COMBINATION THEOREMS FOR GEOMETRICALLY FINITE CONVERGENCE GROUPS

ALEC TRAASETH AND THEODORE WEISMAN

ABSTRACT. We prove combination theorems in the spirit of Klein and Maskit in the context of discrete convergence groups acting geometrically finitely on their limit sets. As special cases, we obtain combination theorems for geometrically finite groups of isometries of Hadamard manifolds with pinched negative curvature, and for relatively quasiconvex subgroups of relatively hyperbolic groups.

CONTENTS

1. INTRODUCTION

When a group G acts by homeomorphisms on a compact metrizable space M, we say G is a *discrete convergence group* if the induced action on the space of distinct triples in M is properly discontinuous. Despite their simple definition, convergence groups carry a considerable amount of structure. They arise naturally when considering actions of isometry groups of negatively curved metric spaces on their ideal boundaries, and provide a way to study discrete subgroups of isometries of these spaces from the perspective of topological dynamics.

The action of a convergence group G on a space M is *geometrically finite* if every point of the limit set of G in M is either a *conical limit point* or *bounded parabolic point* for the action. Work of Bowditch [Bow12], [Bow95] implies that, if M is the boundary of a proper geodesic negatively curved metric space X, and the action of G on $M = \partial X$ is induced by an isometric action on X, then this definition essentially agrees with the usual definitions of geometrical finiteness in this geometric context.

In this paper, we prove Klein-Maskit-type combination theorems for convergence groups acting geometrically finitely on their limit sets. Our theorems give sufficient conditions for geometrically finite subgroups of $\mathrm{Homeo}(M)$ to generate a geometrically finite convergence group isomorphic to an amalgamated free product or HNN extension of the original groups. Precisely, we prove the following two theorems:

Theorem A. *Let G_1 and G_2 be discrete convergence groups acting on a compact metrizable space M. Suppose that $J = G_1 \cap G_2$ is geometrically finite, and G_1 and G_2 are in AFP ping-pong position with respect to J. Let $G = \langle G_1, G_2 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_1 *_J G_2$.
(ii) G *is discrete.*
(iii) *Elements of G not conjugate into G_1 nor G_2 are loxodromic.*
(iv) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

Theorem B. *Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f \rangle$ be an infinite cyclic discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in $\mathrm{Homeo}(M)$. Suppose G_0 is in HNN ping-pong position with respect to f, J_1 and J_{-1}. Let $G = \langle G_0, G_1 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_0 *_f$.
(ii) G *is discrete.*
(iii) *Elements of G not conjugate into G_0 are loxodromic.*
(iv) *G is geometrically finite if and only if G_0 is geometrically finite.*

The exact definitions of "AFP ping-pong position" and "HNN ping-pong position" are given at the beginning of Section 4 and Section 6, respectively. They are versions of the "ping-pong" configuration of limit sets required by Maskit's combination theorems for Kleinian groups (see [Mas88]).

1.1. **Special case: M is the boundary of a negatively curved metric space.** If X is a proper geodesic metric space which is hyperbolic in the sense of Gromov, then any discrete subgroup of $\mathrm{Isom}(X)$ acts on both the Gromov boundary ∂X of X and the compactification $X \sqcup \partial X$ as a discrete convergence group. In particular this holds if X is a Hadamard manifold with pinched negative curvature, e.g. a rank-1 symmetric space of noncompact type.

Theorem A and Theorem B both apply in this situation, meaning they imply combination theorems for geometrically finite subgroups of rank-1 semisimple Lie groups. In particular, in the special case $M = \partial \mathbb{H}^n_{\mathbf{R}}$, Theorem A recovers a result of Li-Ohshika-Wang [LOW09], who proved a version of Maskit's combination theorem for amalgamated free products of geometrically finite groups acting on real hyperbolic space of any dimension.

In [LOW15], Li-Ohshika-Wang also proved a version of Maskit's HNN extension theorem in arbitrary-dimensional real hyperbolic space. Theorem B is *not* strong enough to fully recover this result, and both Theorem A and Theorem B fail to fully recover Maskit's analogous combination theorems for Kleinian groups. The reason is that we impose some additional hypotheses on the relative positions of the limit sets of the subgroups we are combining. See Remark 4.2 for more detail.

1.2. **Combination theorems for relatively hyperbolic groups.** Any relatively hyperbolic group acts as a convergence group on its Bowditch boundary, which means that Theorem A and Theorem B also directly imply combination theorems for *relatively quasi-convex* subgroups of relatively hyperbolic groups.

A number of combination theorems along these lines can be found in the literature; see for instance [Git99], [MP09], [MPS12], [Yan12], [MM22]. The combination theorems given by these papers are all *virtual*: they provide conditions guaranteeing that certain finite-index subgroups G_1', G_2' of relatively quasi-convex subgroups G_1, G_2 generate a relatively quasi-convex subgroup, isomorphic to an amalgam of G_1' and G_2'. In contrast, the results in this paper give explicit conditions which can be used to verify that a particular pair of relatively quasi-convex subgroups generates a relatively quasi-convex amalgam.

We can additionally contrast the combination theorems in this paper with *abstract* combination theorems for relatively hyperbolic groups, which provide conditions that guarantee that the fundamental group of some graph of relatively hyperbolic groups is also relatively hyperbolic. Combination theorems of this type were originally proved for hyperbolic groups by Bestvina-Feighn [BF92], and various generalizations have been given for relatively hyperbolic groups (see [Dah03], [Ali05], [MR08], [Gau16], [Tom22a], [Tom22b]).

We give special mention to the work of Dahmani [Dah03] (later generalized by Tomar [Tom22a], [Tom22b]), because the method of proof is particularly relevant to the situation we encounter in this paper. Dahmani and Tomar prove that an amalgam of relatively hyperbolic groups is relatively hyperbolic by giving an abstract construction of an appropriate "limit set" for the amalgam to act on. That is, their strategy is to directly construct the Bowditch boundary of the amalgam, and then prove that the amalgam acts geometrically finitely on this space. Indeed, an alternative approach to the proof of the combination theorems given in this paper would be to show that the "limit sets" constructed in [Dah03], [Tom22a], [Tom22b] appear embedded in the space M on which our subgroups all act. However, in this paper we prefer a direct approach, which we feel is more self-contained and straightforward.

1.3. **Possible generalizations: higher rank combination theorems.** Dey-Kapovich-Leeb [DKL19] have previously proved combination theorems for *Anosov* subgroups of higher-rank Lie groups, along the lines of the combination theorems for hyperbolic groups proved by Gitik in [Git99]. Anosov subgroups are discrete subgroups of higher-rank Lie groups which generalize the dynamical behavior of convex cocompact groups in rank one.

More recently, Dey-Kapovich [DK22], [DK23] have proved sharper combination theorems for Anosov subgroups, giving Maskit-type ping-pong criteria (analogous to the ones in this paper) which guarantee that a pair of Anosov subgroups generates a larger Anosov subgroup, isomorphic to an amalgam of the original groups. The Dey-Kapovich results also apply in rank one, which means that they imply Theorem A and Theorem B in the special case where M is the visual boundary of a rank-one symmetric space X and the subgroups generating the amalgam G are all convex cocompact in $\mathrm{Isom}(X)$.

Unlike the Dey-Kapovich results, the theorems in the present paper do not apply directly in the higher rank setting. However, at their core, our arguments only involve the topological dynamics of an action by homeomorphisms on some space M, and do not rely directly on geometric properties of any metric space bounded by M. Consequently, it is reasonable to believe that our methods could be adapted to extend the work of Dey-Kapovich, and prove combination theorems for discrete subgroups of higher-rank Lie groups with "geometrically finite" dynamical behavior—for instance *relative* Anosov subgroups, or the *extended geometrically finite* subgroups considered by the second author in [Wei22].

1.4. **Tools used in the proof.** Although almost all of the arguments in this paper are purely topological, we do use some metric geometry to prove a key technical result at the end of Section 2. The proof of this proposition (Proposition 2.18) uses the following fact, due to Yaman [Yam04]: if G is a geometrically finite convergence group acting on M, then there is a proper δ-hyperbolic metric space X where G acts by isometries, such that ∂X is equivariantly homeomorphic to the limit set of G in M.

We use the coarse geometry of the space X, together with the convergence action of G on M, to establish some further dynamical properties of the action of G on M. Specifically, we prove that, when a subgroup $J < G$ is *fully quasi-convex*, there is a way to modify sequences (g_k) in $G \setminus J$ by elements of J, so that certain sequences of the form $(g_k x)$ for $x \in M$ do not accumulate on the limit set of J. Once we have established this dynamical fact, we apply the conclusion repeatedly while working in M, and no longer need to reference the geometry of any metric space.

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2. Convergence group actions and geometrical finiteness

This section is mostly devoted to background related to convergence group actions. We start by defining convergence groups and geometrical finiteness in Section 2.1 and Section 2.2. In Section 2.3 we recall some background on relatively hyperbolic groups. At the end of this section, we state and prove a key proposition (Proposition 2.18) about relatively quasi-convex subgroups of geometrically finite convergence groups.

2.1. Convergence groups.

We refer to [Tuk94], [Tuk98], [Bow99] for further background on the material in this section.

Definition 2.1. Let G be a group acting on a compact metrizable space M. We say the action is a *convergence action* and call G a *convergence group* if, whenever (g_k) is a sequence of pairwise distinct elements in G, we can take a subsequence so that one of the following two conditions is satisfied:

(1) The sequence (g_k) converges to a homeomorphism g in the compact-open topology on $\mathrm{Homeo}(M)$.
(2) There are points $z_+, z_- \in M$ (not necessarily distinct) so that the maps $g_k|_{M \setminus \{z_-\}}$ converge to the constant map $z \mapsto z_+$ uniformly on compacts.

If G is a convergence group such that only the second condition occurs, we call G a *discrete convergence group* and the action a *discrete convergence action.*

Remark 2.2. Note that when $G < \mathrm{Homeo}(M)$ is a convergence group, G is a discrete convergence group if and only if G is a discrete subgroup of $\mathrm{Homeo}(M)$ with respect to the compact-open topology on $\mathrm{Homeo}(M)$. So, if (g_k) is a divergent sequence (that is, a sequence which leaves every compact subset of $\mathrm{Homeo}(M)$) in a discrete convergence group G, then we can extract a subsequence so that the second condition above holds.

When M is a topological n-sphere, the definition of a convergence group is due to Gehring and Martin [GM87], who observed that the isometry group of $\mathbb{H}^n_{\mathbf{R}}$ always acts as a convergence group on $\partial\mathbb{H}^n_{\mathbf{R}}$. Gehring and Martin also showed (again when M is an n-sphere) that a group G is a discrete convergence group if and only if the induced action of G on the space of distinct triples in M is properly discontinuous; later Bowditch [Bow99] observed that the same holds when M is an arbitrary compact Hausdorff space.

In the setting where M is compact metrizable, convergence groups were studied systematically by Tukia [Tuk94]. In particular Tukia showed that any group of isometries acting properly discontinuously on a proper geodesic Gromov-hyperbolic metric space X acts as a discrete convergence group on both the boundary ∂X and the compactification $\overline{X} = X \sqcup \partial X$ (see also Freden [Fre95]).

Definition 2.3. Following [Tuk94], if (g_k) is a sequence in $G < \mathrm{Homeo}(M)$ such that the second condition of Definition 2.1 holds *without* extracting a subsequence, then we say that (g_k) is a *convergence sequence*.

In this case, the (uniquely defined) points z_+ and z_- are respectively called the *attracting point* and *repelling point* of the sequence (g_k). It follows immediately that if (g_k) is a convergence sequence with attracting and repelling points z_+, z_-, then (g_k^{-1}) is also a convergence sequence, with attracting and repelling points z_-, z_+.

When G is a discrete convergence group, an *arbitrary* sequence of pairwise distinct elements in G is not necessarily a convergence sequence, but it always has a subsequence which is.

One easy consequence of the definitions above is the following:

Proposition 2.4. *Let (g_k) be a convergence sequence in a discrete convergence group G acting on a compact metrizable space M containing at least 3 points. If U is any open neighborhood of the repelling point z_- of (g_k), then $(g_k\overline{U})$ converges to M in the topology on closed subsets of M induced by Hausdorff distance.*

Proof. If $\overline{U} = M$ the result is immediate, so assume that $M \setminus U$ is nonempty. Then, since $M \setminus U$ is a nonempty compact subset of $M \setminus \{z_-\}$, the set $g_k(M \setminus U)$ converges to a singleton $\{z_+\}$. So $g_k\overline{U}$ eventually contains every compact in the complement of $\{z_+\}$, and must converge to the closure of $M \setminus \{z_+\}$. In addition, since M contains at least 3 points, there are distinct points $x, y \in M$ so that $(g_k x)$ and $(g_k y)$ both converge to z_+. This implies z_+ is not an isolated point of M and so the closure of $M \setminus \{z_+\}$ is M. $\square$

The set of attracting points (or equivalently, the set of repelling points) of sequences in a discrete convergence group G acting on a space M is called the *limit set* of G in M, and is denoted $\Lambda(G)$. The limit set is always a closed G-invariant subset of M. In fact, if G is neither finite nor virtually cyclic, then the limit set of G is the unique minimal nonempty closed G-invariant subset of M.

The complement of $\Lambda(G)$ in M is denoted $\Omega(G)$, and is called the *domain of discontinuity* for G since (as in the setting of Kleinian groups) it is the maximal open subset of M on which G acts properly discontinuously. Recall that a group G acts properly discontinuously on a space X if for any compact $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite.

We say that a discrete convergence group G is *elementary* if $|\Lambda(G)| \leq 2$; this turns out to be equivalent to asking for $|\Lambda(G)|$ to be finite. When G is non-elementary, one can also view $\Lambda(G)$ as the set of accumulation points of any G-orbit in M.

The classification of isometries in hyperbolic space also generalizes to a classification of the elements of a group G acting as a convergence group on M:

Proposition 2.5 ([Tuk94])**.** *Let G act as a convergence group on a compact metrizable space M. Every $g \in G$ satisfies exactly one of the following:*

- *The closure of the cyclic group $\langle g \rangle$ is compact in* $\mathrm{Homeo}(M)$*, in which case we say g is elliptic.*
- *g is not elliptic and g fixes exactly one point in M, in which case we say g is parabolic.*
- *g is not elliptic and g fixes exactly two points in M, in which case we say g is loxodromic.*

Moreover, if g is parabolic or loxodromic, then (g^n) is a convergence sequence, and the set of attracting and repelling points $\{z_\pm\}$ of (g^n) is precisely the set of fixed points of g.

