, if $||u|| > 1$ and $|v|/||u|| > D$, then (u, v) is contained in the neighborhood U of p_C.

The function $f_{\rho'}$ is strictly convex, and we can assume that it is nonnegative and uniquely minimized at the origin. So, if $f_{\rho'}(u)/||u|| > D$ on $\{u \in \mathbb{R}^{d-2} : ||u|| = M\}$ for some constant M, then $f_{\rho'}(u)/||u|| > D$ for all u with $||u|| \geq M$.

So, as long as $\mathcal{W}$ is sufficiently small, there is a fixed ball $B \subset \mathbb{R}^{d-1}$ so that if $u \in \mathbb{R}^{d-1} - B$, then

$$(u, f_{\rho'}(u)) \in U$$

for any $\rho' \in \mathcal{W}$. As both the constants D, M above vary continuously as x varies in the compact subset K, we can choose $\mathcal{W}$ so that the above holds for any $f_{\rho'}$ determined by any $x \in K$.

The ball B contains at most finitely many elements of $\Gamma_1 - F$. So we can in fact choose $\mathcal{W}$ small enough so that for any $\rho' \in \mathcal{W}$, the set $\rho'(\Gamma_1 - F)K$ lies in U. Then since Γ_1 has finite index in $\pi_1 C$, we can shrink $\mathcal{W}$ even further to guarantee that for any $\rho' \in \mathcal{W}$,

$$\rho'(\pi_1 C - F)K$$

lies in U as well. □

5.5. **Peripheral stability.**

Proof of Theorem 1.5. Let $\rho : \pi_1 M \to \mathrm{PGL}(d, \mathbb{R})$ be the holonomy of a finite-volume convex projective manifold M, and let Ω be a ρ-invariant strictly convex domain such that $M =$

$\Omega/\rho(\pi_1 M)$. Write $\Gamma = \pi_1 M$, and let $\mathcal{H}$ be the collection of cusp groups, so $(\Gamma, \mathcal{H})$ is a relatively hyperbolic pair, and for each $H \in \mathcal{H}$, $\rho|_H$ is the holonomy of a hyperbolic cusp.

Since Ω is strictly convex, [CLT15, Theorem 11.6] also implies that Ω has C^1 boundary. So there is a ρ-equivariant homeomorphism $\partial\Omega \to \partial\Omega^*$ assigning the point $z \in \partial\Omega$ to the unique supporting hyperplane of Ω at z. We let $\partial\hat{\Omega}$ denote the space

$$\{(x, w) \in \mathcal{F}(d) : x \in \partial\Omega, w \in \partial\Omega^*\}.$$

There is an equivariant homeomorphism $\psi : \partial(\Gamma, \mathcal{H}) \to \partial\hat{\Omega}$, with the parabolic points in $\partial(\Gamma, \mathcal{H})$ corresponding to the fixed flags of the cusp groups. The inverse map $\phi : \partial\hat{\Omega} \to \partial(\Gamma, \mathcal{H})$ extends the convergence action of Γ on $\partial(\Gamma, \mathcal{H})$. For each parabolic point $p \in \partial(\Gamma, \mathcal{H})$, the open set C_p is

$$\mathrm{Opp}(\psi(p)) = \{\xi \in \mathcal{F} : \xi \text{ is opposite to } \psi(p)\}.$$

Let $\pi : \mathcal{F}(d) \to \mathbb{P}(\mathbb{R}^d)$ be the canonical projection map. It suffices to show that for any compact set $K \subset \pi(C_p)$, any open neighborhood U of $\pi(\psi(p))$ in $\mathbb{P}(\mathbb{R}^d)$, and any cofinite subset $F \subset \Gamma_p = \mathrm{Stab}_\Gamma(p)$ such that

$$\rho(F) \cdot K \subset U,$$

we can find a neighborhood $\mathcal{W} \subset \mathrm{Hom}_{\mathrm{VF}}(\Gamma_p, \mathrm{PGL}(d, \mathbb{R}), \mathcal{H})$ containing ρ such that

$$\rho'(F) \cdot K \subset U$$

for any $\rho' \in \mathcal{W}$. However, this is exactly the content of Lemma 5.7. The same argument applied dually then shows that we can upgrade K to a compact subset of $C_p \subset \mathcal{F}$ and U to an open subset in $\mathcal{F}(d)$, giving the required peripheral stability. ☐

6. Symmetric representations

In this section, we construct new examples of extended geometrically finite representations by taking symmetric powers of convex cocompact representations of groups which are hyperbolic relative to virtually abelian subgroups. We also prove Theorem 1.8, which states that these EGF representations are *absolutely* stable in the representation variety.

6.1. **Symmetric powers.** Let V be a finite-dimensional real vector space. We let τ_m denote the symmetric representation

$$\mathrm{SL}(V) \to \mathrm{SL}(\mathrm{Sym}^m(V)).$$

Throughout this section, we will view $\mathrm{Sym}^m(V)$ as a quotient of the space of homogeneous degree-m polynomials in elements of V. We will always leave this quotient implicit. That is, if $v_1, \ldots, v_k \in V$, and $r_1, \ldots, r_k \in \mathbb{N} \cup \{0\}$ with $\sum r_i = m$, we will view the monomial $v_1^{r_1} \cdots v_k^{r_k}$ as an element of $\mathrm{Sym}^m(V)$.

There is a τ_m-equivariant embedding

$$\iota : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^m(V))$$

given by $[v] \mapsto [v^m]$. There is also a corresponding dual embedding

$$\iota^* : \mathbb{P}(V^*) \to \mathbb{P}(\mathrm{Sym}^m(V)^*),$$

using the canonical identification $\mathrm{Sym}^m(V^*) \simeq \mathrm{Sym}^m(V)^*$. We observe that $v \in \mathbb{P}(V)$ and $w \in \mathbb{P}(V^*)$ are transverse if and only if their respective images under ι and ι^* are also transverse. This means that the maps ι and ι^* also give rise to a τ_m-equivariant map

$$\hat{\iota} : \mathcal{F}(V) \to \mathcal{F}(\mathrm{Sym}^m(V))$$

given by $\hat{\iota}(v, w) = (\iota(v), \iota^*(w))$.

6.1.1. *Dynamics in symmetric powers.* The dynamics of 1-divergent sequences in $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are respected in $\mathbb{P}(\mathrm{Sym}^m V)$, in sense given by the following (easily verified) proposition:

Proposition 6.1. *Let $\{g_n\}$ be an infinite sequence in $\mathrm{SL}(V)$ such that for some $w \in \mathbb{P}(V^*)$, $x \in \mathbb{P}(V)$, we have $g_n|_{\mathrm{Opp}(w)} \to x$ uniformly on compacts.*

Then

$$\tau_m(g_n)|_{\mathrm{Opp}(\iota^*(w))} \to \iota(x)$$

uniformly on compacts.

6.2. **Symmetric powers of relatively hyperbolic convex cocompact groups.** Suppose that Γ is a convex cocompact subgroup of $\mathrm{PGL}(V)$, so that there is a properly convex domain $\Omega \subset \mathbb{P}(V)$ with Γ acting convex cocompactly on Ω. We may replace Γ with a finite index-subgroup and lift Γ to a representation $\rho : \Gamma \to \mathrm{SL}(V)$. In this situation we say that the representation $\rho : \Gamma \to \mathrm{SL}(V)$ is *convex cocompact in* $\mathbb{P}(V)$.

Let $(\Gamma, \mathcal{H})$ be a relatively hyperbolic pair such that each group in $\mathcal{H}$ is virtually abelian, and let $\rho : \Gamma \to \mathrm{SL}(V)$ be a convex cocompact representation. Representations of this form have been studied extensively by Islam-Zimmer [IZ19b], [IZ19a], who proved a number of strong structural results. In particular, Islam-Zimmer showed that in this situation, for each $H \in \mathcal{H}$, the image $\rho(H)$ acts cocompactly on a properly embedded k-simplex $\Delta_H \subset \Omega$, with $k = \mathrm{rank}(H)$. (A simplex $\Delta \subset \Omega$ is *properly embedded* if $\partial\Delta \subset \partial\Omega$.) In fact, the action of $\rho(H)$ on Ω is convex cocompact, and the full orbital limit set of $\rho(H)$ in Ω is $\partial\Delta_H$.

Conversely, every properly embedded maximal k-simplex in the convex hull of $\Lambda_\Omega(\Gamma)$ always has a cocompact action by some $H \in \mathcal{H}$ with rank k.

We let

$$\rho^m : \Gamma \to \mathrm{SL}(\mathrm{Sym}^m V)$$

denote the composition $\tau_m \circ \rho$. We have two main goals in this section. The first is to prove that the representation ρ^m is always EGF. This result does *not* follow directly from the fact that convex cocompact representations in $\mathbb{P}(V)$ are EGF (Theorem 1.2), because we do not know that the representations ρ^m are convex cocompact in $\mathbb{P}(\mathrm{Sym}^m V)$. In fact, Jeff Danciger and Fanny Kassel have indicated in personal communication to the author that ρ^m should *never* be convex cocompact in $\mathbb{P}(\mathrm{Sym}^m V)$ unless the collection $\mathcal{H}$ is empty: while $\rho^m(\Gamma)$ does preserve a properly convex domain in $\mathbb{P}(\mathrm{Sym}^m V)$, the convex hull of the full orbital limit set in any such domain seems "too big" for $\rho^m(\Gamma)$ to act cocompactly.