If G is a discrete convergence group, then the elliptic elements of G are precisely those with finite order. The classification also implies that if G is a virtually cyclic discrete convergence group, then G is elementary (but note that the converse need not hold).

2.2. **Geometrical finiteness.** When X is a Hadamard manifold with pinched negative curvature, the geometrically finite subgroups of $\mathrm{Isom}(X)$ are the subgroups G such that the quotient X/G is topologically tame in a precise sense. Geometrical finiteness was originally defined in real hyperbolic spaces of dimension 2 and 3, where the definition concerned the existence of a well-behaved fundamental domain for the action of G on X. This definition proved to be unsatisfactory in hyperbolic spaces of higher dimension and in other negatively curved Hadamard manifolds, however.

In [Bow95], Bowditch gave several different definitions of geometrical finiteness for groups of isometries of a Hadamard manifold X with pinched negative curvature, and proved that they are all equivalent. One of Bowditch's definitions (definition GF5), based on work of Beardon and Maskit [BM74], can be expressed entirely in terms of the convergence action of G on its limit set in ∂X, and therefore generalizes readily to the situation where G is a convergence group acting on an arbitrary compact metrizable space M.

Before giving the definition we recall some essential terminology:

Definition 2.6. Let G be a discrete convergence group acting on a compact metrizable space M.

i) A point $x \in \Lambda(G)$ is a *conical limit point* if there is a sequence (g_k) in G of distinct elements such that for every $z \in M \setminus \{x\}$, the pair $(g_k x, g_k z)$ stays inside a compact subset of $(M \times M) \setminus \Delta$, where $\Delta \subset M \times M$ is the diagonal subspace. We will call the sequence (g_k) a *conical limiting sequence* for the point x.
ii) A point $x \in \Lambda(G)$ is a *parabolic point* if it is the fixed point of a parabolic isometry in G. A *parabolic subgroup* of G is the stabilizer in G of a parabolic point in $\Lambda(G)$. A parabolic point x is *bounded* if $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}_G(x)$ is compact.

Remark 2.7. Tukia [Tuk98] showed that no point in M can be both a parabolic point and a conical limit point. By using the convergence group condition and extracting subsequences, one can also see that a point $x \in M$ is a conical limit point if and only if there are distinct points $a, b \in M$ and a conical limiting sequence (g_k) in G such that $(g_k x)$ converges to a and $(g_k y)$ converges to b for all $y \neq x$. The sequence (g_k) is then a convergence sequence, with $z_+ = b$ and $z_- = x$.

Furthermore, we could just as well ask that the defining condition for a conical limiting sequence holds only for $z \in \Lambda(G) \setminus \{x\}$, and then the discrete convergence dynamics imply this also holds in $\Omega(G)$.

Definition 2.8. Let G be a discrete convergence group acting on a compact metrizable space M. We say that G is *geometrically finite* if every point of $\Lambda(G)$ is either a conical limit point or a bounded parabolic point.

Remark 2.9. Unfortunately, the standard definitions of “geometrically finite” in the geometric and dynamical contexts do not exactly agree. According to the definitions in e.g. [Bow12], [Dah03], a convergence group G acting on M is “geometrically finite” if every point of M (not just of $\Lambda(G)$) is a conical limit point or bounded parabolic point. With this convention, if X is a Hadamard manifold with pinched negative curvature, and G is a geometrically finite subgroup of $\mathrm{Isom}(X)$ (according to the definitions in [Bow93], [Bow95]), then the action of G on ∂X is not a "geometrically finite convergence action" if $\Lambda(G)$ is a proper subset of ∂X.

In this paper, we adopt the convention that a convergence group acting on M is geometrically finite if and only if it acts geometrically finitely (in the sense of [Bow12], [Dah03]) on its limit set in M. So for us, when G acts by isometries on a hyperbolic space X, "geometrically finite" means the same thing regardless of whether we consider the isometric action on X or the induced action by homeomorphisms on ∂X.

We conclude this subsection with another simple but useful criterion which can be used to guarantee that a point $x \in M$ is a conical limit point.

Lemma 2.10. *Let G be a discrete convergence group acting on a compact metrizable space M. Let Y be a subset of M containing at least two points, let K_1, K_2 be disjoint compact subsets of M, and let $x \in M$. If there exists a sequence (g_k) of pairwise distinct elements of G such that for all k we have $g_k x \in K_2$ and $g_k Y \subset K_1$, then x is a conical limit point for G.*

Proof. Since G is a discrete convergence group we can extract a subsequence so that, for points $z_\pm \in M$, the sequence (g_k) converges in $\mathrm{Homeo}(M)$ to the constant map z_+ uniformly on compacts. In particular, for any $y \neq z_-$, $(g_k y)$ converges to z_+. Since Y contains at least two points, it contains at least one point y not equal to z_-. Then since $g_k y \in g_k Y \subset K_1$ we must have $z_+ \in K_1$. Since $g_k x \in K_2$, $(g_k x)$ cannot converge to z_+, hence $x = z_-$. Then for any $y \in M$ with $y \neq x$, $(g_k y)$ converges to z_+. The characterization of conical limit points described in Remark 2.7 implies that x is a conical limit point. ☐

2.3. **Relatively hyperbolic groups.** For most of this paper, we will only ever need to work with the dynamical definition of geometrical finiteness given above. However, our proof of one key technical lemma (Proposition 2.18) does rely on a geometric interpretation of the definition, which is best understood via the connection between geometrically finite groups and *relative hyperbolicity.* We refer to [Bow12], [Hru10] for further background on relatively hyperbolic groups.

The definition of geometrical finiteness we will use is given in Proposition 2.12 below. As in the classical (Kleinian) case, the definition says that, if G is a geometrically finite convergence group acting on a compact metrizable space M, then an appropriately defined "convex core" for the G-action has a "thick-thin" decomposition into a compact piece and some standard "cusps." When M is the boundary of a δ-hyperbolic metric space X, this "convex core" can be defined via the following. For any closed subset Z of ∂X, we let $\mathrm{join}(Z)$ denote the union of all bi-infinite geodesics in X joining distinct points in Z.

Proposition 2.11 (see e.g. [Bow12], section 5)**.** *Suppose that X is a proper geodesic δ-hyperbolic metric space, and $Z \subset \partial X$ is a closed subset containing at least two points. Then* $\mathrm{join}(Z)$ *(with the metric induced by X) is the image of a quasi-isometrically embedded proper geodesic metric space, and its ideal boundary is precisely Z.*

When G is a Kleinian group, $\mathrm{join}(\Lambda(G))$ is within uniformly bounded Hausdorff distance of the convex hull of the limit set of G, i.e. the minimal closed G-invariant convex subset of $\mathbb{H}^3_{\mathbf{R}}$ whose closure in $\overline{\mathbb{H}^3_{\mathbf{R}}}$ contains $\Lambda(G)$. So in the general setting, we can think of the quotient $\mathrm{join}(\Lambda(G))/G$ as a "convex core" for X/G.

Proposition 2.12 (see [Bow12], section 6). *Let X be a proper geodesic δ-hyperbolic metric space and let G be an infinite discrete subgroup of* $\mathrm{Isom}(X)$. *Then the following are equivalent:*

- *The induced action of G on ∂X is geometrically finite in the sense of Definition 2.8.*
- *There exists a G-invariant system of pairwise disjoint horoballs $\mathcal{B}$ in X, such that the stabilizer in G of each $B \in \mathcal{B}$ is a parabolic subgroup, and G acts cocompactly on the set*

$$C(G, \mathcal{B}) := \mathrm{join}(\Lambda(G)) \setminus \bigcup_{B \in \mathcal{B}} B.$$

Moreover, if $|\Lambda(G)| > 1$, then for any *G-invariant system of pairwise disjoint horoballs $\mathcal{B}$ in X, the action of G on $C(G, \mathcal{B})$ is cocompact if and only if the set of centers of horoballs in $\mathcal{B}$ is precisely the set of parabolic points in $\Lambda(G)$.*

Proof. Since G is infinite and discrete, $\Lambda(G)$ cannot be empty. If $|\Lambda(G)| = 1$, then the first bullet point is trivial because the unique point in $\Lambda(G)$ is trivially bounded parabolic, and the second bullet point is trivial because $\mathrm{join}(\Lambda(G))$ is empty. So assume $|\Lambda(G)| > 1$.

The space $Y = \mathrm{join}(\Lambda(G))$ is a *taut* hyperbolic metric space (i.e. every point in Y lies within uniformly bounded distance of a bi-infinite geodesic in Y). Furthermore, horoballs in Y (which can be viewed as a proper geodesic hyperbolic metric space via Proposition 2.11) are at a uniformly bounded Hausdorff distance away from horoballs in X intersected with Y. The result now follows from Proposition 6.12 and Proposition 6.13 in [Bow12] after replacing X with Y. $\square$

If $|\Lambda(G)| > 1$ in the situation above, then we say G is a *relatively hyperbolic group*, and the stabilizers of horoballs in $\mathcal{B}$ are called the *peripheral subgroups*. We say G is *hyperbolic relative to* the collection $\mathcal{P}$ of peripheral subgroups. We also say that any countably infinite group G is hyperbolic relative to $\{G\}$, and that any finite group is hyperbolic relative to an empty collection of peripheral subgroups.

In the special case where X is taut and $\Lambda(G) = \partial X$, we say that X is a *cusped space* for the data of the relatively hyperbolic group G and the peripheral subgroups $\mathcal{P}$. If $|\Lambda(G)| > 1$ we can always find a cusped space by replacing X with $\mathrm{join}(\Lambda(G))$.

The cusped space is in general *not* uniquely determined, even up to quasi-isometry. However, its ideal boundary is a well-defined G-space once the peripheral subgroups of G have been specified (see section 9 in [Bow12]). This space is called the *Bowditch boundary* of G and we denote it ∂G (the notation ignores the dependence on $\mathcal{P}$). When $\mathcal{P} = \{G\}$, then the Bowditch boundary of G is defined to be a singleton, and when G is finite its Bowditch boundary is empty.

When $|\partial G| \leq 2$, then we say G is *elementary*. The Bowditch boundary of a non-elementary relatively hyperbolic group is always *perfect*, i.e. it contains no isolated points. In particular if $|\partial G| \geq 3$, then ∂G is infinite.

A result of Yaman shows that the action of G on its Bowditch boundary can actually be used to completely recover the definition of G as a relatively hyperbolic group:

Theorem 2.13 ([Yam04]). *Let G be a discrete convergence group acting on a perfect compact metrizable space M. If every point of M is either a conical limit point or a bounded parabolic point (equivalently, if G is geometrically finite and $\Lambda(G) = M$), then there is a proper geodesic δ-hyperbolic metric space X, an embedding $G \to \mathrm{Isom}(X)$, and a G-equivariant homeomorphism from M to ∂X.*

The theorem implies in particular that a geometrically finite convergence group is exactly the same thing as a relatively hyperbolic group.

Remark 2.14. Some definitions of relative hyperbolicity explicitly require either the group G or the peripheral subgroups in $\mathcal{P}$ to be finitely generated. We do not make this assumption in this paper, since both Proposition 2.12 and Theorem 2.13 hold without it. Our setup does always force the groups in $\mathcal{P}$ to be infinite, since they are parabolic subgroups of a convergence group.

2.3.1. *Accumulation in geometrically finite subgroups.* Yaman's theorem means that we can always understand a non-elementary discrete convergence group G which is geometrically finite in the sense of Definition 2.8 using its isometric action on a cusped space X. In the case $|\partial G| = 0$ or $|\partial G| = 2$, we can also find a cusped space by taking X to be either a point or a line; if $|\partial G| = 1$ and G is finitely generated, then we can take the cusped space to be a "horoball" modeled on G (see [GM08], [Hru10]).

We take advantage of the existence of the cusped space to prove some properties of subgroups of G which act geometrically finitely on $\Lambda(G)$. A convenient notation we will use here and many times later whenever we have a group G acting on M is

$$H(U) = \bigcup_{g \in H} gU$$

for some $U \subset M, H \subset G$. For the orbit of a point, we will just write Hx.

Definition 2.15. Let G be a relatively hyperbolic group, with Bowditch boundary ∂G. A subgroup $H \leq G$ is *relatively quasi-convex* if H acts geometrically finitely on ∂G (i.e. if every point of $\Lambda(H) \subseteq \partial G$ is either a conical limit point or a bounded parabolic point for the H-action).

Following [Dah03], we say that a relatively quasi-convex subgroup H is *fully quasi-convex* if for all but finitely many left cosets gH, we have $gH(\Lambda(H)) \cap \Lambda(H) = \varnothing$.

Observe that, if G is elementary, then any fully quasi-convex subgroup of G is either finite or has finite index in G.

Lemma 2.16. *Let G be a non-elementary relatively hyperbolic group with associated cusped space $X = X(G)$, and let $H \leq G$ be a fully quasi-convex subgroup of G.*

Fix $x \in X$, and suppose that (g_k) is an infinite sequence in $G \setminus H$ such that

$$d_X(g_k x, x) = d_X(g_k x, Hx) \tag{2.1}$$

for all k. Then no attracting point of g_k in ∂X lies in $\Lambda(H)$.

Proof. Suppose for a contradiction that g_k has an attracting point $z \in \Lambda(H) \subset \partial X$. It follows that H is infinite, since $\Lambda(H)$ is nonempty. Since G acts as a convergence group on both ∂X and $X \sqcup \partial X$, we see that $(g_k x)$ converges to z in $X \sqcup \partial X$.

For each k, we let $c_k : [0, r_k] \to X$ be a geodesic ray in X from x to $g_k x$; since (g_k) is divergent we have $r_k \to \infty$. We may extend each c_k to a map $[0, \infty) \to X$ by setting $c_k(t) = c_k(r_k)$ for all $t \geq r_k$. Up to subsequence, these maps converge uniformly on compacts to a geodesic ray $c_z : [0, \infty) \to X$, whose ideal endpoint must be z.

By Proposition 2.12, there is a G-invariant family $\mathcal{B}_G$ of pairwise disjoint horoballs in X such that the parabolic subgroups of G are precisely the stabilizers of the horoballs in $\mathcal{B}_G$, and the quotient of

$$C(G, \mathcal{B}_G) = X \setminus \bigcup_{B \in \mathcal{B}_G} B \tag{2.2}$$

by the action of G is compact. By shrinking the horoballs in $\mathcal{B}_G$ if necessary, we can also assume that $x \in C(G, \mathcal{B}_G)$.

We claim that z is the center of some horoball $B \in \mathcal{B}_G$. If H is an infinite subgroup of a parabolic subgroup P in G, this is immediate, because then the unique point in $\Lambda(H)$ is the center of the unique horoball in $\mathcal{B}_G$ fixed by P. Otherwise, $\Lambda(H)$ contains at least two points, and we can consider the space $\mathrm{join}(\Lambda(H)) \subset X$.

Let $\mathcal{B}_H$ be the horoballs in $\mathcal{B}_G$ whose centers are parabolic points in $\Lambda(H)$. By Proposition 2.12 again, H acts cocompactly on the set

$$C(H, \mathcal{B}_H) := \mathrm{join}(\Lambda(H)) \setminus \bigcup_{B \in \mathcal{B}_H} B.$$

Since the endpoint of the geodesic c_z lies in $\Lambda(H)$, there is some uniform $R > 0$ so that every point in the image of c_z lies within distance R of $\mathrm{join}(\Lambda(H))$.

Now, suppose that for arbitrarily large t, the point $c_z(t)$ lies in an open R-neighborhood of the set $C(H, \mathcal{B}_H)$. But then for some $k = k(t)$, the point $c_k(t)$ also lies in an R-neighborhood of $C(H, \mathcal{B}_H)$. Since H acts cocompactly on $C(H, \mathcal{B}_H)$, this means that $c_k(t)$ is within uniform distance of hx for some $h \in H$. But this contradicts assumption (2.1).

So, for all sufficiently large times t, $c_z(t)$ must lie in some horoball in $\mathcal{B}_H$. Since the horoballs in $\mathcal{B}_H$ are pairwise disjoint, there is in fact a single horoball $B \in \mathcal{B}_H$ so that $c_z(t)$ is in the interior of B for all large enough t. The center of this horoball must be z.

Since (c_k) converges to c_z, for all sufficiently large k, the geodesic c_k enters B. However, since we have assumed $x \in C(G, \mathcal{B}_G)$, we know that $g_k x \in C(G, \mathcal{B}_G)$, and thus c_k must also leave the horoball B after it enters it. So, let w_k denote the last point where c_k leaves B. The distances $d_X(x, w_k)$ must tend to infinity as $k \to \infty$, since c_z never leaves B. See Figure 2.1.

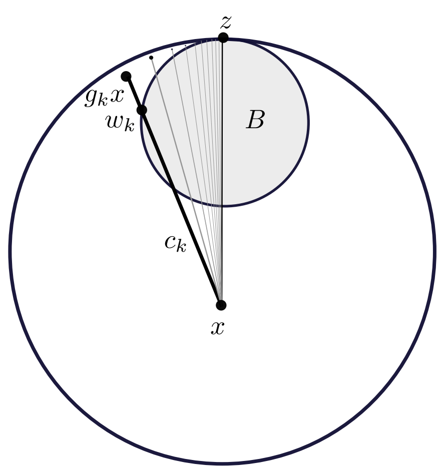

FIGURE 2.1. Illustration for the proof of Lemma 2.16. The geodesic c_k from x to $g_k x$ must enter B, and leave B far from x.

Since c_k is a geodesic we know that $d_X(x, g_k x) = d_X(x, w_k) + d_X(w_k, g_k x)$. Then, because $d_X(x, w_k)$ tends to infinity, our assumption (2.1) implies that $d_X(Hx, w_k)$ tends to infinity as well. But, we also know that the stabilizer of B in G acts cocompactly on ∂B. Then since $w_k \in \partial B$, there is some constant $D > 0$ so that for every k, we have $s_k \in G$ preserving

B such that $d_X(x, s_k^{-1}w_k) < D$, hence $d_X(s_k x, w_k) < D$. It follows that the elements in the sequence (s_k) cannot lie in finitely many left cosets of H. However, since s_k preserves B, each s_k also fixes the point $z \in \Lambda(H)$, which contradicts the full quasi-convexity of H. $\square$

The geometric statement of the lemma above has the following (completely dynamical) consequence:

Lemma 2.17. *Let G be a relatively hyperbolic group with Bowditch boundary ∂G, and let J_1, J_2 be fully quasi-convex subgroups of G.*

For any sequence (g_k) in G, there exists $j_k \in J_1, j_k' \in J_2$ such that the sequence $(j_k g_k j_k')$ has no attracting points in $\Lambda(J_1) \subset \partial G$ and no repelling points in $\Lambda(J_2) \subset \partial G$.

Proof. If $g_k \in J_1 \cup J_2$, then we can choose j_k and j_k' so that $j_k g_k j_k'$ is the the identity. A bounded sequence has no attracting or repelling points. So, we may assume $g_k \in G \backslash (J_1 \cup J_2)$ for all k.

If G is elementary, then J_1 and J_2 are both either finite or finite-index subgroups of G. In this case the result is immediate, so we can assume G is non-elementary and let X be a cusped space for G. Fix $x \in X$. For each k, we choose $j_k \in J_1$, $j_k' \in J_2$ so that

$$d_X(g_k(J_2 x), J_1 x) = d_X(g_k j_k' x, j_k^{-1} x).$$

We know such j_k, j_k' exist because $J_i x$ are discrete subsets of X for $i = 1, 2$. Let $g_k' = j_k g_k j_k'$. We will show that g_k' has no repelling points in $\Lambda(J_2)$; the argument that g_k' has no attracting points in $\Lambda(J_1)$ is completely symmetric, after replacing g_k' with its inverse.

Since $jJ_i x = J_i x$ for any $j \in J_i$, we know that for all k we have

$$d_X(g_k' x, x) = d_X(g_k J_2 x, J_1 x) = d_X(g_k' J_2 x, J_1 x).$$

By definition, we know that

$$d_X(g_k' J_2 x, J_1 x) \le d_X(g_k' J_2 x, x) = d_X(J_2 x, (g_k')^{-1} x),$$

so combining this with the previous equality we conclude

$$d_X(x, (g_k')^{-1} x) = d_X(g_k' x, x) \le d_X(J_2 x, (g_k')^{-1} x)$$

so in fact $d_X(x, (g_k')^{-1} x) = d_X(J_2 x, (g_k')^{-1} x)$ for every k. Then Lemma 2.16 implies that $((g_k')^{-1})$ has no attracting points in $\Lambda(J_2)$, or equivalently (g_k') has no repelling points in $\Lambda(J_2)$. $\square$

Our main application of these lemmas is the technical proposition below. Roughly, this proposition tells us that in certain circumstances, it is possible to strengthen the "ping-pong" combinatorics of geometrically finite convergence groups. That is, the proposition gives us a way to modify a "ping-pong" element $g \in \mathrm{Homeo}(M)$, so that instead of nesting the closure of an open subset $U \subset M$ inside of another open subset $V \subset M$, g takes the closure of U inside of a fixed compact subset $K \subset V$. This "strong nesting" property will be useful throughout the paper.

Proposition 2.18. *Let G be a geometrically finite convergence group acting on a compact metrizable space M, let H be a subgroup of G, and let $J_1, J_2 \le H$ be fully quasi-convex subgroups of G.*

Let U_1, U_2 be open subsets of M such that, for $i \in \{1, 2\}$, we have $J_i(U_i) = U_i$ and $\Lambda(H) \setminus \Lambda(J_i) \subset U_i$. Suppose that for every $g \in H \setminus J_2$, we have $g(M \setminus U_2) \subset U_1$.

Then, there exists a compact set $K \subset U_1$ such that for all $g \in H \setminus J_2$, we can find $j \in J_1$ such that $jg(M \setminus U_2) \subset K$.

Proof. Suppose that the claim does not hold. This means that we can find a sequence of group elements (g_k) in $H \setminus J_2$ such that for any sequence (j_k) in J_1, there is a sequence (x_k) in $M \setminus U_2$ such that the sequence $(j_k g_k x_k)$ accumulates in $M \setminus U_1$.

Fix this sequence (g_k). Lemma 2.17 gives a pair of sequences (j_k) in J_1 and (j_k') in J_2 so that any attracting points of the sequence $(g_k') = (j_k g_k j_k')$ do not lie in $\Lambda(J_2)$, and any repelling points do not lie in $\Lambda(J_1)$. Then, since U_2 is J_2-invariant, there is a sequence (x_k) in $M \setminus U_2$ so that $(j_k g_k j_k' x_k)$ accumulates in $M \setminus U_1$. After taking a subsequence, we may assume that $(j_k g_k j_k' x_k)$ has a unique limit $z \in M \setminus U_1$.

Again using the fact that U_1 and U_2 are invariant under J_1 and J_2 respectively, we know that for every k, we have $g_k'(M \setminus U_2) \subset U_1$. So, if only finitely many different elements appear in the sequence (g_k'), we can find a fixed compact set $K \subset U_1$ so that $g_k'(M \setminus U_2) \subset K$ for every k, hence $g_k' x_k \in K$ for every k. This is impossible if $g_k' x_k \to z \in M \setminus U_1$.

So, we may extract a subsequence so that the elements in (g_k') are pairwise distinct. After taking a further subsequence, we can find a pair of points $z_+ \in M \setminus \Lambda(J_1)$, $z_- \in M \setminus \Lambda(J_2)$ so that (g_k') converges uniformly to the constant map z_+, uniformly on compacts in $M \setminus \{z_-\}$. Both of $z_\pm$ lie in $\Lambda(H)$, so in fact $z_+ \in U_1$ and $z_- \in U_2$.

Since $M \setminus U_2$ is closed, x_k cannot accumulate on z_-, which means $(g_k' x_k)$ converges to $z_+ \in U_1$, which contradicts the fact that $g_k' x_k \to z$. □

3. Combinatorial group theory: amalgamated free products

Our first main result deals with *amalgamated free products*, so we set up the notation and basic facts here. Our reference throughout is [Mas88]. As before, M will continue to denote a compact metrizable space, although the results in this section are purely set-theoretic. We further assume throughout this section that G_1, G_2 are subgroups of $\mathrm{Homeo}(M)$, and $G_1 \cap G_2 = J$, where J is a proper subgroup of both G_1 and G_2. We let G denote $\langle G_1, G_2 \rangle$, the subgroup generated by G_1 and G_2.

The following definition will be convenient in this section as well as later in the paper:

Definition 3.1. We say a subset $U \subset M$ is *precisely invariant* under J in G if U is J-invariant, and for every $g \in G \setminus J$, we have $gU \cap U = \varnothing$.

More generally, given subgroups $J_1, \cdots, J_n < G$, we say a tuple of subsets $(U_1, \cdots, U_n)$ is *precisely invariant* under $(J_1, \cdots, J_n)$ in G if each U_i is precisely invariant under J_i in G, and if for $i \neq j$ and for every $g \in G$, we have $gU_i \cap U_j = \varnothing$.

Given a word $g = g_1 \cdots g_n$ in the elements of G_1 and G_2, we call g a *normal form* when the elements g_i alternate between $G_1 \setminus J$ and $G_2 \setminus J$. We say two normal forms $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$ are *equivalent* if g can be obtained from h by inserting finitely many words of the form jj^{-1} for $j \in J$. We set

$$G_1 *_J G_2 = J \cup \{\text{equivalence classes of normal forms}\}.$$

We have a group operation on $G_1 *_J G_2$ given by concatenation, which is well-defined on normal forms up to equivalence. The abstract group $G_1 *_J G_2$ is called the *free product of G_1 and G_2 amalgamated over J*.

The normal form $g = g_1 \cdots g_n$ is called an *(i, j)-form* if $g_1 \in G_i$ and $g_n \in G_j$. The *length* of the normal form is defined as $|g| = n$. By convention, we will say that elements of J have length 0. Note that if g is an (i, j)-form, then its formal inverse g^{-1} is a (j, i)-form.

There is group homomorphism

$$\begin{aligned}\varphi : G_1 *_J G_2 &\to G\\ g_1 \cdots g_n &\mapsto g_1 \circ \cdots \circ g_n,\end{aligned}$$

where on the right we are just composing the corresponding elements in $\mathrm{Homeo}(M)$. This map is always surjective, but its kernel need not be trivial. When φ is an isomorphism, we will abuse notation and leave it implicit, writing $G = G_1 *_J G_2$; then we can view elements of the subgroup G as (equivalence classes of) normal forms in the abstract amalgamated free product $G_1 *_J G_2$.

Using a ping-pong technique (Proposition 3.6 below), we can give a sufficient condition which guarantees that φ is actually an isomorphism.

Definition 3.2. A pair of disjoint nonempty J-invariant sets $U_1, U_2 \subset M$ is called an *interactive pair* for G_1 and G_2 if for every $g \in G_i \setminus J$, we have $gU_i \subset U_{3-i}$.

If, in addition, $gU_i \subset U_{3-i}$ is a proper inclusion for every $g \in G_i \setminus J$ for at least one of $i \in \{1, 2\}$, then we call (U_1, U_2) a *proper interactive pair.*

Remark 3.3. Maskit's convention is to call an interactive pair U_1, U_2 *proper* if the G_i-translates of U_i do not cover U_{3-i} for at least one $i \in \{1, 2\}$. Our assumption is slightly weaker, but does not change any of the standard arguments.

It is immediate that if (U_1, U_2) is an interactive pair, then U_i is precisely invariant under J in G_i for $i = 1, 2$.

We observe the following:

Proposition 3.4. *If (U_1, U_2) is a proper interactive pair for G_1 and G_2, then both U_1 and U_2 are infinite sets.*

Proof. Since J is a proper subgroup of G_i for $i = 1, 2$, there is at least one element $g_1 \in G_1 \setminus J$ and at least one element $g_2 \in G_2 \setminus J$. We know that at least one inclusion $g_1U_1 \subset U_2$ or $g_2U_2 \subset U_1$ is proper, so $g_2g_1U_1$ is a proper subset of U_1. Therefore U_1 is infinite, and since $g_1U_1 \subset U_2$, so is U_2. □

Via the map φ, normal forms in $G_1 *_J G_2$ act in a "ping-pong" manner on the sets in an interactive pair.