Our second goal is to show that the *entire* space $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$ is peripherally stable about ρ^m, meaning (by Theorem 1.1) an open neighborhood of ρ^m consists of EGF representations. In particular this shows that small perturbations of ρ^m still have finite kernel and discrete image, giving new examples of discrete subgroups of higher-rank Lie groups which are stable (as discrete groups).

6.2.1. *Proof strategy.* To show that ρ^m is EGF, we will give an explicit description of the boundary set $\hat{\Lambda}_m \subset \mathcal{F}(\mathrm{Sym}^m V)$. The naive choice is to just take $\hat{\Lambda}_m$ to be $\hat{\iota}(\hat{\Lambda}_\Omega(\Gamma))$, where $\hat{\Lambda}_\Omega(\Gamma)$ is the flag-valued full orbital limit set giving the EGF boundary set for $\rho : \Gamma \to \mathrm{SL}(V)$ (see Section 4). While there is a equivariant surjective map from this set to $\partial(\Gamma, \mathcal{H})$, it turns out that we will have to *enlarge* it in order to ensure that the relevant dynamics hold.

The idea is the following: for each parabolic point $p \in \partial(\Gamma, \mathcal{H})$, with stabilizer H, we take the fiber over p in $\hat{\Lambda}_m$ to be the space of flags in the boundary of a simplex $S_H \subset \mathbb{P}(\mathrm{Sym}^m V)$, constructed using the simplex $\Delta_H \subset \Omega$ on which $H = \mathrm{Stab}_\Gamma(p)$ acts cocompactly. The simplex S_H is chosen so that if γ_n is a sequence in Γ converging to p, then a face of S_H spans a minimal attracting subspace of $\rho^m(\gamma_n)$. (Here, and throughout this section, *attracting* and *repelling* subspaces are understood in the sense defined in Section 3.1.3).

We also want to ensure that the simplex S_H is *stable*, i.e. if ρ_t^m is a small deformation of ρ^m in $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$, then $\rho_t^m(H)$ preserves a simplex S_H^t close to S_H. We verify that S_H has these properties by analyzing the relationship between the weights of ρ and ρ^m on the virtually abelian group H.

The other main steps in the proof involve checking that the boundary set $\hat{\Lambda}_m$ we construct is actually a compact space surjecting continuously onto $\partial(\Gamma, \mathcal{H})$, and constructing the open sets $\hat{C}_p$ also required by the definition of an EGF representation. For the latter, we make heavy use of the fact that the *dual* action of $\rho(\Gamma)$ on $\mathbb{P}(V^*)$ is also projectively convex cocompact, which allows us to construct a stable *dual* simplex S_H^* for each $H \in \mathcal{H}$. The vertices of S_H^* are thought of as hyperplanes in $\mathbb{P}(\mathrm{Sym}^m(V))$, cutting out a region C_p of $\mathbb{P}(\mathrm{Sym}^m(V))$ on which $\rho^m(H)$ attracts points towards S_H.

6.2.2. *Example: symmetric squares of convex projective 3-manifold groups.* We illustrate the general idea of our approach with a specific example. Let $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ be a properly convex domain, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$ be a discrete group acting cocompactly on Ω. In [Ben06], Benoist produced examples of such groups which are hyperbolic relative to a nonempty collection $\mathcal{H}$ of virtually abelian subgroups of rank 2. Further examples were constructed by Ballas-Danciger-Lee in [BDL15]. Up to finite index, each $H \in \mathcal{H}$ acts diagonalizably on $\mathbb{P}(\mathbb{R}^4)$, preserving a projective tetrahedron $T_H \subset \mathbb{P}(\mathbb{R}^4)$ and acting cocompactly on a properly embedded triangle $\Delta_H \subset \Omega$. Each edge of Δ_H is contained in a unique supporting hyperplane of Ω. The common intersection of these hyperplanes is the fourth vertex of T_H.

More explicitly, up to finite index, each H acts diagonally on $\mathbb{R}^4$ in the basis $\{v_1, v_2, v_3, v_4\}$, where the v_i are the vertices of T_H, and v_1, v_2, v_3 are the vertices of Δ_H. We can consider the situation where (in this basis) H is the discrete group

$$\left\{ \begin{pmatrix} 2^a & & & \\ & 2^b & & \\ & & 2^c & \\ & & & 1 \end{pmatrix} : a, b, c \in \mathbb{Z},\ a + b + c = 0 \right\}.$$

The dual of H preserves the corresponding dual basis $\{v_1^*, v_2^*, v_3^*, v_4^*\}$, and acts cocompactly on a projective triangle $\Delta_H^* \subset \mathbb{P}((\mathbb{R}^4)^*)$ with vertices v_1^*, v_2^*, v_3^*. The kernels $[\ker v_i^*]$ for $i = 1, 2, 3$ give three supporting hyperplanes of Ω which cut out a region R_H of projective space containing Ω. In fact, R_H also contains $\partial\Omega - \partial\Delta_H$.

Now let $\rho^2 : \Gamma \to \mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^4)) \simeq \mathrm{SL}(10, \mathbb{R})$ be the composition of the inclusion $\Gamma \hookrightarrow \mathrm{SL}(4, \mathbb{R})$ with the symmetric square $\tau_2 : \mathrm{SL}(4, \mathbb{R}) \to \mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^4))$. In this case, the induced map $\iota : \mathbb{P}(\mathbb{R}^4) \to \mathbb{P}(\mathbb{R}^{10})$ is the Veronese embedding.

For each $H \in \mathcal{H}$, $\rho^2(H)$ preserves a 9-simplex in $\mathbb{P}(\mathbb{R}^{10})$, with vertices

$$\{v_1^2, v_2^2, v_3^2, v_4^2, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}.$$

In particular $\rho^2(H)$ also preserves the 5-simplex S_H with vertices

$$\{v_1^2, v_2^2, v_3^2, v_1v_2, v_2v_3, v_1v_3\}.$$

Any divergent sequence in $\rho^2(H)$ always has an attracting subspace spanned by a face of S_H, since the eigenvalues of elements of H on v_4 are always dominated by some eigenvalue of that element on either v_1, v_2, or v_3. For instance, if we consider the sequence

$$a_n = \begin{pmatrix} 2^{2n} & & & \\ & 2^{-n} & & \\ & & 2^{-n} & \\ & & & 1 \end{pmatrix},$$

then $\rho^2(a_n)$ attracts towards the subspace spanned by $\{v_1^2\}$. On the other hand, if a_n is the sequence

$$\begin{pmatrix} 2^n & & & \\ & 2^n & & \\ & & 2^{-2n} & \\ & & & 1 \end{pmatrix},$$

then $\rho^2(a_n)$ attracts towards the subspace spanned by $\{v_1^2, v_2^2, v_1v_2\}$.

Moreover, the subspaces spanned by faces of S_H are transverse to $\iota(\partial\Omega - \partial\Delta_H)$, so large elements of H attract points in $\iota(\partial\Omega)$ that are "far from" ∂S_H towards ∂S_H. In fact, this dynamical behavior extends to an entire open subset of $\mathbb{P}(\mathbb{R}^{10})$: namely, a region cut out by the hyperplanes corresponding to the vertices of the dual 5-simplex $S_H^* \subset \mathbb{P}((\mathbb{R}^{10})^*)$ with vertices

$$\{(v_1^*)^2, (v_2^*)^2, (v_3^*)^2, v_1^*v_2^*, v_2^*v_3^*, v_1^*v_3^*\}.$$

So, the simplex S_H serves as the "parabolic point" for the action of the peripheral subgroup H on $P(\mathbb{R}^{10})$—and moreover, this behavior is stable under perturbations of $\rho^2(H)$ in $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^2 V))$. This means that we can construct our candidate boundary set for the representation ρ^m by taking

$$\hat{\iota}(\hat{\Lambda}_\Omega(\Gamma)) \cup \bigcup_{H \in \mathcal{H}} \partial \hat{S}_H,$$

where $\partial\hat{S}_H \subset \mathcal{F}(\mathrm{Sym}^2 \mathbb{R}^4)$ is a closed subset of the space of flags projecting to the boundary of the simplex S_H.

6.3. **Generalized weight spaces.** To carry out the general construction of the simplex S_H identified in the previous example, we need some description of *attracting subspaces* for the groups $\rho^m(H) \subset \mathrm{SL}(\mathrm{Sym}^m V)$. We obtain this description by recalling some of the properties of *weights* of representations of free abelian groups.

Definition 6.2. Let $\rho : H \to \mathrm{GL}(V)$ be a representation of a free abelian group H, and let $\rho_{\mathbb{C}} : H \to \mathrm{GL}(V \otimes_{\mathbb{R}} \mathbb{C})$ be the complexification of ρ.