Lemma 3.5 ([Mas88] VII.A.9)**.** *Suppose we have an interactive pair (U_1, U_2). Then if $g \in G_1 *_J G_2$ is an (i, j)-form, we have $\varphi(g)U_j \subset U_{3-i}$. Further, this inclusion is proper if (U_1, U_2) is proper and $|g| \geq 2$.*

The lemma can be proved via a straightforward combinatorial argument; see the reference for details. To illustrate the idea, suppose the G_1-translates of U_1 are all properly contained in U_2, and that g has length 2. If $g = g_1g_2$ is a $(2, 1)$-form, then $g_2(U_1) \subset U_2$ is already proper, and hence $\varphi(g)U_1 \subset U_1$ is also a proper inclusion. If g is a $(1, 2)$-form, then $g_2U_2 \subset U_1$ need not be a proper inclusion, but then applying g_1 will cause the next inclusion $\varphi(g)U_2 = g_1g_2U_2 \subset U_2$ to be proper.

Proposition 3.6 (Ping-pong for amalgamated free products; see [Mas88] VII.A.10)**.** *Suppose (U_1, U_2) is a proper interactive pair for G_1 and G_2. Set $G = \langle G_1, G_2\rangle$. Then $G = G_1 *_J G_2$.*

Proof. We will show the surjective group homomorphism $\varphi : G_1 *_J G_2 \to G$ has trivial kernel. The only length 0 element sent to the identity is the identity, and length 1 elements are all nontrivial in G_1 or G_2, so it suffices to show $\varphi(g) \neq 1$ when $|g| \geq 2$. Suppose g is an (i, j)-form. We now note that because we have a proper interactive pair, $\varphi(g)U_j \subset U_{3-i}$ is a proper inclusion by Lemma 3.5, and so $\varphi(g)$ cannot be the identity. The result follows. $\square$

4. Theorem A

We now introduce the main definition for Theorem A.

Definition A (AFP ping-pong position)**.** Let G_1 and G_2 act as discrete convergence groups on a compact metrizable space M, and suppose that $G_1 \cap G_2 = J$ is a geometrically finite group distinct from both G_1 and G_2. We say G_1 and G_2 are in *AFP ping-pong position* (with respect to J) if there exist closed sets $B_1, B_2 \subset M$ with nonempty disjoint interiors satisfying the following:

(1) For $i \in \{1, 2\}$, B_i is J-invariant.
(2) For $i \in \{1, 2\}$, and for each $g \in G_i \setminus J$, $gB_i \subset \mathrm{Int}(B_{3-i})$.
(3) For $i \in \{1, 2\}$, $\Lambda(G_i) \setminus \Lambda(J) \subset \mathrm{Int}(B_{3-i})$.

The definition above for the most part mimics the setup in Maskit's original combination theorem for amalgamated free products of Kleinian groups. It may be helpful to consider the following concrete example.

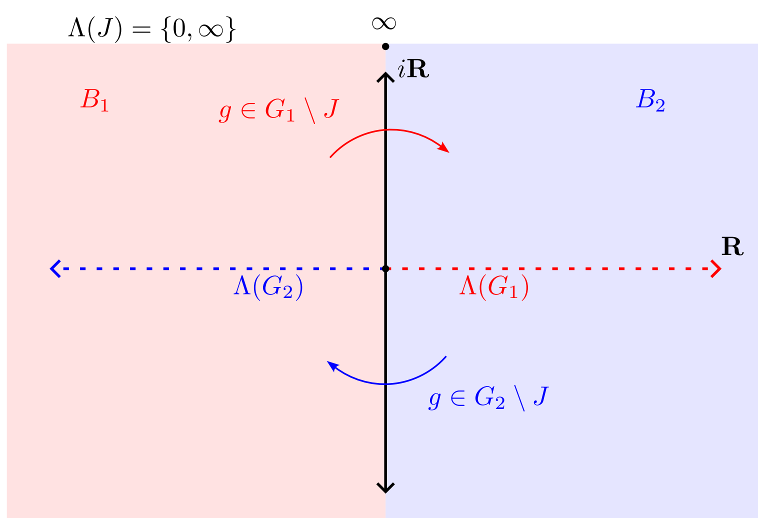

Figure 4.1. Illustration for the example. The limit sets $\Lambda(G_i)$ are Cantor sets.

Example 4.1. Let $X = \mathbb{H}^3_{\mathbf{R}}$, and let M be the visual boundary $\partial\mathbb{H}^3_{\mathbf{R}}$, viewed as the one-point compactification $\widehat{\mathbf{C}}$ of $\mathbf{C}$. We let G be a Fuchsian genus 2 surface group, embedded in $\mathrm{PSL}_2(\mathbf{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$ via the inclusion $\mathrm{PSL}_2(\mathbf{R}) < \mathrm{PSL}_2(\mathbf{C})$.

Then G has the presentation $\langle a, b, c, d \mid [a, b][c, d] = 1\rangle$. Set $j = [a, b] = [c, d]^{-1}$. We then take $G_1 = \langle a, b\rangle \cong F_2$, and $G_2 = \langle c, d\rangle \cong F_2$, the free group on 2 letters, and $J = G_1 \cap G_2 = \langle j\rangle \cong \mathbb{Z}$. We can arrange our generators so $\Lambda(G_1) \subset \mathbf{R}_{\geq 0} \cup \{\infty\}$ and

$\Lambda(G_2) \subset \mathbf{R}_{\leq 0} \cup \{\infty\}$, where $\Lambda(J) = \{0, \infty\}$. These will be the fixed points of $j = (x \mapsto \lambda x)$ where $\lambda > 1$. See Figure 4.1.

This is precisely the picture one gets when gluing the sides of an octagon in $\mathbb{H}^2_{\mathbf{R}}$ to form a surface of genus 2, and then isometrically embedding this picture into the standard $\mathbb{H}^2_{\mathbf{R}}$ sitting inside $\mathbb{H}^3_{\mathbf{R}}$ whose boundary is $\mathbf{R} \cup \{\infty\}$. The limit set of the surface group coincides with $\partial\mathbb{H}^2_{\mathbf{R}}$, and then the octagon with sides identified appears once in each connected component of $\partial\mathbb{H}^3_{\mathbf{R}} \setminus \partial\mathbb{H}^2_{\mathbf{R}}$. Our J-invariant sets B_1 and B_2 are then the closed left and right closed half-planes respectively, including ∞, and their common intersection is $i\mathbf{R} \cup \infty$. If we identify $\widehat{C}$ with S^2, then B_1 and B_2 are complementary hemispheres.

We recall the main result of this section:

Theorem A. *Let G_1 and G_2 be discrete convergence groups acting on a compact metrizable space M. Suppose that $J = G_1 \cap G_2$ is geometrically finite, and G_1 and G_2 are in AFP ping-pong position with respect to J. Let $G = \langle G_1, G_2 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_1 *_J G_2$.
(ii) G is discrete.
(iii) Elements of G not conjugate into G_1 nor G_2 are loxodromic.
(iv) G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.

Remark 4.2. The hypotheses for Theorem A are different from the hypotheses for Maskit's original combination theorems in $\mathbb{H}^3_{\mathbf{R}}$ in two respects. First, Maskit insists that the sets B_1, B_2 in Definition A are topological balls in $M = \partial\mathbb{H}^3_{\mathbf{R}}$, satisfying $\partial B_1 = \partial B_2$ and $B_1 \cup B_2 = M$. This requirement is unnatural in our setting, since M may not even be a manifold, and it is not needed in any of our arguments.

Second, and more significantly, Maskit's version of condition (2) in Definition A is weaker than what we have given here. Our condition implies in particular that if $g \in G_i \setminus J$, then $gB_i \cap B_i = \varnothing$. This means that if $P < J$ is a maximal parabolic subgroup in J, then P must also be a maximal parabolic subgroup in G_i.

Maskit's original statement in $\mathbb{H}^3_{\mathbf{R}}$ allows gB_i to intersect B_i in limit points of J, which means his theorem allows for amalgamations along subgroups $J < G_i$ whose parabolic subgroups are *not* maximal in G_i. This means our theorem is not strong enough to recover Maskit's original result in the case $M = \partial\mathbb{H}^3_{\mathbf{R}}$. However, most of the examples constructed in Maskit's book satisfy the stronger hypothesis we have given above.

Below, we give a quick proof of the first three parts of Theorem A. The arguments are standard, but we provide them for convenience.

Proof of (i) - (iii) in Theorem A. (i) Let B_1, B_2 be the closed subsets of M from Definition A. We note that since $gB_i \subset \mathrm{Int}(B_{3-i})$ for any $g \in G_i \setminus J$, it follows that $g\mathrm{Int}(B_i) \subset \mathrm{Int}(B_i)$ is a proper inclusion for every $g \in G_i \setminus J$. Hence $(\mathrm{Int}(B_1), \mathrm{Int}(B_2))$ form a proper interactive pair for G_1 and G_2 by conditions (1) and (2), so we are done by Proposition 3.6.

(ii) It suffices to show no sequence in G accumulates at the identity. Let (g_k) be a sequence of distinct elements in G. Since G_1, G_2 are discrete we can assume $|g_k| > 1$. If the length of g_k is odd, then g_k maps one of the sets B_1, B_2 into the interior of the other and hence is far from the identity, so assume the lengths are all even. Without loss of generality, we may assume every g_k is a $(2,1)$-form. We have $g_k B_1 \subset \mathrm{Int}(B_1)$ for every k.

Suppose for a contradiction that (g_k) converges to the identity. Then $g_k B_1$ converges to B_1. Write $g_k = h_k g'_k$ where $|g'_k| = |g_k| - 1$ and $h_k \in G_2 \setminus J$. Then $g_k B_1 \subset h_k B_2 \subset \mathrm{Int}(B_1)$

for every k since $g'_k B_1 \subset B_2$. It now also follows that $(h_k B_2)$ converges to B_1. Now, in general, when $g, h \in G_2 \setminus J$, we will have gB_2 and hB_2 either disjoint or equal. Indeed, if $gB_2 \cap hB_2 \neq \varnothing$, then $h^{-1}g$ sends a point in B_2 back into B_2, hence $h^{-1}g = j \in J$. Then $gB_2 = hjB_2 = hB_2$ as desired.

Since $h_1 B_2 \subset \mathrm{Int}(B_1)$ has nonempty interior and $h_k B_2 \subset \mathrm{Int}(B_1)$ converges to B_1, it follows that for some fixed large k, we will have $h_k B_2 \cap h_1 B_2 \neq \varnothing$, and also $h_k B_2 \neq h_1 B_2$. This gives our contradiction, so we conclude G is discrete.

(iii) Assume $g \in G$ is not conjugate into G_1 nor G_2. Take g to have minimal length in its conjugacy class. If g is an (i, i)-form (that is, $|g|$ is odd) then we can conjugate by an element of G_i to reduce its length, hence g has even length. Without loss of generality suppose g is a $(2, 1)$-form. Since $g^n B_1 \subset \mathrm{Int}(B_1)$ is a proper inclusion for every n, we see that g has infinite order, hence is parabolic or loxodromic since G is discrete. At least one fixed point of g is an attracting point z_+ for the convergence sequence (g^n) (see Proposition 2.5). Since B_1 has nonempty interior, there is some $w \in B_1$ so that $g^n w \to z_+$. But for every $n \geq 1$, the set $g^n B_1$ is a subset of the fixed compact $gB_1 \subset \mathrm{Int}(B_1)$, so we must have $z_+ \in \mathrm{Int}(B_1)$. An identical argument applied to g^{-1} (a $(1, 2)$-form) gives a fixed point for g in $\mathrm{Int}(B_2)$, hence g is loxodromic by Proposition 2.5. ☐

4.1. **Limit sets of amalgamated free products.** The rest of the section is devoted to the proof of part (iv) of Theorem A, so for the rest of the section, we fix groups G_1, G_2, J, G and sets $B_1, B_2 \subset M$ satisfying the conditions of Definition A. We will prove each direction of the theorem separately, but we start by making some general observations about the positioning of the limit sets of subgroups of G.

Proposition 4.3. *Each of the following holds.*

(i) $\Lambda(J) \subset \partial B_1 \cap \partial B_2$. *In particular, if J is infinite, then $\partial B_1 \cap \partial B_2$ is nonempty.*
(ii) For $i \in \{1, 2\}$, $\Lambda(G_i) \subset B_{3-i}$.
(iii) For $i \in \{1, 2\}$, and any $g \in G \setminus G_i$, we have $g(\Lambda(G_i)) \cap \Lambda(G_i) = \varnothing$.

Proof. (i) Note that since J preserves the closed set B_1 and $\mathrm{Int}(B_1)$ is an infinite set by Proposition 3.4, we have $\Lambda(J) \subset B_1$. Similarly, $\Lambda(J) \subset B_2$, hence $\Lambda(J) \subset B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ since these sets have disjoint interiors.

(ii) This is an immediate consequence of condition (3) in Definition A along with (i) above.

(iii) For concreteness, take $i = 1$, and let $g \in G \setminus G_1$. In particular $g \notin J$, so g has a normal form with positive length. We can always find some $h, h' \in G_1$ so that $g' = hgh'$ is a $(1, 2)$-form. Then, applying (ii), we know that $g'\Lambda(G_1) \subset \mathrm{Int}(B_1)$ and so

$$g'\Lambda(G_1) \cap \Lambda(G_1) = \varnothing.$$

Now, since $\Lambda(G_1)$ is invariant under G_1, we see that $g'\Lambda(G_1) = hg\Lambda(G_1)$, and therefore

$$hg\Lambda(G_1) \cap \Lambda(G_1) = \varnothing.$$

But then $h^{-1}(hg\Lambda(G_1) \cap \Lambda(G_1)) = g\Lambda(G_1) \cap h^{-1}\Lambda(G_1) = g\Lambda(G_1) \cap \Lambda(G_1)$ is empty as well. ☐

4.2. **AFP ping-pong and contraction.** Both directions of the proof of Theorem A rely crucially on a key contraction property of the ping-pong action of G on the sets B_1 and B_2, stated as Lemma 4.6 below. This contraction lemma gives a sufficient condition for a sequence of sets $(g_k B_i)$ to converge to a singleton in M.

The proof of the contraction lemma relies on an application of Proposition 2.18 to the subgroups we are currently considering. Recall that this proposition gives us control over the topological behavior of the action of fully quasi-convex subgroups on certain subsets of M. So, in order to apply the proposition, we first need to check:

Lemma 4.4. *Let H be one of G, G_1, or G_2. If H is geometrically finite, then J is a fully quasi-convex subgroup of H.*

Proof. We know J is relatively quasi-convex since it is a geometrically finite subgroup of M, so we just need to prove that for all but finitely many $h \in H \setminus J$ we have $h\Lambda(J) \cap \Lambda(J) = \varnothing$. In fact, we will see that this is true for *all* $h \in H \setminus J$.

First, if $H = G_i$ for $i = 1$ or 2, by assumption we know that for any $h \in H \setminus J$ we have $h\Lambda(J) \subset hB_i \subset \mathrm{Int}(B_{3-i})$, hence $h\Lambda(J) \cap \Lambda(J) = \varnothing$ by part (i) of Proposition 4.3. If $H = G$, then any $h \in H \setminus J$ is an (i, j)-form, so that $h\Lambda(J) \subset hB_j \subset \mathrm{Int}(B_{3-i})$ and again $h\Lambda(J) \cap \Lambda(J) = \varnothing$. ☐

Now, we can specialize Proposition 2.18 to the current setting.

Lemma 4.5. *Suppose that either G is geometrically finite, or both G_1 and G_2 are geometrically finite. For $i \in \{1, 2\}$, there exists a compact $K_i \subset \mathrm{Int}(B_{3-i})$ so that for any $g \in G_i \setminus J$, there is $j \in J$ so that $jgB_i \subset K_i$.*

Proof. This follows directly from Proposition 2.18, taking the ambient geometrically finite group G to be either G or G_i for $i \in \{1, 2\}$, H to be G_i, $J_1 = J_2 = J$, U_1 to be $\mathrm{Int}(B_{3-i})$, and U_2 to be $M \setminus B_i$. By assumption we know that $\Lambda(G_i) \setminus \Lambda(J) \subset \mathrm{Int}(B_{3-i}) \subset M \setminus B_i$, so in fact $\Lambda(G_i) \setminus \Lambda(J) \subset U_1 \cap U_2$ and the hypotheses of the proposition are satisfied. ☐

Finally, we can establish the contraction property for sequences in G.

Lemma 4.6 (Contraction for amalgamated free products)**.** *Suppose that either G is geometrically finite, or both G_1 and G_2 are geometrically finite. If (h_k) is a sequence of (i, j)-forms (for fixed i and j) lying in distinct left cosets of J, then, up to subsequence, $(h_k B_j)$ converges to a singleton $\{x\}$.*

It is not hard to verify directly that the subgroup $\{g \in G : gB_j = B_j\}$ is exactly J. So, asking for the sequence of cosets $(h_k J)$ to be pairwise distinct is equivalent to asking for the sequence of translates $(h_k B_j)$ to be pairwise distinct.

Proof. We first prove the following:

Claim. *There exists a compact subset $K \subset \mathrm{Int}(B_{3-j})$ and a sequence (j_k) in J such that $j_k h_k^{-1} B_i \subset K$ for all k.*

To prove the claim, first observe that if $|h_k| = 1$ for every k, then $i = j$ and $h_k \in G_i \setminus J$ for all k. Then the claim follows directly from Lemma 4.5. Otherwise, suppose that $|h_k| > 1$, and write a normal form for h_k:

$$h_k = g_{k,1} \cdots g_{k,n}.$$

Although n can depend on k, we ignore this in the notation. The word $h_k^{-1} = g_{k,n}^{-1} \cdots g_{k,1}^{-1}$ is a (j, i)-form, and the word

$$g_{k,n} h_k^{-1} = g_{k,n-1}^{-1} \cdots g_{k,1}^{-1}$$

is a $(3-j, i)$-form. This means that $g_{k,n} h_k^{-1} B_i \subset B_j$. Then, we can apply Lemma 4.5 again to find a fixed compact $K \subset \mathrm{Int}(B_{3-j})$ and $j_k \in J$ so that $j_k g_{k,n}^{-1} B_j \subset K$ for every k, and therefore

$$j_k h_k^{-1} B_i = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_i \subset j_k g_{k,n}^{-1} B_j \subset K.$$

This proves the claim, so now we consider the sequence $(h_k j_k^{-1})$. Since the left cosets $h_k J$ are all distinct, it follows that the sequence of group elements $(h_k j_k^{-1})$ is divergent in G, and therefore we can extract a convergence subsequence: we can find attracting and repelling points $z_+, z_- \in M$ so that $(h_k j_k^{-1} y)$ converges to z_+ whenever $y \neq z_-$. Equivalently, $(j_k h_k^{-1} y)$ converges to z_- whenever $y \neq z_+$.

By Proposition 3.4, the set B_j is infinite, so there is at least one point $y \in B_j \setminus \{z_+\}$. Since $j_k h_k^{-1} B_j \subset K$, we must have $z_- \in K$. In particular, z_- must lie in $\mathrm{Int}(B_{3-j})$, which means that B_j is a compact subset of $M \setminus \{z_-\}$. Thus, $(h_k j_k^{-1} B_j) = (h_k B_j)$ converges to the singleton $\{z_+\}$ as desired. □

4.3. **Geometrical finiteness of the product.** We now turn to the proof of the implication (G_1 and G_2 geometrically finite) $\implies$ (G geometrically finite), which is one of the directions of Theorem A (iv).

The proof of this direction of the theorem relies on the fact that limit points of G fall into one of two classes: either they are G-translates of limit points of G_1 or G_2, or else they are limit points of sequences of (i,j)-forms in G whose length tends to infinity. The essential step in the proof is to show that any limit point x of the latter form can be "coded" by a sequence of nested translates of B_1 or B_2.

Precisely, we prove the following:

Proposition 4.7 (AFP coding for G-limit points). *Suppose that G_1 and G_2 are geometrically finite, and let x be a point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$. Then there exists a sequence (g_k) in $(G_1 \cup G_2) \setminus J$ so that for every k,*

$$h_k = g_1 \cdots g_k$$

has length k, and if $g_k \in G_j$, then $x \in h_k B_j$.

To prove this proposition, we follow Maskit's strategy, and consider a sequence of "ping-pong" sets in M, defined inductively as follows. We let $T_0 = B_1 \cup B_2$. Then, for every $n > 0$, and $i \in \{1, 2\}$, we define

$$T_{n,i} = \bigcup_{g \in G_i \setminus J} g(B_i \cap T_{n-1}).$$

Then we define

$$T_n = T_{n,1} \cup T_{n,2}.$$

The set T_1 is just the union of the $G_1 \setminus J$ translates of B_1 and the $G_2 \setminus J$ translates of B_2. More generally, T_n is the union of translates of B_1 by $(i,1)$-forms of length n and the translates of B_2 by $(i,2)$-forms of length n. See Figure 4.2 for a depiction of T_1 and T_2. We see that these sets are decreasing, so let

$$T = \bigcap_{n=0}^{\infty} T_n.$$

We will see that limit points of G which are not translates of limit points of G_1 nor G_2 are in T, which allows us to construct the sequence given by the conclusion of the proposition above.

We observe:

Lemma 4.8. *The set T is G-invariant and nonempty. In particular, since G is non-elementary, we have $\Lambda(G) \subset \overline{T} \subset B_1 \cup B_2$.*

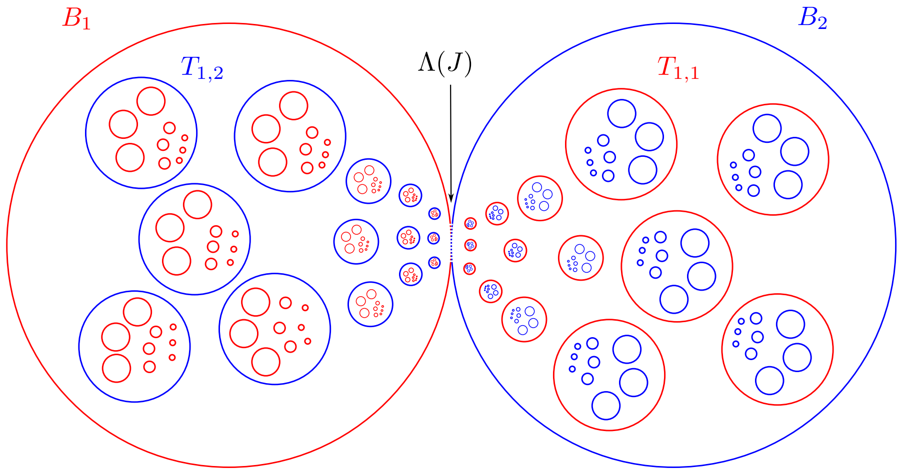

FIGURE 4.2. Part of the sets T_1 and T_2.

Proof. We know T is nonempty because it is the intersection of a decreasing sequence of nonempty subsets of the compact space M. The definition of T_n implies that if $x \in T_n$ and $g \in G_1 \cup G_2$, then $gx \in T_{n-1}$. Inductively, we see that if $g \in G$ has $|g| = k$, and $x \in T_{n+k}$, then $gx \in T_n$. It follows that if $x \in T$ then $gx \in T$ for any $g \in G$. ☐

Lemma 4.9. *If G_1 and G_2 are geometrically finite, we have $\Lambda(G) \setminus (\Lambda(G_1) \cup \Lambda(G_2)) \subset T_1$.*

Proof. We will prove that if $y \in \Lambda(G) \setminus T_1$, then $y \in \Lambda(G_1) \cup \Lambda(G_2)$, so suppose that $y \in \Lambda(G)$ does not lie in T_1. Using the above lemma, we know $y \in B_1 \cup B_2$, so without loss of generality assume $y \in B_2$. Since y is in the limit set of G, we can find a sequence (g_k) in G so that $(g_k w)$ converges to y for all but a single point in M. If $y \in \Lambda(J)$ we are done, so we can assume that $g_k \notin J$ for infinitely many k.

Then, after extracting a subsequence, we can assume that for every k, g_k is an (i, j)-form for i, j fixed, and then find $w \in B_j$ so that $g_k w \to y$.

Since g_k is an (i, j)-form and $w \in B_j$, we have $g_k w \in G_i(B_i)$ for every k. So, we may write $g_k w = g'_k z_k$ for $g'_k \in G_i \setminus J$ and $z_k \in B_i$. Note that g'_k is just the first letter in the (i, j)-form g_k. In particular, we know $g'_k z_k \in T_1$ for every k, so $g'_k z_k$ is never equal to y. If, up to subsequence, there are only finitely many distinct translates $g'_k B_i$, then we would have $g'_k z_k \in \bigcup g'_k B_i$, a compact set in the complement of T_1, which contradicts the fact that $g'_k z_k \to y \in T_1$. Hence we may assume that the translates $g'_k B_i$ are all distinct, which means that the left cosets $g'_k J$ are all distinct.

Now, Lemma 4.6 implies that $(g'_k B_i)$ converges to a singleton. This singleton must be y since $g'_k z_k \to y$. It follows that $g'_k z \to y$ for any $z \in B_i$, and since B_i is an infinite set it follows that $y \in \Lambda(G_i)$ as desired. ☐

Proof of Proposition 4.7. We first claim that $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a subset of T. So, fix $z \in \Lambda(G)$, and suppose $z \notin T$. We will show $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$.

By Lemma 4.8 we know that $\Lambda(G) \subset B_1 \cup B_2 = T_0$, so there is some $n > 0$ such that $z \in T_{n-1} \setminus T_n$. In particular, because $z \in T_{n-1}$, there is an (i,j)-form $g \in G$, with $|g| = n-1$, such that $gy = z$ for $y \in B_j$. We must have $y \notin T_1$, since otherwise we would have $y = hw$ for $w \in B_{3-j}$ and $h \in G_{3-j} \setminus J$, and then $z = ghw$ would lie in T_n. Then, since $\Lambda(G)$ is G-invariant we see that $y \in \Lambda(G)$ but $y \notin T_1$, so by the previous lemma we have $y \in \Lambda(G_1) \cup \Lambda(G_2)$, hence $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$.

We have now seen that $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a subset of T, so we just need to show that for any $x \in T$, there is a sequence of (i,j)-forms (h_k) satisfying the conclusions of the proposition. We construct this sequence inductively. Take h_0 to be the identity. For $k > 0$, assume that $x \in h_{k-1}B_j$ for an (i,j)-form

$$h_{k-1} = g_1 \cdots g_{k-1}.$$

By Lemma 4.8, T is G-invariant, so $h_{k-1}^{-1}x \in B_j \cap T$. In particular, $h_{k-1}^{-1}x$ lies in $T_1 \cap B_j = T_{1,j}$, so there is some $g_k \in G_{3-j} \setminus J$ so that $h_{k-1}^{-1}x \in g_k B_{3-j}$. Then if h_k is the $(i, 3-j)$-form

$$g_1 \cdots g_k,$$

we have $x \in h_k(B_{3-j})$ and $|h_k| = |h_{k-1}| + 1$, as required. □

The next step is to use the "coding" of limit points given by Proposition 4.7 to prove that there is a conical limit sequence for every point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$.

Lemma 4.10. *If G_1 and G_2 are geometrically finite, every point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a conical limit point for G.*

Proof. Let $x \in \Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$. We know $x \in B_1 \cup B_2$ from Lemma 4.8, so to simplify notation assume $x \in B_2$. We let (g_k) be the sequence in $(G_1 \cup G_2) \setminus J$ from Proposition 4.7, so that, for every k, we have $|g_1 \cdots g_k| = k$ and if $g_k \in G_j$, then $x \in g_1 \cdots g_k B_j$.

For each k, we let $h_k = g_1 \cdots g_{2k}$, so that h_k is an (i,j)-form for fixed $i \neq j$. Since $h_k B_j \subset \mathrm{Int}(B_{3-i})$, and $x \in B_2$, we have $i = 1$ and thus h_k is a $(1,2)$-form for every k. This means that (g_{2k}) is a sequence in $G_2 \setminus J$. So, using Lemma 4.5, we find a fixed compact subset $K \subset \mathrm{Int}(B_1)$ and a sequence (j_k) in J so that $j_k g_{2k}^{-1} B_2 \subset K$.