A *complex weight* of ρ is a homomorphism $\mu_{\mathbb{C}} : H \to \mathbb{C}$ such that the *weight space*

$$V_{\mu_{\mathbb{C}}} = \ker(\rho_{\mathbb{C}}(h) - \exp(\mu_{\mathbb{C}}(h))I)$$

is nontrivial for evey $h \in H$. A *generalized complex weight* is similarly a homomorphism $\mu_{\mathbb{C}} : H \to \mathbb{C}$ such that the *generalized weight space*

$$V_{\mu_{\mathbb{C}}} = \bigcup_{n=1}^{\dim V} \ker(\rho_{\mathbb{C}}(h) - \exp(\mu_{\mathbb{C}}(h))I)^n$$

is nontrivial.

For any generalized weight $\mu_{\mathbb{C}}$, the *nilpotence degree* of $\mu_{\mathbb{C}}$ is the minimal $\ell \in \mathbb{N}$ such that $V_{\mu_{\mathbb{C}}} = \ker(\rho_{\mathbb{C}}(h) - \exp(\mu_{\mathbb{C}}(h))I)^\ell$.

Given a representation $\rho : H \to \mathrm{GL}(V)$, the generalized complex weight spaces of $\rho_{\mathbb{C}}$ give a $\rho_{\mathbb{C}}$-invariant decomposition of $V \otimes_{\mathbb{R}} \mathbb{C}$. This in turn gives a ρ-invariant decomposition of V, since when $\mu_{\mathbb{C}}$ is a weight which takes on complex values, the direct sum $V_{\mu_{\mathbb{C}}} \oplus V_{\overline{\mu_{\mathbb{C}}}}$ is a ρ-invariant real subspace of V. By a slight abuse of terminology we refer to this as the *generalized weight space decomposition* for the representation ρ. The associated *real weights* of the representation are homomorphisms $\mu : H \to \mathbb{R}$ of the form $\log|\exp \mu_{\mathbb{C}}|$, where $\mu_{\mathbb{C}}$ is a (generalized) complex weight. For the rest of the section, unless otherwise indicated, when we refer to (generalized) *weights* of a representation into $\mathrm{SL}(V)$, we will mean the (generalized) *real* weights, and similarly for weight spaces.

Generalized weight spaces of ρ are stable under deformations of ρ. To be precise, we observe the following:

Proposition 6.3. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation of a free abelian group, and let*

$$V^0_{\mu_1} \oplus \ldots \oplus V^0_{\mu_s}$$

be the generalized weight space decomposition of V for ρ. Let ρ_t be a continuous family of representations in $\mathrm{Hom}(\Gamma, \mathrm{GL}(V))$, with $\rho = \rho_0$.

For all sufficiently small t, there is a ρ_t-invariant decomposition

$$V^t_1 \oplus \ldots \oplus V^t_s$$

such that V^t_i is a sum of generalized weight spaces for ρ_t, with V^t_i varying continuously with t, and each of the weights associated to V^t_i also varying continuously with t.

Proof. The weights vary continously as a set with multiplicity, because the roots of the characteristic polynomial of $\rho(\gamma)$ vary continuously in ρ for fixed $\gamma \in \Gamma$. And, if μ is a complex weight with multiplicity k, then for small t there are complex weights $\mu^t_1, \ldots, \mu^t_k$ of ρ_t (possibly with repeats) close to μ such that the sum of the kernels $\ker(\rho_{\mathbb{C}}(\gamma) - \exp(\mu^t_i(\gamma))I)^{\dim V}$ is close to $\ker(\rho_{\mathbb{C}}(\gamma) - \exp(\mu(\gamma))I)^{\dim V}$. □

Definition 6.4. Let $\rho : H \to \mathrm{SL}(V)$ be a representation of a free abelian group H, and let Φ be the set of generalized weights of ρ. For any subset $\theta \subseteq \Phi$, we let $V_\theta \subseteq V$ denote the span of the generalized weight spaces V_μ for $\mu \in \theta$, and we let $V^{\mathrm{opp}}_\theta \subseteq V$ denote the span of the generalized weight spaces $V_{\mu'}$ for $\mu' \in \Phi - \theta$.

6.3.1. *Faces in the convex hull of the weights.* Whenever $\rho : H \to \mathrm{SL}(V)$ is a representation of a free abelian group with rank k, we can extend any real (generalized) weight $\mu : H \to \mathbb{R}$ to a homomorphism $\mu : H \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$, and view it as an element of $(\mathbb{R}^k)^* \simeq \mathbb{R}^k$.

Definition 6.5. Let $\rho : H \to \mathrm{SL}(V)$ be a representation of a free abelian group, and let Φ be the set of generalized weights of ρ. We denote the closed convex hull of Φ in $(H \otimes_{\mathbb{Z}} \mathbb{R})^* \simeq (\mathbb{R}^k)^*$ by $\mathcal{C}(\rho)$; since Φ is a finite subset of $(\mathbb{R}^k)^*$, $\mathcal{C}(\rho)$ is a convex polytope in $(\mathbb{R}^k)^*$.

The convex polytope $\mathcal{C}(\rho)$ is important for our purposes because it tells us how to find attracting subspaces for $\rho(H)$. In particular, there is a correspondence between the *faces* of $\mathcal{C}(\rho)$ and attracting subspaces of $\rho(H)$.

Definition 6.6. Let $\rho : H \to \mathrm{SL}(V)$ be a represention of a free abelian group with generalized weight set Φ. Let F be a closed face of $\mathcal{C}(\rho)$. We let $\Phi(F)$ denote the set of generalized weights of ρ lying in F.

For a face F of $\mathcal{C}(\rho)$, we write V_F and V^{opp}_F for $V_{\Phi(F)}$ and $V^{\mathrm{opp}}_{\Phi(F)}$, respectively.

Below, we will prove the following:

Proposition 6.7. *Let $\rho : H \to \mathrm{SL}(V)$ be a representation of a free abelian group. For any divergent sequence $h_n \in H$, there is a face F of $\mathcal{C}(\rho)$ such that V_F and V^{opp}_F are respectively attracting and repelling subspaces for $\rho(h_n)$.*

Conversely, for any face F of $\mathcal{C}(\rho)$, V_F is an attracting subspace for some sequence $\rho(h_n)$ with $h_n \in H$ divergent.

To prove Proposition 6.7, we first establish some estimates which will later help us show that the convergence to the spaces V_F is both uniform and stable. To help make our estimates explicit, we choose a norm $|\cdot|$ on $H \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^k$. We also fix a norm $||\cdot||$ on $\mathbb{C}^d$, which induces an operator norm $||\cdot||$ on $\mathrm{SL}(d, \mathbb{C})$. We use $\mathbf{m}(\cdot)$ to denote the conorm $\mathbf{m}(g) = ||g^{-1}||^{-1}$.

Lemma 6.8. *Let $U(d,\mathbb{C})$ be the group of upper-triangular matrices in $\mathrm{SL}(d,\mathbb{C})$, and let H be a free abelian group. For any $\rho \in \mathrm{Hom}(H, U(d,\mathbb{C}))$, there exists $D(\rho) > 0$ (varying continuously with ρ) so that for any $h \in H$, we have*

$$\frac{1}{D}|h|^{1-d} \cdot r_\rho^-(h) \leq \mathbf{m}(\rho(h)) \leq ||\rho(h)|| \leq D|h|^{d-1} \cdot r_\rho^+(h),$$

where $r_\rho^+(h)$ and $r_\rho^-(h)$ are respectively the maximum and minimum modulus of any eigenvalue of $\rho(h)$.

Proof. We can extend each $\rho \in \mathrm{Hom}(H, U(d,\mathbb{C}))$ to a representation $\rho : H \otimes_{\mathbb{Z}} \mathbb{R} \to U(d,\mathbb{C})$. For each $h \in H$, we write

$$\rho(h) = S^\rho(h) + N^\rho(h),$$

where $S^\rho(h)$ is diagonal and $N^\rho(h)$ is strictly upper-triangular. The entries of $S^\rho(h)$ are the eigenvalues of $\rho(h)$, which means $||S^\rho(h)|| = r_\rho^+(h)$ and $||S^\rho(-h)||^{-1} = r_\rho^-(h)$.