Consider the sequence (f_k) given by $f_k = h_k j_k^{-1}$. Since $|f_k| \to \infty$, a subsequence of (f_k^{-1}) consists of pairwise distinct elements of G. Since f_k^{-1} is a $(2,1)$-form, we know that

$$f_k^{-1} B_1 = j_k g_{2k}^{-1} \cdots g_1^{-1} B_1 \subset K.$$

On the other hand, by construction, we know that

$$h_k^{-1} x = g_{2k}^{-1} \cdots g_1^{-1} x \in B_2.$$

Since B_2 is J-invariant, we also see that $f_k^{-1} x = j_k h_k^{-1} x \in B_2$ for every k. By Proposition 3.4, $\mathrm{Int}(B_1)$ is an infinite set. Then, since B_2 and K are disjoint compact subsets of M, we can apply Lemma 2.10 (with $Y = \mathrm{Int}(B_1)$, $K_1 = K$, and $K_2 = B_2$) to complete the proof. □

Next we deal with parabolic points.

Lemma 4.11. *If both G_1 and G_2 are geometrically finite, then every parabolic point of G in $\Lambda(G_1) \cup \Lambda(G_2)$ is a bounded parabolic point for the action of G on $\Lambda(G)$.*

Proof. Fix a parabolic point $p \in \Lambda(G_1)$, and let $P < G$ be the parabolic subgroup stabilizing p. We will show that there is a compact set $K \subset \Lambda(G) \setminus \{p\}$ so that $P(K) = \Lambda(G) \setminus \{p\}$, which implies the action is cocompact. The main idea here is to apply Proposition 2.18 to the parabolic subgroup P, which gives us a way to use elements of P to position certain points in M far away from $\Lambda(P) = \{p\}$. Our strategy is to decompose the set $\Lambda(G) \setminus \{p\}$ into pieces. We will show that every point in $\Lambda(G) \setminus \{p\}$ is either far away from p to begin with, or else it is in a piece of $\Lambda(G) \setminus \{p\}$ which can be translated far away from p using either Proposition 2.18 or the boundedness of p in $\Lambda(G_1)$.

We consider two cases. For both cases, in order to apply Proposition 2.18, we need to know that J and P are fully quasi-convex subgroups of G_1; for J this follows from Lemma 4.4, and for P this is true because P is exactly the stabilizer of its limit set $\{p\} \subset \Lambda(G_1)$ in G_1.

Case 1: $p \in \Lambda(G_1) \setminus G_1(\Lambda(J))$. Using Lemma 4.9, we can see that every point in $\Lambda(G) \setminus \{p\}$ lies in one of the sets $\Lambda(G_1), \Lambda(G_2)$, or T_1. Since $\Lambda(G_2) \subset B_1$, and $T_1 \subset B_1 \cup B_2$, this means that every point in $\Lambda(G)$ lies in one of the sets

$$L_1 = \Lambda(G_1), \quad L_2 = B_1, \quad L_3 = T_1 \cap B_2.$$

Now, for each i, we will find a compact set $K_i \subset M \setminus \{p\}$ so that $P(K_i)$ contains $(\Lambda(G) \setminus \{p\}) \cap L_i$. Then we can define $K = (K_1 \cup K_2 \cup K_3) \cap \Lambda(G)$, so that $P(K) = \Lambda(G) \setminus \{p\}$.

Since p is a bounded parabolic point for the action of G_1 on $\Lambda(G_1)$, and $\Lambda(G_1)$ is locally compact, we already know that there is a compact $K_1 \subset \Lambda(G_1) \setminus \{p\}$ so that $P(K_1) = \Lambda(G_1) - \{p\}$. And, by part 3 of Definition A, we know $p \in \mathrm{Int}(B_2)$, so B_1 is already a compact subset of $M \setminus \{p\}$ and we can take $K_2 = B_1$. So, we just need to construct the compact set K_3.

For this, we apply Proposition 2.18, with $G = H = G_1$, $J_1 = P$, $J_2 = J$, $U_1 = M \setminus \{p\}$, and $U_2 = M \setminus B_1$. To verify that the hypotheses of the proposition are satisfied, we need to check that $gB_1 \subset M \setminus \{p\}$ for every $g \in G_1 \setminus J$. But, since $\Lambda(G_1)$ is G_1-invariant we can only have $p \in gB_1$ if $g^{-1}p \in B_1 \cap \Lambda(G_1) = \Lambda(J)$, which is impossible since we assume $p \in \Lambda(G_1) \setminus G_1(\Lambda(J))$.

So, we know there is a compact subset $K' \subset M \setminus \{p\}$ so that for any $g \in G_1 \setminus J$, we can find $h \in P$ so that $hgB_1 \subset K'$. But by definition, any $y \in T_1 \cap B_2$ lies in $(G_1 \setminus J)(B_1)$, so we can take $K_3 = K'$ and we are done.

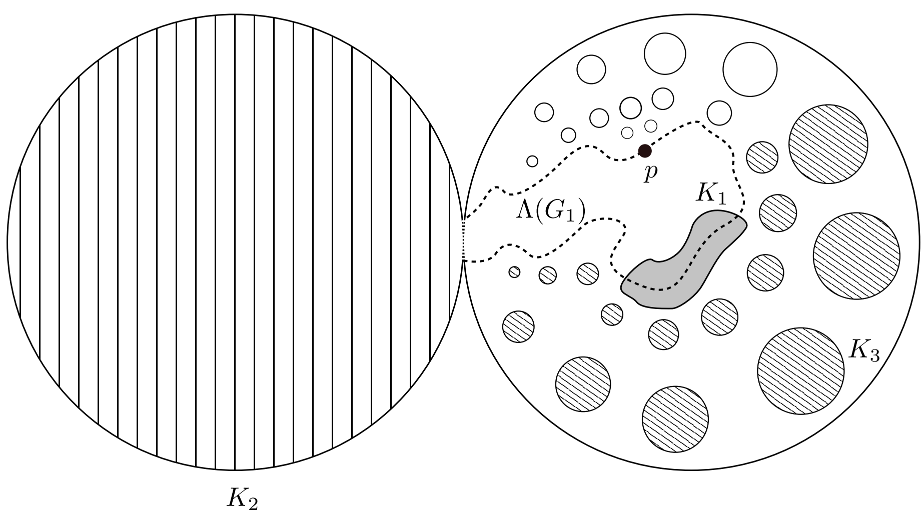

FIGURE 4.3. The sets K_1, K_2, and K_3 proving that $p \in \Lambda(G_1)$ is a bounded parabolic point (Case 1).

Case 2: $p \in G_1(\Lambda(J))$. Since G acts by homeomorphisms on $\Lambda(G)$ it suffices to consider the case $p \in \Lambda(J)$. For this case, we again use Lemma 4.9 to see that every point in $\Lambda(G)$ lies in one of the three sets

$$L_1 = \Lambda(G_1), \quad L_2 = \Lambda(G_2), \quad L_3 = T_1.$$

As in the previous case, for each of these sets, we will find a compact set $K_i \subset M \setminus \{p\}$ so that $P(K_i)$ contains $(\Lambda(G) \setminus \{p\}) \cap L_i$.

For $i = 1, 2$, as in Case 1, we can use the fact that p is a bounded parabolic point for the G_i-action on $\Lambda(G_i)$, to find compact sets $K_i \subset \Lambda(G_i) \setminus \{p\}$ such that $P(K_i) = \Lambda(G_i) \setminus \{p\}$.

To find K_3, we apply Proposition 2.18 twice: for $i = 1, 2$, we take $G = H = G_i$, $J_1 = P$, $J_2 = J$, $U_1 = M \setminus \{p\}$, and $U_2 = M \setminus B_i$. As in the previous case we need to verify that $gB_i \subset M \setminus \{p\}$ for every $g \in G_i \setminus J$, but this follows because $gB_i \subset \mathrm{Int}(B_{3-i})$, which is disjoint from $\Lambda(J)$ and hence does not contain p.

This gives us a pair of compact set $K_{3,1}$ and $K_{3,2}$, such that for any $g \in G_i \setminus J$, we can find $h \in P$ so that $hgB_i \subset K_{3,i}$. Then, since any $y \in T_1$ lies in $(G_1 \setminus J)(B_1) \cup (G_2 \setminus J)(B_2)$ by definition, we can take $K_3 = K_{3,1} \cup K_{3,2}$ and we are done. ☐

Finally, we can complete the proof of this direction of Theorem A part (iv).

Proposition 4.12. *If G_1 and G_2 are geometrically finite, then G is geometrically finite.*

Proof. Let $x \in \Lambda(G)$. We must show x is either a conical limit point or a bounded parabolic point for G. First, if x is not a translate of a limit point of G_1 nor G_2, then x is a conical limit point by Lemma 4.10. So, assume $x \in G(\Lambda(G_1) \cup \Lambda(G_2))$. Acting by elements of G preserves the properties we are trying to show, so in fact we may assume $x \in \Lambda(G_1) \cup \Lambda(G_2)$. If x is a parabolic point of G, we are done by Lemma 4.11. Otherwise, x is necessarily a conical limit point for G_1 or G_2 since these are geometrically finite, and again we are done since x will also be a conical limit point for G. ☐

4.4. **Geometrical finiteness of the factors.** The last thing to do in this section is prove the other direction of Theorem A part (iv), and show that G_1 and G_2 are geometrically finite if G is geometrically finite. The first step is the following lemma, which makes use of the contraction property proved earlier in this section.

Lemma 4.13. *Assume that G is geometrically finite. Let $x \in \Lambda(G_i)$ for $i \in \{1, 2\}$, and suppose that (h_k) is a conical limit sequence in G for x. Then, after extracting a subsequence, we can find some $h \in G$ so that $h_k \in hG_i$ for every k.*

Proof. Without loss of generality take $x \in \Lambda(G_1)$. Let (h_k) be a conical limit sequence for x. This means that there are distinct points $a, b \in M$ such that $h_k x \to a$ and $h_k z \to b$ for any $z \in M \setminus \{x\}$.

If there is some $h \in G$ so that $h_k \in hJ$ for infinitely many k, then we are done. So we may assume that, after taking a subsequence, each h_k is an (i, j)-form for i, j fixed, and each h_k represents a different left J-coset in G. There are two cases to consider: either every h_k is an $(i, 1)$-form or every h_k is an $(i, 2)$-form.

First suppose that h_k is an $(i, 2)$-form. By Lemma 4.6, after extraction the sets $(h_k B_2)$ converge to a singleton. Since $x \in \Lambda(G_1) \subset B_2$, and $h_k x \to a$, we must have $h_k B_2 \to \{a\}$. Since B_2 is an infinite set by Proposition 3.4, there is some point $z \in B_2 \setminus \{x\}$, which must satisfy $h_k z \to a$. But this is impossible if (h_k) is a conical limit sequence for x.

We conclude that each h_k must be $(i, 1)$-form. If $h_k \in G_1$ for infinitely many k then we are done, so assume that this is not the case. Then after taking a subsequence we have

$|h_k| > 1$ for every k. We write h_k as an (i,j)-form of length $n \geq 2$:

$$h_k = g_{k,1} \cdots g_{k,n}.$$

Note that although n can depend on k, we omit this from the notation. Since h_k is an $(i,1)$-form, we have $g_{k,n} \in G_1$, and since $\Lambda(G_1)$ is G_1-invariant, $g_{k,n}x$ lies in $\Lambda(G_1) \subset B_2$. Then

$$(h_k g_{k,n}^{-1}) = (g_{k,1} \cdots g_{k,n-1})$$

is a sequence of $(i,2)$-forms. If the elements in this sequence lie in infinitely many different left J-cosets in G, then we extract a subsequence and apply Lemma 4.6 to see that $(h_k g_{k,n}^{-1} B_2)$ again converges to a singleton. This singleton contains the limit of $(h_k g_{k,n}^{-1} g_{k,n} x) = (h_k x)$, so it is again equal to $\{a\}$. But then for any $z \in B_1 \setminus \{x\}$, we have $g_{k,n} z \in B_2$ and thus $h_k z = h_k g_{k,n}^{-1} g_{k,n} z \to a$, again giving a contradiction. We conclude that a subsequence of $(h_k g_{k,n}^{-1})$ lies in a single coset hJ for $h \in G$, hence $h_k \in hJg_{k,n} \subset hG_1$. □

Proposition 4.14. *If G is geometrically finite, then G_1 and G_2 are geometrically finite.*

Proof. Let $x \in \Lambda(G_1)$. We will show that x is either a conical limit point or a bounded parabolic point for G_1. Since G is geometrically finite, we know x is either a conical limit point for G or a bounded parabolic point for G.

If x is a conical limit point for G, then it has a conical limit sequence (h_k) in G, i.e. a sequence such that $(h_k x, h_k z)$ lies in a compact subset of $(M \times M) \setminus \Delta$ for any $z \neq x$ in M. By Lemma 4.13, we know that, up to subsequence, $h_k = hg_k$ for $g_k \in G_1$ and h fixed. Then (g_k) is a conical limit sequence for x in G_1 and we are done.

Otherwise, suppose x is a bounded parabolic point for G. Let P be the stabilizer of x in G. As we observed in the proof of Lemma 4.11, part (iii) of Proposition 4.3 implies that P is a subgroup of G_1.

Since x is a bounded parabolic point for G, again applying local compactness of $\Lambda(G) \setminus \{x\}$, there is a compact $K \subset \Lambda(G) \setminus \{x\}$ so that $P(K) = \Lambda(G) \setminus \{x\}$. Let $K_1 = K \cap \Lambda(G_1)$. Since $\Lambda(G_1)$ is closed, K_1 is compact, and since $\Lambda(G_1)$ is G_1-invariant (hence P-invariant), we have

$$P(K_1) = P(K \cap \Lambda(G_1)) = P(K) \cap \Lambda(G_1) = \Lambda(G_1) \setminus \{x\}.$$

Thus x is bounded parabolic for G_1 and we are done. □

5. Combinatorial group theory: HNN extensions

In this section we establish notation and give some basic facts about *HNN extensions*, in preparation for our second combination theorem. Our main reference is again [Mas88]. In this section, M is again an arbitrary compact metrizable space, but as in Section 3, these results are purely set-theoretic. We further assume throughout this section that G_0, G_1 are subgroups of $\mathrm{Homeo}(M)$, where $G_1 = \langle f \rangle$ is infinite cyclic, and J_1, J_{-1} are subgroups of G_0 with $fJ_{-1}f^{-1} = J_1$. We let G denote $\langle G_0, G_1 \rangle$, the subgroup of $\mathrm{Homeo}(M)$ generated by G_0 and G_1. Note that conjugation by f induces an abstract isomorphism $J_{-1} \to J_1$, which we denote f_*. The indices are chosen to make notation more convenient later.

As was the case for amalgamated free products, we can define HNN extensions using equivalence classes of *normal forms*.