For any fixed $h \in H \otimes_{\mathbb{Z}} \mathbb{R}$ and $k \in \mathbb{N}$, if we binomially expand the expression

$$\rho(h)^k = (S^\rho(h) + N^\rho(h))^k,$$

then all monomials containing at least d terms equal to $N^\rho(h)$ must vanish. This gives us an inequality of the form

$$||\rho(h)^k|| \leq ||S^\rho(h)||^k \cdot p(k, ||N^\rho(h)||), \tag{1}$$

where $p(x,y)$ is a polynomial with nonnegative coefficients depending continuously on ρ and h, with degree at most $d-1$. Using the identity $\mathbf{m}(\rho(h)^k) = ||\rho(-h)^k||^{-1}$, we also see that

$$\mathbf{m}(\rho(h)^k) \geq ||S^\rho(-h)||^{-k}/p(k, ||N^\rho(-h)||). \tag{2}$$

Now, since the unit sphere in $H \otimes_{\mathbb{Z}} \mathbb{R}$ is compact, and $N^\rho(h)$ varies continuously with ρ and h, we can find D_0 varying continuously with ρ so that $||N^\rho(s)|| \leq D_0$ for all s with $|s| = 1$. Now suppose that $h = ks$ for $k \in \mathbb{N}$ and $s \in H \otimes_{\mathbb{Z}} \mathbb{R}$ with $|s| = 1$. Then $k = |h|$ and from (1) we see

$$||\rho(h)|| = ||\rho(s)^k|| \leq r_\rho^+(s)^k \cdot p(|h|, ||N^\rho(s)||) = r_\rho^+(h) \cdot p(|h|, ||N^\rho(s)||).$$

Since $0 \leq ||N^\rho(s)|| < D_0$, we can bound the polynomial term beneath $D|h|^{d-1}$ for a constant D depending continuously on ρ, giving us the desired upper bound. For the lower bound, we can argue similarly using (2). A priori, these bounds only holds for $h \in H \otimes_{\mathbb{Z}} \mathbb{R}$ which are positive integer multiples of elements on the unit sphere, but every $h \in H$ is bounded distance from such an element, so we get the desired bounds everywhere. ☐

For the next estimate, we choose an inner product on our real vector space V, which induces a norm $|| \cdot ||$ on V and a smooth metric $d_{\mathbb{P}}$ on $\mathbb{P}(V)$. Specifically, for any transverse subspaces $W, W' \subset V$, we define

$$\angle(W, W') = \inf_{\substack{w \in W - \{0\}, \\ w' \in W' - \{0\}}} \angle(w, w'),$$

and then take $d_{\mathbb{P}}([u],[v]) = \sin(\angle([u],[v]))$. As before, the norm $|| \cdot ||$ induces an operator norm and conorm $\mathbf{m}(\cdot)$ on $\mathrm{SL}(V)$.

Lemma 6.9. *Let $W, W^\perp$ be transverse subspaces of V with $W \oplus W^\perp = V$. If $g \in \mathrm{SL}(V)$ preserves both W and $W^\perp$, then for any $x \in \mathbb{P}(V) - \mathbb{P}(W^\perp)$, we have*

$$\frac{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W))}{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W^\perp))} \leq \frac{1}{\sin^2 \angle(W, W^\perp)} \frac{||g|_{W^\perp}||}{\mathbf{m}(g|_W)} \cdot d(x, \mathbb{P}(W^\perp))^{-1}.$$

Proof. Let $x = [v]$ for $v \in V$, and uniquely write $v = w + w^\perp$ for $w \in W$, $w^\perp \in W^\perp$, with $w^\perp \neq 0$. Then we have

$$\frac{||w^\perp||}{||w||} \sin \angle(W, W^\perp) \leq d_{\mathbb{P}}(x, \mathbb{P}(W)) \leq \frac{||w^\perp||}{||w||},$$

and similarly

$$\frac{||w||}{||w^\perp||} \sin \angle(W, W^\perp) \leq d_{\mathbb{P}}(x, \mathbb{P}(W^\perp)) \leq \frac{||w||}{||w^\perp||}.$$

So in particular we have

$$\sin \angle(W, W^\perp) \frac{||w^\perp||}{||w||} \leq \frac{d_{\mathbb{P}}(x, \mathbb{P}(W))}{d_{\mathbb{P}}(x, \mathbb{P}(W^\perp))} \leq \frac{1}{\sin \angle(W, W^\perp)} \frac{||w^\perp||}{||w||}. \tag{3}$$

Since g preserves the decomposition $V = W \oplus W^\perp$, we see that

$$\frac{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W))}{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W^\perp))} \leq \frac{1}{\sin \angle(W, W^\perp)} \frac{||g \cdot w^\perp||}{||g \cdot w||}.$$

We know that $||g \cdot w^\perp|| \leq ||g|_{W^\perp}|| \cdot ||w^\perp||$ and $||g \cdot w|| \geq \mathbf{m}(g|_W) \cdot ||w||$, so we get the inequality

$$\frac{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W))}{d_{\mathbb{P}}(g \cdot x, \mathbb{P}(W^\perp))} \leq \frac{1}{\sin \angle(W, W^\perp)} \frac{||g|_{W^\perp}||}{\mathbf{m}(g|_W)} \cdot \frac{||w^\perp||}{||w||}.$$

Then we apply the left-hand inequality of (3) and use the fact that $d_{\mathbb{P}}(x, \mathbb{P}(W)) \leq 1$ to finish the proof. □

Proof of Proposition 6.7. Let $\rho : H \to \mathrm{SL}(V)$ be a representation of a free abelian group, let Φ be the set of generalized weights, and let h_n be a divergent sequence in H. Up to subsequence, the sequence $h_n/|h_n|$ converges to some $h_\infty \in H \otimes_{\mathbb{Z}} \mathbb{R}$ with $|h_\infty| = 1$.

We can view h_∞ as a linear functional on the space $(H \otimes_{\mathbb{Z}} \mathbb{R})^*$. Since $\mathcal{C}(\rho) \subset (H \otimes_{\mathbb{Z}} \mathbb{R})^*$ is a convex polytope, this means there is a face F of $\mathcal{C}$ so that for any $\mu \in \Phi(F)$ and any $\mu^{\mathrm{opp}} \in \Phi - \Phi(F)$, we have $\mu(h_\infty) > \mu^{\mathrm{opp}}(h_\infty)$. Then for sufficiently large n we also have $\mu(h_n/|h_n|) > \mu^{\mathrm{opp}}(h_n/|h_n|)$. In fact, since Φ is finite, there is a constant $M > 0$ such that $\mu(h_n) - \mu^{\mathrm{opp}}(h_n) > M|h_n|$ for every $\mu \in \Phi(F)$ and every $\mu^{\mathrm{opp}} \in \Phi - \Phi(F)$. In particular, if $r(h_n)$ is the eigenvalue of $\rho(h_n)$ on V_F with smallest modulus, and $r^{\mathrm{opp}}(h_n)$ is the eigenvalue of $\rho(h_n)$ on V_F^{opp} with largest modulus, we have $|r(h_n)|/|r^{\mathrm{opp}}(h_n)| > \exp(M|h_n|)$.

We can choose an identification of $V \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}^{\dim V}$ so that the complexification $\rho_{\mathbb{C}} : H \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{SL}(V \otimes_{\mathbb{R}} \mathbb{C})$ lies in the group of upper-triangular matrices, and the eigenvectors of $\rho_{\mathbb{C}}$ are standard basis vectors. The norm $|| \cdot ||$ we have chosen on V induces a norm on $V \otimes_{\mathbb{R}} \mathbb{C}$ which agrees with the standard norm on $\mathbb{C}^{\dim V}$ up to bounded multiplicative error. So, we can apply Lemma 6.8 to see that the quantity

$$\frac{\mathbf{m}(\rho(h_n)|_{V_F})}{||\rho(h_n)|_{V_F^{\mathrm{opp}}}||}$$

tends to infinity as $n \to \infty$. Then Lemma 6.9 implies that for any $x \in \mathbb{P}(V) - \mathbb{P}(V_F^{\mathrm{opp}})$, the distance

$$d_{\mathbb{P}}(\rho(h_n)x, \mathbb{P}(V_F)) \leq \frac{d_{\mathbb{P}}(\rho(h_n)x, \mathbb{P}(V_F))}{d_{\mathbb{P}}(\rho(h_n)x, \mathbb{P}(V_F^{\mathrm{opp}}))}$$

tends to 0 as $n \to \infty$, so V_F and V_F^{opp} must respectively be attracting and repelling subspaces for $\rho(h_n)$.

Conversely, if F is any face of $\mathcal{C}(\rho)$, we can choose $h \in H \otimes_{\mathbb{Z}} \mathbb{R}$ so that $\mu(h) > 0$ and $\mu(h) > \mu^{\mathrm{opp}}(h)$ for any $\mu \in \Phi(F)$ and $\mu^{\mathrm{opp}} \in \Phi - \Phi(F)$. Then if $h_n \in H$ is any divergent sequence with $h_n/|h_n| \to h$ in $H \otimes_{\mathbb{Z}} \mathbb{R}$, we can similarly apply Lemma 6.8 to see that the ratio $||\rho(h_n)|_{V_F}||/||\rho(h_n)|_{V_F^{\mathrm{opp}}}||$ tends to infinity, and again use Lemma 6.9 to see that V_F is an attracting subspace for h_n. □

6.4. **Weights of peripheral subgroups in convex cocompact groups.** For the rest of this section, we fix a relatively hyperbolic pair $(\Gamma, \mathcal{H})$, where each $H \in \mathcal{H}$ is virtually abelian with rank at least 2. We also fix a representation $\rho : \Gamma \to \mathrm{SL}(V)$ which is convex cocompact in $\mathbb{P}(V)$, and let $\Omega \subset \mathbb{P}(V)$ be a properly convex domain where $\rho(\Gamma)$ acts convex cocompactly.