Definition 5.1. A word $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ in f and elements g_k of G_0 is a *normal form* if:

(1) Each $g_k \in G_0$ is nontrivial for $k < n$;

(2) Each α_k is an integer, with $\alpha_k \neq 0$ whenever $k > 1$;
(3) If $\alpha_k < 0$ and $g_{k-1} \in J_{-1} \setminus \{1\}$, then $\alpha_{k-1} < 0$;
(4) If $\alpha_k > 0$ and $g_{k-1} \in J_1 \setminus \{1\}$, then $\alpha_{k-1} > 0$.

Two words $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ and $h = f^{\beta_1} h_1 \cdots f^{\beta_n} h_n$ are *equivalent* if we can obtain g from h by inserting finitely many conjugates and inverses of words of the form $fjf^{-1}(f_*(j))^{-1}$ for $j \in J_{-1}$ (words of this form are the identity in G). Every word of the form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is equivalent either to a normal form or to the identity, which means that every word in f and elements of G_0 is equivalent to either a normal form or the identity.

The *length* of a normal form $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is defined to be $|g| = \sum_{i=1}^n |\alpha_i|$. Note that, in contrast to normal forms for amalgamated free products, the length of a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is *not* necessarily n. Length-0 normal forms correspond by definition to elements of G_0.

If a normal form g has positive length, $i \in \{0, \pm 1\}$ and $j \in \{\pm 1\}$, then we say $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is an (i,j)-form if α_1 is positive (resp. negative, zero) and $i = 1$ (resp. -1, 0), and α_n is positive (resp. negative) and $j = 1$ (resp. -1). Our notation differs slightly from Maskit's, which will make some of our later arguments less cumbersome.

We set

$$G_0 *_f = \{\mathrm{id}\} \cup \{\text{equivalence classes of normal forms}\}.$$

This set forms a group, with operation given by concatenation followed by reduction to a normal form. It is called the *HNN extension of* G_0 *by* f. Note that it is *not* in general true that the formal inverse of a normal form g is also a normal form (see Lemma 5.6 below), but it is a formal product of normal forms, which tells us that $G_0 *_f$ contains inverses.

We again have a natural surjective homomorphism

$$\begin{aligned} \varphi : G_0 *_f &\to G \\ f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n &\mapsto f^{\alpha_1} \circ g_1 \circ \cdots \circ f^{\alpha_n} \circ g_n. \end{aligned}$$

The map φ may or may not be an isomorphism. As was the case for amalagamated free products, if φ is an isomorphism, we will abuse notation and say that $G = G_0 *_f$. In this situation, we implicitly identify elements of G with equivalence classes of normal forms in $G_0 *_f$.

As in Section 3, we want a "ping-pong" condition ensuring that φ actually is an isomorphism.

Definition 5.2. Let $U_1, U_{-1} \subset M$ be nonempty disjoint sets, with $A = M \setminus (U_1 \cup U_{-1})$ nonempty. We call (A, U_1, U_{-1}) an *interactive triple* for G_0 and G_1 if the following hold:

(1) The pair (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}) in G_0.
(2) For $i \in \{\pm 1\}$, and for every $g \in G_0$, $gU_i \subset A \cup U_i$.
(3) We have $f(A \cup U_1) \subset U_1$ and $f^{-1}(A \cup U_{-1}) \subset U_{-1}$.

We say an interactive triple is *proper* if the set $A \setminus (G_0(U_1 \cup U_{-1}))$ is nonempty.

Note that these conditions imply that in particular $gU_i \subset A$ for $g \in G_0 \setminus J_i$. Similarly to Section 3, we can observe:

Proposition 5.3. *If (A, U_1, U_{-1}) is a proper interactive triple for G_1 and G_2, then A, U, and U_{-1} are all infinite sets.*

Proof. Since J_1 is a proper subgroup G_0, there is some element $g \in G_0 \setminus J_1$, and by precise invariance we have $gU_1 \subset A$. By properness of the triple, the inclusion is proper, which means that fgU_1 is a proper subset of U_1. We conclude that U_1 is infinite. Since $gU_1 \subset A$ and $f^{-1}gU_1 \subset U_{-1}$ the other two sets are infinite as well. □

We have a description of the way normal forms in G_0*_f act on certain sets in the interactive triple, in analogy to the way normal forms in an amalgamated free product act on sets in an interactive pair.

Lemma 5.4 ([Mas88] VII.D.11)**.** *Suppose there is an interactive triple* (A, U_1, U_{-1}) *for* G_0 *and* G_1*, and set* $A_0 = A \setminus G_0(U_1 \cup U_2)$*. Let* $g = f^{\alpha_1}g_1 \cdots f^{\alpha_n}g_n \in G_0*_f$ *be a normal form with* $|g| > 0$*. Then the following hold.*

- *i) if* g *is an* (i, j)*-form for* $i, j \in \{\pm 1\}$*, then* $\varphi(g)(A_0 \cup U_j) \subset U_i$*.*
- *ii) If* g *is a* $(0, j)$*-form for* $j \in \{\pm 1\}$*, then there is* $h \in G_0$ *so* $\varphi(g)(A_0 \cup U_j) \subset hU \subset A$*, where* $U = U_{-1}$ *if* $\alpha_2 < 0$ *and* $U = U_1$ *if* $\alpha_2 > 0$*.*

The combinatorics in this case are slightly more complicated than for amalgamated free products, but the basic idea is the same. To illustrate the idea, consider a $(1, 1)$-form of length 2, for example $g = fg_1fg_2$. Then $g_2(A_0 \cup U_1) \subset A \cup U_1$ by definition (in fact A_0 is G_0-invariant by our conditions). Then we have

$$\begin{aligned} g(A_0 \cup U_1) &\subset fg_1f(A \cup U_1) \\ &\subset fg_1(U_1) \\ &\subset f(A \cup U_1) \\ &\subset U_1. \end{aligned}$$

Conditions (3) and (4) in Definition 5.1 ensure that when we iteratively apply a normal form to $A_0 \cup U_i$, we always can say where each set is mapped to next. We have chosen our notation so that if g is an (i, j)-form with $i \neq 0$, then $gU_j \subset U_i$. This is consistent with the convention for amalgamated free products.

The proposition below gives the combinatorial condition we need to ensure that φ is actually an isomorphism.

Proposition 5.5 (Ping-pong for HNN extensions; [Mas88] VII.D.12)**.** *Suppose* (A, U_1, U_{-1}) *is a proper interactive triple for* G_0 *and* G_1*. Then* $G = G_0*_f$*.*

Proof. We just need to show that $\varphi : G_0*_f \to G$ is injective. This map is already injective on G_0, so suppose $g \in G_0*_f$ has $|g| > 0$. Then by Lemma 5.4 we have $\varphi(g)x \neq x$ for any $x \in A_0 = A \setminus G_0(U_1 \cup U_{-1})$, showing that $\varphi(g)$ is not the identity. □

5.1. **More combinatorics of normal forms.** Normal forms in an HNN extension are slightly more complicated than normal forms for an amalgamated free product, so here we collect some results which will later make working with these normal forms a little easier.

5.1.1. *Formal inverses.* In several situations later in the paper, we will want to work with formal inverses of (i, j)-forms. These inverses may not themselves be normal forms, but it is still useful to work with them directly, rather than with an equivalent normal form. To that end, we prove:

Lemma 5.6. *Let* $g = f^{\alpha_1}g_1 \cdots f^{\alpha_n}g_n$ *be an* (i, j)*-form for* $i, j \in \{\pm 1\}$ *(so in particular,* g *has positive length). Then the formal inverse*

$$g^{-1} = g_n^{-1}f^{-\alpha_n} \cdots g_1^{-1}f^{-\alpha_1}$$

is a $(0, -i)$*-form if* $g_n \in G_0 \setminus (J_1 \cup J_{-1})$*. The word* $f^{-\alpha_n} \cdots g_1^{-1}f^{-\alpha_1}$ *is a* $(-j, -i)$*-form, regardless of* g_n*.*

Proof. We set $\beta_0 = 0$ and $\beta_k = -\alpha_{n+1-k}$ for $1 \le k < n$, so that g^{-1} is the word

$$f^{\beta_0} g_n^{-1} f^{\beta_1} g_{n-1}^{-1} \cdots f^{\beta_{n-1}}.$$

We need to verify that this word is a normal form. The only conditions in Definition 5.1 which could possibly fail are the technical requirements (3) and (4).

For (3), we must show that, for $k \ge 0$, if $\beta_{k+1} < 0$ and $g_{n-k}^{-1} \in J_{-1} \setminus \{\mathrm{id}\}$, then $\beta_k < 0$. Equivalently, we need to show that if $\alpha_{n+1-k} > 0$ and $g_{n-k} \in J_{-1}$, then $\alpha_{n-k} < 0$. When $k \ge 1$ this follows from condition (4) on our original normal form g, and when $k = 0$ the condition is vacuous because we assume $g_n \notin J_{-1}$. The argument for condition (4) is nearly identical.

The same reasoning implies that $f^{\beta_1} g_{n-1}^{-1} \cdots f^{\beta_{n-1}}$ is a normal form, with $\beta_1 = -\alpha_n$ and $\beta_{n-1} = -\alpha_1$. ☐

5.1.2. *Ping-pong for normal forms.* When we have an interactive triple (A, U_1, U_{-1}) for an HNN extension G, Lemma 5.4 above gives us a way to locate sets of the form gU_i when g is a normal form in G. However, the statement of the lemma is often a little unwieldy to work with directly, so to simplify some arguments later on, we introduce some additional terminology.

Definition 5.7. Let $G = G_0 *_f$ be the HNN extension of G_0 along $J_1 = f^{-1} J_{-1} f$. We say that a normal form

$$g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$$

is an *HNN ping-pong form of type 1* (or just a *type-1 form*) if either $g_n \in G_0 \setminus J_1$, or $\alpha_n > 0$. Similarly a normal form is an *HNN ping-pong form of type -1* if either $\alpha_n < 0$ or $g_n \in G_0 \setminus J_{-1}$.

Note that if $|g| = 0$, then g has type i if and only if $g \in G_0 \setminus J_i$. An (i, j)-form is always type j, and it may or may not also be type $-j$. If (A, U_1, U_{-1}) is an interactive triple for G_0, $\langle f \rangle$, then a normal form g has type k when the dynamics of the triple allow us to locate the set gU_k. That is, we have the following immediate consequence of Lemma 5.4:

Lemma 5.8. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f \rangle$. If g is an (i, j)-form of type k, and $i \neq 0$, then $gU_k \subset U_i$.*

Frequently we will want to apply inductive arguments to normal forms, which means that we want some control over the ping-pong behavior of a prefix of an (i, j)-form. The lemma below gives one way to do this. Here (and elsewhere), a "prefix" h' of a normal form h is a normal form which appears as an initial subword of h. That is, if h is a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$, then a prefix h' is a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_k} g_k$ for some $1 \le k \le n$.

Lemma 5.9. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f \rangle$, and let g be a type-i normal form of length $m \ge 1$. Then for some $j \in \{-1, 1\}$, there is a length-$(m-1)$ prefix g' of g and $g_0 \in G_0$ so that $g = g' f^j g_0$ and $f^j g_0 U_i \subset U_j$. If $|g'| \ge 1$, then g' is type j.*

Proof. When $m = 1$ we can just take $g' = \mathrm{id}$, so assume $m > 1$. We let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ be a type-i normal form. Without loss of generality assume $\alpha_n > 0$, and consider the normal form

$$g' = g g_n^{-1} f^{-1} = f^{\alpha_1} g_1 \cdots f^{\alpha_{n-1}} g_{n-1} f^{\alpha_n - 1}.$$

This is a normal form with positive length $m - 1$. It is also type 1: if $\alpha_n > 1$ or $g_{n-1} \in G \setminus J_1$, then this follows directly from the definition; if $\alpha_n = 1$ and $g_{n-1} \in J_1$, then, since $f^{\alpha_1} g_1 \cdots f^{\alpha_{n-1}} g_{n-1} f^{\alpha_n} g_n$ is a normal form, we must have $\alpha_{n-1} > 0$, which again means the above form has type 1.

We need to verify that $fg_nU_i \subset U_1$, which will show the lemma holds with $g_0 = g_n$. If $i = 1$, then $fg_nU_i = fg_nU_1 \subset U_1$. On the other hand, if $i = -1$, then since $\alpha_n > 0$ and g has type i, we must have $g_n \in G \setminus J_{-1}$. Then $fg_nU_i = fg_nU_{-1} \subset U_1$. □

We will also sometimes want to characterize elements in G via their action on sets in an interactive triple (A, U_1, U_{-1}). This can again be expressed using the ping-pong type of normal forms for these elements. The lemma below is a precise statement of this form, and generalizes the fact that in G_0, (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}):

Lemma 5.10. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f\rangle$. Let g be a ping-pong form of type i and let h be a ping-pong form of type k.*

Suppose that $|g| = |h|$. Then either $i = k$, $gU_i = hU_i$, and $g = hj$ for $j \in J_i$, or $gU_i \cap hU_k = \varnothing$.

Proof. We proceed by induction on the length m of g and h; the main idea is to use the previous lemma to find prefixes of g and h where we can assume that the statement holds, and then apply precise invariance of (U_1, U_{-1}) under (J_1, J_{-1}) for the inductive step.

First observe that, if g, h are elements of G_0, and if $gU_i \cap hU_k \neq \varnothing$, then the fact that (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}) implies $i = k$ and $g = hj$ for $j \in J_i$. Now, let $m \geq 1$, let g, h be normal forms with $|g| = |h| = m$, and suppose that g has type i, h has type k, and $gU_i \cap hU_k \neq \varnothing$.

By Lemma 5.9 we can find prefixes g', h' of type i', k' respectively, with $|g'| = |h'| = m-1$ and $g = g'f^{i'}g_0$, $h = h'f^{k'}h_0$ for $g_0, h_0 \in G_0$ satisfying $f^{i'}g_0U_i \subset U_{i'}$ and $f^{k'}h_0U_k \subset U_{k'}$. Then we know that both $gU_i \subset g'U_{i'}$ and $hU_k \subset h'U_{k'}$ hold, which means that $h'U_{k'} \cap g'U_{i'} \neq \varnothing$. By induction (or by precise invariance if $n = 1$), we know that $i' = k'$ and $h' = g'j'$ for $j' \in J_{i'}$. Without loss of generality take $i' = k' = 1$, so $j' \in J_1$.