Our goal now is to describe the convex polytope in $(H \otimes_{\mathbb{Z}} \mathbb{R})^*$ associated to the restriction of ρ to each $H \in \mathcal{H}$, which we can use to understand the dynamics of both $\rho(H)$ and $\rho^m(H)$.

Definition 6.10. For each $H \in \mathcal{H}$, we let $\mathcal{V}_H \subset \mathbb{P}(V)$ denote the set of vertices of Δ_H.

Proposition 6.11. *Let $H \in \mathcal{H}$ be a peripheral subgroup of rank $k \geq 2$, and let $H_0 \subseteq H$ be a finite-index free abelian subgroup. Consider the restriction $\rho_0 = \rho|_{H_0}$. Then, the convex polytope $\mathcal{C}(\rho_0)$ is a k-simplex in $(H_0 \otimes_{\mathbb{Z}} \mathbb{R})^*$, and each vertex of $\mathcal{C}(\rho_0)$ is a weight μ whose associated weight space is a vertex of Δ_H.*

Moreover, every weight of ρ_0 which is not a vertex of $\mathcal{C}(\rho_0)$ lies in the interior of $\mathcal{C}(\rho_0)$.

Proof. Each vertex $v \in \mathcal{V}_H$ lies in a weight space of ρ_0, with an associated weight μ_v. Let Φ denote the weights of ρ_0, and let $\Phi(\mathcal{V}_H) \subseteq \Phi$ be the set of weights of the form μ_v for $v \in \mathcal{V}_H$. We claim that for any $\mu \in \Phi - \Phi(\mathcal{V}_H)$ and any $h \in H_0 \otimes_{\mathbb{Z}} \mathbb{R}$, there is a vertex $v \in \mathcal{V}_H$ such that

$$\mu_v(h) > \mu(h).$$

Suppose for a contradiction that the claim does not hold, i.e. there exists $h \in H_0 \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mu \in \Phi - \Phi(\mathcal{V}_H)$ such that $\mu(h) \geq \mu_v(h)$ for all $v \in \mathcal{V}_H$. We choose a subspace V'_μ of the weight space V_μ, so that the restriction of ρ_0 to V'_μ is (complex) diagonalizable. Then we let $W_H = \mathrm{span}(\Delta_H)$, and let V' be the ρ_0-invariant subspace $V'_\mu \oplus W_H$. Then ρ_0 induces a representation $\rho'_0 : H_0 \to \mathrm{SL}(V')$. Since each vertex in $\mathcal{V}_H$ is an eigenspace for ρ_0, this representation is (complex) diagonalizable.

We may choose our norm on V' so that the eigenspaces for ρ'_0 are pairwise orthogonal. Then, for any $h \in H_0$, the norm of $\rho'_0(h)$ restricted to any weight space is given by the modulus of the corresponding weight. So we have

$$||\rho_0(h)|_{W_H}|| \leq \mathbf{m}(\rho_0(h)|_{V'_\mu}).$$

Now, if h_n is any divergent sequence in H with $h_n/|h_n| \to h$, Lemma 6.9 implies that for any $x \in \mathbb{P}(V') - \mathbb{P}(W_H)$, the ratio

$$\frac{d_{\mathbb{P}}(\rho(h_n)x, \mathbb{P}(V_\mu))}{d_{\mathbb{P}}(\rho(h_n)x, \mathbb{P}(W_H))}$$

does not tend to infinity as $n \to \infty$. In particular this is true for some $x \in \Omega$, since $\mathbb{P}(V') \cap \Omega$ is relatively open and nonempty. But since $\Delta_H \subset \mathbb{P}(W_H)$, this contradicts the fact that $\partial\Delta_H$ is the full orbital limit set of $\rho(H_0)$ in Ω.

We have now proved our claim, which implies that any extreme point of the convex polytope $\mathcal{C}(\rho_0)$ is a weight μ_v for $v \in \mathcal{V}_H$. On the other hand, by Corollary 4.8, we may assume that each vertex $v \in \mathcal{V}_H$ is an extreme point in $\partial\Omega$, and by Proposition 3.5, this means that for each $v \in \mathcal{V}_H$, there is a sequence $h_n \in H_0$ such that v is an attracting subspace for $\rho_0(h_n)$. Since $\rho_0(H_0)$ acts diagonalizably on W_H, this implies that μ_v is an extreme point of $\mathcal{C}(\rho_0)$. The last assertion of the proposition follows directly from the claim. □

Proposition 6.11 tells us that we can combinatorially identify the k-simplex Δ_H and the k-simplex $\mathcal{C}(\rho_0)$ for $\rho_0 = \rho|_{H_0}$. We write this identification explicitly:

Definition 6.12. Let $H \in \mathcal{H}$, and let H_0 be a finite-index free abelian subgroup. For each face F of Δ_H with vertices $\mathcal{V}(F)$, we let $\tilde{F}$ denote the face of $\mathcal{C}(\rho_0)$ whose vertices are the weights μ_v for $v \in \mathcal{V}(F)$.

6.5. **Invariant simplices in the symmetric power.** Our next step is to describe the simplices $S_H \subset \mathbb{P}(\mathrm{Sym}^m V)$ which give rise to the fibers in $\hat{\Lambda}_m$ over parabolic points, for our EGF boundary extension $\hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$.

Let $H \in \mathcal{H}$ have rank k, and let $H_0 \subseteq H$ be a finite-index free abelian subgroup. We let ρ_0, ρ_0^m respectively denote the restrictions of ρ, ρ^m to H_0, and let Φ, Φ^m denote the sets of weights of ρ_0 and ρ_0^m. We observe the following:

Lemma 6.13. *The convex polytope $\mathcal{C}(\rho_0^m)$ is the k-simplex $m\mathcal{C}(\rho_0)$. Moreover, for every face $\tilde{F}$ of $\mathcal{C}(\rho_0)$, the weights in $\Phi^m \cap m\tilde{F}$ are exactly the vertices of the mth barycentric subdivision of $m\tilde{F}$, and each such weight has a one-dimensional generalized weight space.*

Proof. The weights of ρ_0^m are exactly the set of homomorphisms of the form

$$\sum_{\mu \in \Phi} a_\mu \mu,$$

where $a_\mu \in \mathbb{N} \cup \{0\}$ and $\sum a_\mu = m$. In particular, the set of rescaled weights $\frac{1}{m}\Phi^m$ consists entirely of convex combinations of weights of ρ_0, and contains every weight in Φ. This (together with Proposition 6.11) implies that $\mathcal{C}(\rho_0^m)$ is a k-simplex.

Further, every (rescaled) weight in the boundary of the rescaled simplex $\frac{1}{m}\mathcal{C}(\rho_0^m)$ must be a convex combination of weights lying in a single face of the simplex $\mathcal{C}(\rho_0)$. But Proposition 6.11 says that every weight in $\Phi \cap \partial\mathcal{C}(\rho_0)$ is a vertex of $\mathcal{C}(\rho_0)$. So, if F is a face of the simplex Δ_H with vertices $\mathcal{V}(F)$, the weights in $\tilde{F} \cap \frac{1}{m}\Phi^m$ are exactly the convex combinations of the form

$$\frac{1}{m} \sum_{v \in \mathcal{V}(F)} a_v \mu_v, \tag{4}$$

where $a_v \in \mathbb{N} \cup \{0\}$ and $\sum a_v = m$. These are exactly the vertices in the mth barycentric subdivision of $\tilde{F}$, and in fact each such vertex has *unique* expression of the form (4). Since each weight μ_v for $v \in \mathcal{V}(F)$ has a one-dimensional generalized weight space, it follows that the weights in $\Phi^m \cap mF$ do as well. ☐

6.5.1. *The simplices $S_H \subset \mathbb{P}(\mathrm{Sym}^m V)$.* Using Lemma 6.13, we can define the vertices of the simplex S_H: they are exactly the weight spaces for the weights μ lying in $\Phi^m \cap \partial\mathcal{C}(\rho_0^m)$.

To define S_H as a subset of $\mathbb{P}(\mathrm{Sym}^m V)$, we choose lifts in $\mathrm{Sym}^m V$ of each vertex of S_H, and then take convex combinations. Our lifts are chosen as follows: we first pick a lift $\tilde{v} \in V$ of each vertex $v \in \mathcal{V}_H$, so that Δ_H is the projectivization of the convex hull in V of $\{\tilde{v} : v \in \mathcal{V}_H\}$. The weight space of μ for each $\mu \in \partial\mathcal{C}(\rho_0^m)$ is spanned by a unique vector in $\mathrm{Sym}^m V$ of the form

$$\tilde{v}_\mu = \prod_{v \in \mathcal{V}_H} \tilde{v}^{a_v},$$

where $a_v \in \mathbb{N} \cup \{0\}$ and $\sum a_v = m$. Then we can define S_H to be the projectivization of the convex hull in $\mathrm{Sym}^m V$ of the $\tilde{v}_\mu$'s.

6.5.2. *Dynamics on the simplices S_H.* By definition, the vertices of S_H are exactly the weight spaces for the weights in the boundary of the simplex $\mathcal{C}(\rho_0^m) \subset (H_0 \otimes_{\mathbb{Z}} \mathbb{R})^*$. So, Proposition 6.7 immediately implies the following:

Corollary 6.14. *Let $H \in \mathcal{H}$. For every divergent sequence $h_n \in H$, there is a face F of S_H which spans an attracting subspace for the sequence $\rho^m(h_n)$.*

6.6. **Dual simplices in symmetric powers.** As discussed in Section 4, [DGK17, Proposition 5.6] says that since $\rho : \Gamma \to \mathrm{SL}(V)$ is convex cocompact in $\mathbb{P}(V)$, the dual representation $\Gamma \to \mathrm{SL}(V^*)$ is convex cocompact in $\mathbb{P}(V^*)$, and in fact there is a domain $\Omega \subset \mathbb{P}(V)$ so that Γ acts convex cocompactly on both Ω and the dual domain Ω^*. By the work of Islam-Zimmer [IZ19b], each virtually abelian subgroup $H \in \mathcal{H}$ must act cocompactly on a properly embedded *dual* simplex $\Delta_H^* \subset \Omega^*$. And, for each vertex w of Δ_H^*, the projective hyperplane $[\ker w]$ is a supporting hyperplane of Ω at $\partial\Delta_H$.

6.6.1. *The simplices $S_H^* \subset \mathbb{P}(\mathrm{Sym}^m V^*)$.* For each $H \in \mathcal{H}$, we can define an H-invariant *dual* simplex $S_H^* \subset \mathbb{P}(\mathrm{Sym}^m V^*)$, by carrying out the construction we used to find S_H (but this time for the dual representation $\rho^* : \Gamma \to \mathrm{SL}(V^*)$). We can describe the relationship between the simplices S_H and S_H^* a little more explicitly. For a finite-index free abelian subgroup $H_0 \subseteq H$, we let $\rho_0^* : H_0 \to \mathrm{SL}(V^*)$ be the dual of the restriction of ρ to H_0, and similarly define $(\rho_0^m)^* : H_0 \to \mathrm{SL}(V^*)$. Then the weights of ρ_0^* are the negative weights of ρ_0, and the weights of $(\rho_0^m)^*$ are the negative weights of ρ_0^m.

Suppose μ^m is a weight of ρ_0^m with a one-dimensional generalized weight space v^m. Then, the negative weight $-\mu^m$ also has a one-dimensional generalized weight space $w^m \in \mathbb{P}(\mathrm{Sym}^m V^*)$, and $[\ker w^m]$ is the hyperplane spanned by the weight spaces of the weights in $\Phi^m - \{\mu^m\}$. In particular, we can consider the case where μ^m is a weight lying in the boundary of $\mathcal{C}(\rho_0^m)$. In this case, v^m is a vertex of S_H, w^m is a vertex of S_H^*, and $[\ker w^m]$ is a hyperplane intersecting S_H in a codimension-1 face of S_H.

This allows us to define a simultaneous lift of the *boundaries* of the simplices S_H, S_H^* in the space of flags $\mathcal{F}(\mathrm{Sym}^m V)$.

Definition 6.15. For a peripheral subgroup $H \in \mathcal{H}$, we let $\partial\hat{S}_H$ denote the set

$$\partial\hat{S}_H = \{(v, w) \in \mathcal{F}(\mathrm{Sym}^m(V)) : v \in \partial S_H, w \in \partial S_H^*\}.$$

The discussion above shows that $\partial\hat{S}_H$ is a nonempty closed invariant subset of $\mathcal{F}(\mathrm{Sym}^m V)$, projecting to ∂S_H and ∂S_H^* under the canonical maps $\mathcal{F}(\mathrm{Sym}^m V) \to \mathbb{P}(\mathrm{Sym}^m V)$ and $\mathcal{F}(\mathrm{Sym}^m V^*) \to \mathbb{P}(V^*)$.

6.7. **Defining the boundary set.** Using the sets $\partial\hat{S}_H$, we can define our candidate for the EGF boundary set $\hat{\Lambda}_m \subset \mathcal{F}(\mathrm{Sym}^m V)$ as follows. We let $\hat{\phi} : \hat{\Lambda}_\Omega(\Gamma) \to \partial(\Gamma, \mathcal{H})$ denote the boundary extension for the EGF representation ρ coming from the proof of Theorem 1.2. For each $z \in \partial(\Gamma, \mathcal{H})$, we define the set $\hat{\psi}_m(z) \subset \mathcal{F}(\mathrm{Sym}^m(V))$ by:

$$\hat{\psi}_m(z) = \begin{cases} \hat{\iota}(\hat{\phi}^{-1}(z)), & z \in \partial_{\mathrm{con}}(\Gamma, \mathcal{H}) \\ \partial\hat{S}_H, & z \in \partial_{\mathrm{par}}(\Gamma, \mathcal{H}). \end{cases}$$

We define

$$\hat{\Lambda}_m = \bigcup_{z \in \partial(\Gamma, \mathcal{H})} \hat{\psi}_m(z),$$

and observe that $\hat{\iota}(\hat{\Lambda}_\Omega(\Gamma)) \subset \hat{\Lambda}_m$.

The set $\hat{\Lambda}_m$ is $\rho^m(\Gamma)$-invariant, since $\hat{\iota}$ is τ_m-equivariant and the construction of the set $\partial\hat{S}_H$ is invariant. Ultimately we want to see that $\hat{\Lambda}_m$ is compact, and that there is a well-defined transverse map $\hat{\phi}_m : \hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$ giving us our EGF boundary extension.

6.8. **Defining the boundary extension.** Our next immediate goal is to show:

Proposition 6.16. *For distinct $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$, the sets*

$$\hat{\psi}_m(z_1), \quad \hat{\psi}_m(z_2)$$

are transverse (in particular, disjoint). Consequently, the map $\hat{\phi}_m : \hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$ given by

$$\hat{\phi}_m(\xi) = z \iff \xi \in \hat{\psi}_m(z)$$

is well-defined, equivariant, surjective, and transverse.

Lemma 6.17. *Let X be a closed subset of $\Lambda_\Omega(\Gamma)$, and let $\mathcal{H}_X \subset \mathcal{H}$ be the set $\{H \in \mathcal{H} : \partial\Delta_H \cap X \neq \emptyset\}$.*

Then the set

$$X_{\mathcal{H}} = X \cup \bigcup_{H \in \mathcal{H}_X} \Delta_H$$

is closed.

Proof. Let x_n be a sequence in $X_{\mathcal{H}}$. By compactness of $\Lambda_\Omega(\Gamma)$, we can choose a subsequence so that $x_n \to x \in \Lambda_\Omega(\Gamma)$. We wish to show that $x \in X_{\mathcal{H}}$. Since X is closed and $X \subset X_{\mathcal{H}}$, we may assume that for each n, we have $x_n \in \partial\Delta_{H_n}$ for some $H_n \in \mathcal{H}_X$.

Up to subsequence, the sets $\partial\Delta_{H_n}$ converge to a closed set $\partial\Delta_\infty$ which is a connected finite union of (possibly degenerate) projective simplices. We must have $x \in \partial\Delta_\infty \subset \Lambda_\Omega(\Gamma)$. By definition, $\partial\Delta_{H_n}$ intersects X nontrivially, and since X is closed we must also have $\partial\Delta_\infty \cap X \neq \emptyset$.

Suppose for a contradiction that $x \notin X_{\mathcal{H}}$. Then in particular $x \notin X$. Since $\partial\Delta_\infty$ intersects X, it must contain at least two points, which means that every point in $\partial\Delta_\infty$ lies in a nontrivial closed projective segment (since $\partial\Delta_\infty$ is a connected finite union of projective simplices). But then by Corollary 4.8, $\partial\Delta_\infty \subset \partial\Delta_H$ for some $H \in \mathcal{H}$, and since $\partial\Delta_\infty \cap X \neq \emptyset$ we have $H \in \mathcal{H}_X$ and therefore $x \in X_{\mathcal{H}}$, contradiction. ☐

Proposition 6.18. *For each $H \in \mathcal{H}$, there is a connected subset C_H in*

$$\mathrm{Opp}(\partial S_H^*) = \{x \in \mathbb{P}(\mathrm{Sym}^m V) : x \perp w \text{ for every } w \in \partial S_H^*\}$$

such that for every closed subset $X \subset \Lambda_\Omega(\Gamma) - \Delta_H$, C_H contains the closure of

$$\iota(X) \cup \bigcup_{H' \in \mathcal{H}_X} S_{H'},$$

where $\mathcal{H}_X = \{H \in \mathcal{H} : \partial\Delta_H \cap X \neq \emptyset\}$.

Proof. Let $\mathcal{V}_H$, $\mathcal{V}_H^*$ denote the vertex sets of Δ_H and Δ_H^*, respectively. Using the convexity of Ω, we can find lifts $\tilde{w} \in V^*$ for each vertex $w \in \mathcal{V}_H^*$, a continuous lift $\tilde{\Lambda} \subset V$ of $\Lambda_\Omega(\Gamma)$, and a continuous lift $\tilde{\Delta}_H \subset V$ of Δ_H so that

$$\tilde{w}(\tilde{\Lambda} - \tilde{\Delta}_H) > 0 \tag{5}$$

for every $w \in \mathcal{V}_H^*$.

The lifts $\tilde{w}$ induce lifts $\tilde{w}^m \in (\mathrm{Sym}^m V)^*$ of each vertex w^m of S_H^*. We take the set C_H to be the projectivization of

$$\{v \in \mathrm{Sym}^m V : \tilde{w}^m(v) > 0 \text{ for all vertices } w^m \text{ of } S_H^*\}.$$

Every point in ∂S_H^* is the projectivization of a convex combination of the lifts $\tilde{w}^m$. This tells us that C_H is a connected subset of $\mathbb{P}(\mathrm{Sym}^m V) - \bigcup_{w \in \partial(S_H^*)} [\ker w]$.

Now let $X \subset \Lambda_\Omega(\Gamma) - \Delta_H$ be closed and let Y be the set

$$\iota(X) \cup \bigcup_{H' \in \mathcal{H}_X} S_{H'}.$$

We wish to show that $\overline{Y} \subset C_H$. Let $X_{\mathcal{H}}$ be the set

$$X_{\mathcal{H}} = X \cup \bigcup_{H' \in \mathcal{H}_X} \Delta_{H'}.$$

By Lemma 6.17, we can find a *compact* lift $\tilde{X}_{\mathcal{H}}$ of $X_{\mathcal{H}}$ in V so that $\tilde{w}(\tilde{x}) > 0$ for every $\tilde{x} \in \tilde{X}_{\mathcal{H}}$ and every $w \in \mathcal{V}_H^*$. We consider the set

$$\mathrm{Sym}^m \tilde{X}_{\mathcal{H}} = \{\tilde{x}^m \in \mathrm{Sym}^m V : \tilde{x}^m = \prod_{i=1}^m \tilde{x}_i \text{ for } \tilde{x}_i \in \tilde{X}_{\mathcal{H}}\}.$$

This set is the image of the m-fold Cartesian product $(\tilde{X}_{\mathcal{H}})^m$ under the continuous map $V^m \to \mathrm{Sym}^m V$ given by $(v_1, \ldots, v_m) \mapsto v_1 \cdots v_m$, so it is compact. Moreover, since $\mathrm{Sym}^m \tilde{X}_{\mathcal{H}}$ contains a lift of every vertex of every $S_{H'}$ for $H' \in \mathcal{H}_X$, the projectivization of the convex hull of $\mathrm{Sym}^m \tilde{X}_{\mathcal{H}}$ contains Y, hence $\overline{Y}$.

But from (5), we know that $\tilde{w}^m(\tilde{x}^m) > 0$ for every vertex w^m of S_H^* and every $\tilde{x}^m \in \tilde{X}_{\mathcal{H}}$, so we see that C_H contains the projectivization of the convex hull of $\mathrm{Sym}^m \tilde{X}_{\mathcal{H}}$. ☐

Proof of Proposition 6.16. Let $z_1, z_2 \in \partial(\Gamma, \mathcal{H})$ be distinct. If both z_1 and z_2 are conical limit points, the proposition follows from the transversality of the EGF boundary extension $\hat{\phi} : \hat{\Lambda}_\Omega(\Gamma) \to \partial(\Gamma, \mathcal{H})$ and the fact that $\hat{\iota}$ preserves transversality. On the other hand, if z_1 is a parabolic point, this follows from Proposition 6.18 (and the equivalent dual statement). ☐

6.9. **Dynamics on S_H.** We have now defined an equivariant transverse surjective map $\hat{\phi}_m : \hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$, but we do not yet know that the set $\hat{\Lambda}_m$ is compact, or even that $\hat{\phi}_m$ is continuous. However, it turns out that it is easier to verify these two facts after proving that $\hat{\phi}_m$ has certain dynamical properties.

Lemma 6.19. *For each $H \in \mathcal{H}$, there exists an open set $\hat{C}_H \subset \mathcal{F}(\mathrm{Sym}^m V)$ containing $\hat{\Lambda}_m - \partial \hat{S}_H$, such that for any infinite sequence $h_n \in H$ and $\xi \in \hat{C}_H$, we have*

$$\rho^m(h_n)\xi \to \partial \hat{S}_H.$$

Proof. For each $H \in \mathcal{H}$, we let $C_H \subset \mathbb{P}(\mathrm{Sym}^m V)$ be the set coming from Proposition 6.18. Let h_n be a divergent sequence in some $H \in \mathcal{H}$, and let H_0 be a finite-index free abelian subgroup. Corollary 6.14 says that some face F of S_H spans an attracting subspace for $\rho^m(h_n)$. The corresponding repelling subspace is a direct sum of weight spaces for the restriction $\rho^m|_{H_0}$, so it is contained in $[\ker w^m]$ for a vertex w^m of the dual simplex S_H^*. So, for any $x \in C_H$, any subsequence of $\rho^m(h_n)x$ subconverges to a point in $[\mathrm{span}(F)]$. In fact, $\rho^m(h_n)x$ subconverges to a point in $\overline{F} \subset \partial S_H$, since C_H is $\rho^m(H)$-invariant and $C_H \cap \mathrm{supp}(F) = \overline{F}$.

Then, we can dually define a set $C_H^* \subset \mathbb{P}(\mathrm{Sym}^m V^*)$, and take

$$\hat{C}_H = \{(x, w) \in \mathcal{F}(\mathrm{Sym}^m V) : x \in C_H, w \in C_H^*\}.$$

☐

6.10. **Continuity and compactness.**

Lemma 6.20. *The set $\hat{\Lambda}_m$ is closed.*

Proof. Let (x_n, w_n) be a sequence in $\hat{\Lambda}_m$, and let $z_n = \hat{\phi}_m(x_n, w_n)$. Up to subsequence, z_n converges to $z \in \partial(\Gamma, \mathcal{H})$.

If z is a conical limit point, let γ_n be a sequence limiting conically to z, chosen so that for any $z' \neq z$, we have $\gamma_n^{-1} z' \to b$ and $\lim \gamma_n^{-1} z_n = a \neq b$.

Then $\hat{\phi}^{-1}(\gamma_n^{-1} z_n)$ converges to $\hat{\phi}^{-1}(a)$, and thus $\hat{\phi}^{-1}(\gamma_n^{-1} z_n)$ lies in a fixed compact subset X of $\mathrm{Opp}(\hat{\phi}^{-1}(b)) \cap \hat{\Lambda}_\Omega(\Gamma)$. By definition, this means that for every n, $\hat{\phi}_m^{-1}(\gamma_n^{-1} z_n)$ lies in the set

$$\iota(X) \cup \bigcup_{H' \in \mathcal{H}_X} S_{H'},$$

Arguing as in Proposition 6.18, we see that this set is compact. So by antipodality of $\hat{\phi}_m$, the sets of flags $\hat{\phi}_m^{-1}(\gamma_n^{-1} z_n)$ lie in a fixed compact subset of $\mathrm{Opp}(\hat{\phi}_m^{-1}(b)) = \mathrm{Opp}(\hat{\iota}(\hat{\phi}^{-1}(b)))$ and by Proposition 6.1,

$$(x_n, w_n) \in \rho^m(\gamma_n) \hat{\phi}_m^{-1}(\gamma_n^{-1} z_n)$$

converges to $\hat{\iota}(\hat{\phi}^{-1}(z))$.

If z is a parabolic point, we let $H = \mathrm{Stab}_\Gamma(z)$, and choose $h_n \in H$ so that $h_n^{-1} z_n \in K$ for a fixed compact $K - \{z\}$. By Proposition 6.18, we know that for all n, $\hat{\phi}_m^{-1}(h_n^{-1} z_n)$ lies in a fixed compact subset of $\hat{C}_H$. Then, Lemma 6.19 implies that

$$(x_n, w_n) \in \rho^m(h_n) \hat{\phi}_m^{-1}(h_n^{-1} z_n)$$

subconverges to a point in $\partial \hat{S}_H$. □

Proposition 6.21. *The map $\hat{\phi}_m$ is continuous.*

Proof. Let (x_n, w_n) be a sequence in $\hat{\Lambda}_m$, converging to (x, w) (which we know lies in $\hat{\Lambda}_m$ by the previous proposition). Let $z_n = \hat{\phi}_m(x_n, w_n)$, and suppose for a contradiction that up to subsequence $z_n \to z$ for $z \neq \hat{\phi}_m(x, w)$.

Proposition 6.18 then implies that z_n lies in a compact subset $K \subset \partial(\Gamma, \mathcal{H})$ so that the closure of $\hat{\phi}_m^{-1}(K)$ is opposite to (x, w). This contradicts the fact that (x_n, w_n) converges to (x, w). □

At this point, we have shown that $\hat{\phi}_m : \hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$ is a continuous equivariant surjective transverse map, and that $\hat{\Lambda}_m$ is a compact subset of $\mathcal{F}(\mathrm{Sym}^m V)$. So, we can finish the proof of Theorem 1.7 by showing:

Proposition 6.22. *The map $\hat{\phi}_m : \hat{\Lambda}_m \to \partial(\Gamma, \mathcal{H})$ extends the convergence action of Γ on $\partial(\Gamma, \mathcal{H})$.*

Proof. We apply Proposition 2.11. If γ_n is a sequence limiting conically to z, then the results of Section 4 imply that the sequences $\rho(\gamma_n^{\pm})$ have unique limit points in $\mathcal{F}(V)$, all lying in $\hat{\Lambda}_\Omega(\Gamma)$. Since $\hat{\iota}(\hat{\Lambda}_\Omega(\Gamma))$ is a subset of $\hat{\Lambda}_m$, Proposition 6.1 ensures that the first condition of Proposition 2.11 holds.

On the other hand, for each parabolic point p, we take $\hat{C}_p$ to be the open set $\hat{C}_H$ considered in Lemma 6.19, for $H = \mathrm{Stab}_\Gamma(p)$. Lemma 6.19 implies that $\hat{C}_p$ contains $\hat{\Lambda}_m - \hat{\phi}_m^{-1}(p)$ and that for any $(x, w) \in \hat{C}_p$ and any infinite sequence $h_n \in H$, $\rho^m(h_n)\xi$ subconverges to a point in $\partial \hat{S}_H$. □

6.11. **Stability.** We have now shown that the representations ρ^m are all extended geometrically finite. Our last goal for the section is the following (which implies Theorem 1.8):

Proposition 6.23. *The space* $\mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$ *is peripherally stable with respect to* $(\rho^m, \hat{\phi}_m)$.

The main step in the proof is the following:

Lemma 6.24. *Let H_0 be a finite-index free abelian subgroup of some $H \in \mathcal{H}$, with $H = \mathrm{Stab}_\Gamma(p)$. For any open set $U \subset \mathbb{P}(\mathrm{Sym}^m V)$ containing S_H and any compact $K \subset C_p$, there exists a cofinite subset $T \subset H_0$ and an open set $\mathcal{W} \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$ containing ρ^m such that for any $\sigma \in \mathcal{W}$, we have $\sigma(h)K \subset U$ for any $h \in T$.*

Proof. We fix U and K as in the proposition, and proceed by contradiction. So, suppose that there exists a sequence of distinct group elements $h_n \in H_0$, a sequence of representations $\sigma_n : \Gamma \to \mathrm{SL}(\mathrm{Sym}^m V)$, and a sequence of points $x_n \in K$ such that $\sigma_n \to \rho^m$ and $\sigma_n(h_n)x_n \notin U$. Up to subsequence we can assume that x_n converges to some $x \in K \subset C_p$. We let Φ^m denote the set of generalized weights of $\rho^m|_{H_0}$, and we let Φ_n^m denote the generalized weights of $\sigma_n|_{H_0}$.

We choose a norm $|\cdot|$ on $H_0 \otimes_{\mathbb{Z}} \mathbb{R}$. Then up to subsequence $h_n/|h_n|$ converges to $h_\infty \in H_0 \otimes_{\mathbb{Z}} \mathbb{R}$ with $|h_\infty| = 1$.

Since Φ^m is finite, there is a face $\tilde{F}$ of the k-simplex $\mathcal{C}(\rho^m|_{H_0})$ and a constant $M > 0$, such that for every weight $\mu \in \Phi^m(F) = \Phi^m \cap \tilde{F}$, and every weight $\mu^{\mathrm{opp}} \in \Phi^m - \Phi(F)$, we have

$$\mu(h_\infty) - \mu^{\mathrm{opp}}(h_\infty) > M.$$

We let $V_F^m \subset \mathrm{Sym}^m V$ denote the span of the weight spaces of the weights in $\Phi^m(F)$; by definition $\mathbb{P}(V_F^m)$ is the projective span of a face of S_H.

Proposition 6.3 implies that as a set with multiplicity, the weights Φ_n^m converge to the weights Φ^m. So, for each n, there is a subset $\theta_n \subset \Phi_n^m$ such that θ_n converges to $\Phi^m \cap \partial\mathcal{C}(\rho^m|_{H_0})$, and a subset $\theta_n(F) \subset \theta_n$ such that $\theta_n(F)$ converges to $\Phi^m(F)$. Proposition 6.3 also implies that for sufficiently large n, all of the weights in θ_n must have one-dimensional generalized weight spaces, converging to the vertices of S_H.

This means that for each n, there are simplices S_H^n and $(S_H^n)^*$, invariant under the action of $\sigma_n(H)$, such that $S_H^n \to S_H$ and $(S_H^n)^* \to S_H^*$. So, we can find a sequence of group elements $g_n \in \mathrm{SL}(V)$, with g_n converging to the identity, so that $\sigma_n' = g_n \sigma_n g_n^{-1}$ preserves the simplices S_H and S_H^*. Moreover, the vertices of S_H are the weight spaces V_μ of σ_n' for $\mu \in \theta_n$, and the space V_F^m is spanned by weight spaces $V_{\mu(F)}$ of σ_n' for $\mu(F) \in \theta_n(F)$. We can also assume that σ_n' preserves the complementary weight space $(V_F^m)^{\mathrm{opp}}$ for V_F.

Now, since $\theta_n(F)$ converges to $\Phi^m(F)$, for sufficiently large n we must have

$$\mu_n(h_\infty) - \mu_n^{\mathrm{opp}}(h_\infty) > M$$

for every $\mu_n \in \theta_n(F)$ and every $\mu_n^{\mathrm{opp}} \in \Phi_n^m - \theta_n(F)$. This also means that for sufficiently large n we have

$$\mu_n(h_n) - \mu_n^{\mathrm{opp}}(h_n) > M|h_n|.$$

Then, letting $r^+(g)$ and $r^-(g)$ respectively denote the largest and smallest modulus of any eigenvalue of g, we see that for sufficiently large n,

$$\frac{r^-(\sigma_n'(h_n)|_{V_F^m})}{r^+(\sigma_n'(h_n)|_{(V_F^m)^{\mathrm{opp}}})} > \exp(M|h_n|).$$

We may choose an inner product on $\mathrm{Sym}^m V$ so that V_F^m and $(V_F^m)^{\mathrm{opp}}$ are orthogonal. And, up to change-of-basis lying in compact subset of $\mathrm{SL}(V_F^m \oplus (V_F^m)^{\mathrm{opp}})$, the restriction of (the complexifications of) $\sigma_n'(H_0)$ to V_F^m and $(V_F^m)^{\mathrm{opp}}$ are both upper-triangular. Then we can apply Lemma 6.8 to see that the ratio

$$\frac{\mathbf{m}(\sigma_n'(h_n)|_{V_F^m})}{||\sigma_n'(h_n)|_{(V_F^m)^{\mathrm{opp}}}||}$$

tends to infinity as $n \to \infty$. Then by Lemma 6.9, for sufficiently large n, $\sigma_n'(h_n)x_n$ lies in a small neighborhood of $\mathbb{P}(V_F)$. Moreover, we know that the set C_p is $\sigma_n'(H)$-invariant, since the simplex S_H^* is $\sigma_n'(H)$-invariant. Since x lies in C_p, $\sigma_n'(h_n)x$ lies in an arbitrarily small neighborhood of $\mathbb{P}(V_F) \cap C_p$. By definition this intersection is a face of S_H, so for large enough n, $\sigma_n(h_n)x$ must lie in an arbitrarily small neighborhood of this face, giving a contradiction. □

Proof of Proposition 6.23. We want to show that if $H = \mathrm{Stab}_\Gamma(p)$ for a parabolic point p, K is a compact subset of $\hat{C}_p$, U is a neighborhood of $\hat{S}_H$, and $T \subset H$ is a cofinite subset such that

$$\rho^m(T) \cdot K \subset U, \tag{6}$$

then we can find a neighborhood $\mathcal{W}$ of $\rho^m|_H$ in $\mathrm{Hom}(H, \mathrm{SL}(\mathrm{Sym}^m V))$ so that for any $\sigma \in \mathcal{W}$,

$$\sigma(T) \cdot K \subset U. \tag{7}$$

For simplicity, we will not work in the space of flags $\mathcal{F}(\mathrm{Sym}^m V)$. Instead we will just show that that if (6) holds for a compact $K \subset C_p \subset \mathbb{P}(\mathrm{Sym}^m V)$ and an open neighborhood U of S_H in $\mathbb{P}(\mathrm{Sym}^m V)$, then (7) holds also.

Fix a finite-index free abelian subgroup $H_0 \subseteq H$. It suffices to show that we can choose an open $\mathcal{W} \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$ so that

$$\sigma(T \cap H_0) \cdot K \subset U$$

for all $\sigma \in \mathcal{W}$. It follows immediately from Lemma 6.24 that we can find a cofinite set $T' \subset H_0$ and an open set $\mathcal{W} \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{Sym}^m V))$ so that for all $\sigma \in \mathcal{W}'$, we have

$$\sigma(T') \cdot K \subset U.$$

But then since $(T \cap H_0) - T'$ is finite, we can just shrink $\mathcal{W}$ to get the desired result. □

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