Then since $gU_i = g'fg_0U_i$ has nonempty intersection with $hU_k = h'fh_0U_k = g'j'fh_0U_k$, the intersection $fg_0U_i \cap j'fh_0U_k$ is also nonempty. Since f conjugates J_1 to J_{-1}, for some $j'' \in J_{-1}$ we have $j'fh_0 = fj''h_0$. Then $fg_0U_i \cap fj''h_0U_k$ is nonempty as well, hence $g_0U_i \cap j''h_0U_k \neq \varnothing$. Then by precise invariance we know $i = k$ and $g_0 = j''h_0j$ for $j \in J_i$.

Finally, we see that

$$g = g'fg_0 = g'fj''h_0j = g'j'fh_0j = h'fh_0j = hj,$$

and we are done. □

6. Theorem B

In this section we prove Theorem B. The proof is very similar in spirit and structure to the proof of Theorem A, but the details are different. Where possible, we have tried to imitate the structure of Section 4, and have indicated the analogies between the proofs.

We start (as in Section 4) by setting up the general ping-pong framework.

Definition B (HNN Ping-Pong Position)**.** Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f\rangle$ be an infinite discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in $\mathrm{Homeo}(M)$. We will say G_0 is in *HNN ping-pong position* (with respect to f, J_1 and J_{-1}) if there exists closed sets $B_1, B_{-1} \subset M$ with nonempty disjoint interiors satisfying the following:

1. (B_1, B_{-1}) is precisely invariant under (J_1, J_{-1}) in G_0 (recall Definition 3.1).
2. If $A = M \setminus (B_1 \cup B_{-1})$, then $f(A \cup B_1) = \mathrm{Int}(B_1)$.
3. For $i \in \{\pm 1\}$, we have $\Lambda(G_0) \cap B_i = \Lambda(J_i)$.

(4) The set $A_0 = M \setminus G_0(B_1 \cup B_{-1})$ is nonempty.

Remark 6.1. Note that our precise invariance assumption forces $B_1 \cap B_{-1} = \varnothing$.

We restate Theorem B here for reference.

Theorem B. *Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f \rangle$ be an infinite cyclic discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in* $\mathrm{Homeo}(M)$*. Suppose G_0 is in HNN ping-pong position with respect to f, J_1 and J_{-1}. Let $G = \langle G_0, G_1 \rangle <$* $\mathrm{Homeo}(M)$*, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_0 *_f$.
(ii) G is discrete.
(iii) Elements of G not conjugate into G_0 are loxodromic.
(iv) G is geometrically finite if and only if G_0 is geometrically finite.

Remark 6.2. As was the case for amalgamated free products, when $M = \partial \mathbb{H}^3_{\mathbf{R}}$, this theorem is not strong enough to recover Maskit's full result, since we ask for stronger hypotheses on our ping-pong configuration. Specifically, we do not allow B_1 and B_{-1} to intersect, and consequently f cannot be parabolic. This condition ensures that our subgroups are fully quasi-convex, and allows us to apply Proposition 2.18.

Before proving the first three parts of Theorem B, we give the following slightly stronger version of Lemma 5.8, which will be useful throughout this section.

Lemma 6.3. *Suppose that g is an (i,j)-form of type k. If $i \neq 0$, then $gB_k \subsetneq \mathrm{Int}(B_i)$, and if $i = 0$, then $gB_k \subsetneq A$.*

Proof. First suppose that $i \neq 0$. For concreteness, assume g is a $(1,j)$-form. We first suppose that $|g| = 1$, so that $g = fg_1$ for $g_1 \in G_0$. If $k = 1$, then g_1B_1 is a subset of $M \setminus B_{-1} = \mathrm{Int}(A \cup B_1)$ by precise invariance. In fact it is a proper subset by properness of the interactive triple, so $fg_1B_1 \subsetneq \mathrm{Int}(B_1)$ by condition (2) in Definition B. If $k = -1$, then since g has type k, we must have $g_1 \in G_0 \setminus J_{-1}$ and $g_1B_{-1} \subset M \setminus B_{-1} = \mathrm{Int}(A \cup B_1)$. Again, the inclusion is proper by properness of the interactive triple, so again we have $fg_1B_{-1} \subsetneq \mathrm{Int}(B_1)$.

When $|g| > 1$, we can apply Lemma 5.9 and induction: we write $g = g'f^jg_n$, where g' is a type-j normal form with length $|g| - 1$, and $f^jg_nB_k \subset B_j$. Via induction we know that $g'B_j \subsetneq \mathrm{Int}(B_1)$, which means $gB_k \subsetneq \mathrm{Int}(B_1)$.

The case $i = 0$ follows from the first case and precise invariance of B_i under J_i, since any $(0,j)$-form g can be written $g = g_1g'$, where g' is an (i,j)-form and $g_1 \in G_0 \setminus J_i$. $\square$

We now prove the first three parts of Theorem B. Again, the arguments are standard.

Proof of (i) - (iii) in Theorem B. (i) Let B_1 and B_{-1} be the sets given by our conditions, and set $A = M \setminus (B_1 \cup B_{-1})$. Note that condition (2) of Definition B implies $f^{-1}(A \cup B_{-1}) = \mathrm{Int}(B_{-1})$. The result now follows from Proposition 5.5 since $(A, \mathrm{Int}(B_1), \mathrm{Int}(B_{-1}))$ form an interactive triple which is proper by condition (4) of Definition B.

(ii) It suffices to show no sequence (g_k) in G can accumulate at the identity. If $|g_k| = 0$, then g_k lies in the discrete group G_0, so assume $|g_k| \geq 1$ for all k. We can consider several cases. If a normal form for g_k ends in a power of f, then $g_kA_0 \subset B_1 \cup B_{-1}$. Otherwise, g_k is either a $(0,1)$ or a $(0,-1)$-form. In the former case, $g_kB_1 \subset A$, and in the latter case $g_kB_{-1} \subset A$. In each of these cases, g_k takes a fixed set with nonempty interior into another fixed disjoint set, which means g_k cannot accumulate on the identity.

(iii) Let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ be a normal form not conjugate into G_0. Conjugating and replacing g with g^{-1} if necessary, we can assume that $\alpha_1 > 0$ (so g is a $(1, j)$-form) and that $|g|$ is minimal in its conjugacy class. Note that if $\alpha_n < 0$ and $g_n \in J_1$, then $f^{-1} g f = f^{\alpha_1 - 1} g_1 \cdots f^{\alpha_n + 1} f^{-1} g_n f$ has a strictly smaller length than g since $f^{-1} g_n f \in J_{-1}$, so we know that either $\alpha_n > 0$ or $g_n \in G_0 \setminus J_1$. That is, g is a $(1, j)$-form of type 1, so by Lemma 6.3, gB_1 is a proper subset of $\mathrm{Int}(B_1)$. Then the same argument as in Theorem A part (iii) implies that g has infinite order, and a fixed point in $\mathrm{Int}(B_1)$.

On the other hand if gB_1 is a proper subset of B_1, then $g^{-1}(M \setminus gB_1)$ is a proper subset of $M \setminus B_1$, so the same argument again shows that g^{-1} has a fixed point in the closure of $M \setminus B_1$. Thus g has two distinct fixed points and is loxodromic. ☐

6.1. **Limit sets of HNN extensions.** The remainder of the section is meant to prove part (iv) of Theorem B, so for the rest of the paper we fix the space M and groups $G_0, J, \langle f \rangle, G$ in $\mathrm{Homeo}(M)$ satisfying the conditions of Definition B. As for Theorem A, we start by establishing some properties of the limit points of G under these assumptions.

Proposition 6.4. *With the above conditions and notation, each of the following holds.*

(i) $B_1 \cap B_{-1} = \varnothing$, *and* f *is loxodromic with attracting fixed point in* $\mathrm{Int}(B_1)$ *and repelling fixed point in* $\mathrm{Int}(B_{-1})$.

(ii) $\Lambda(J_{\pm 1}) \subset \partial B_{\pm 1}$.

(iii) $\Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1})) \subset A_0$.

Proof. (i) The fact that $B_1 \cap B_{-1} = \varnothing$ follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) in G_0. Now, since $f(A \cup B_1) = \mathrm{Int}(B_1)$, we have $f(\partial B_{-1}) = \partial B_1$, and so $fB_1 \subset \mathrm{Int}(B_1)$. Arguing as in the proof of Theorem A part (iii), we know this implies f has a fixed point in $\mathrm{Int}(B_1)$ which is necessarily attracting. The same argument applied to f^{-1} gives a fixed point in $\mathrm{Int}(B_{-1})$ which is necessarily a repelling fixed point for f.

(ii) Since B_i is closed and J_1-invariant, and $\mathrm{Int}(B_i)$ is infinite by Proposition 5.3, it follows that $\Lambda(J_i) \subset B_i$. Further, since f conjugates J_{-1} to J_1, f maps $\Lambda(J_{-1})$ bijectively onto $\Lambda(J_1)$. Hence

$$\Lambda(J_1) = f\Lambda(J_{-1}) \subset fB_{-1} = B_{-1} \cup A \cup \partial B_1.$$

So we conclude $\Lambda(J_1) \subset (B_{-1} \cup A \cup \partial B_1) \cap B_1 = \partial B_1$ as desired. Applying an identical argument using f^{-1} gives $\Lambda(J_{-1}) \subset \partial B_{-1}$.

(iii) Fix $x \in \Lambda(G_0)$, and suppose that $x \notin A_0$, i.e. that $x = gy$ for $g \in G_0$ and $y \in B_i$. Then $y = g^{-1}x \in \Lambda(G_0) \cap B_i$, which means $y \in \Lambda(J_i)$ by condition (3) in Definition B, and therefore $x \in G_0(\Lambda(J_i))$. It follows that $\Lambda(G_0) \setminus A_0$ is contained in $G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, which is equivalent to the desired claim. ☐

6.2. **HNN ping-pong and contraction.** Next, we will establish a contraction lemma for HNN ping-pong sequences, similar to Lemma 4.6 for amalgamated free products. As in the earlier case, the key tool is Proposition 2.18, so we start by establishing that the subgroups J_1 and J_{-1} are fully quasi-convex in some ambient geometrically finite group.

First, we show:

Lemma 6.5. *Fix* $i, j \in \{\pm 1\}$ *and* $g \in G$. *Then* $g\partial B_i \cap \partial B_j \neq \varnothing$ *if and only if either:*

(1) $i = j$ *and* $g \in J_i$, *or*

(2) $i = -j$ *and* $g = f^j h$ *for* $h \in J_i$.

Proof. We induct on the length of g. If $|g| = 0$, then the claim follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) in G_0, so suppose $|g| \geq 1$, and for concreteness, assume $i = 1$. We write a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ for g. If this normal form has type 1, then

Lemma 6.3 implies that $gB_1 \subset \mathrm{Int}(B_1) \cup \mathrm{Int}(B_{-1}) \cup A$, hence $g\partial B_1 \cap \partial B_j = \varnothing$. So we can assume that this normal form does *not* have type 1, which means that $\alpha_n < 0$ and $g_n \in J_1$. In this case, $f^{-1}g_n\partial B_1 = \partial B_{-1}$.

The group element $g' = gg_n^{-1}f$ has strictly smaller length than g, so if $g'\partial B_{-1} \cap \partial B_j \neq \varnothing$ then by induction we know that either $j = -1$ and $g' \in J_{-1}$, or $j = 1$ and $g' = fh$ for $h \in J_{-1}$. In the former case we can rewrite $g = g'f^{-1}g_n = f^{-1}g''g_n$ for $g'' \in J_1$, and in the latter case we can rewrite $g = g'f^{-1}g_n = fhf^{-1}g_n = g''g_n$ for $g'' \in J_1$. Since $g_n \in J_1$ the conclusion follows. $\square$

The lemma above implies in particular that ∂B_i is precisely invariant under J_i in both G and G_0. Then, after applying part (ii) of Proposition 6.4, we see:

Corollary 6.6. *Let H be one of G or G_0. If H is geometrically finite, then J_1 and J_{-1} are fully quasi-convex subgroups of H.*

Now, we can apply Proposition 2.18 to the present setting:

Lemma 6.7. *Suppose that either G or G_0 is geometrically finite. For $i \in \{\pm 1\}$, we can find a compact $K \subset A \cup B_{-i}$ so that both of the following hold:*

(i) For any $g \in G_0 \setminus J_i$, we have $j \in J_i$ so $jgB_i \subset K$.

(ii) For any $g \in G_0$, we have $j \in J_i$ so that $jgB_{-i} \subset K$.

Proof. Take $i = 1$ to simplify notation. We can find a compact for each claim separately and take their union. First, we focus on (i). We can assume $B_1 \subset K$, so the statement follows immediately for $g \in J_1$ by taking j to be the identity. Otherwise, we apply Proposition 2.18 with the ambient geometrically finite group as G or G_0 (depending on which one is geometrically finite) and $H = G_0$ in both cases, and our two fully quasi-convex subgroups J_1 and J_{-1} with corresponding invariant open sets $U_1 = M \setminus B_1$ and $U_{-1} = M \setminus B_{-1}$. Then if $g \in G_0 \setminus J_1$, we have $g(M \setminus U_1) = g(B_1) \subset A \subset U_{-1}$, and so the proposition gives our desired compact subset $K \subset U_{-1} = A \cup B_1$.

For (ii), the proof is identical with J_1 playing the role of both fully quasi-convex subgroups in the statement of Proposition 2.18, and both open sets being $M \setminus B_1 = A \cup B_{-1}$. $\square$

We can now establish the HNN contraction property:

Lemma 6.8 (Contraction for HNN extensions)**.** *Suppose that either G or G_0 is geometrically finite, and let (h_k) be a sequence of type-i forms such that the left cosets h_kJ_i are all distinct. Then up to subsequence, (h_kB_i) converges to a singleton $\{x\}$.*

It follows from Lemma 6.5 that any group element $g \in G$ satisfying $gB_i = B_i$ must lie in J_i. So, asking for the left cosets (h_kJ_i) to be distinct is the same as asking for the translates (h_kB_i) to be distinct.

Proof. To simplify notation, assume $i = 1$. The proof is very similar to the proof of Lemma 4.6. The first step is to show the following:

Claim. *After extracting a subsequence, there is a fixed $\ell = \pm 1$, a compact set $K \subset A \cup B_{-1}$, and a sequence (j_k) in J_1 so that $j_kh_k^{-1}B_\ell \subset K$ for all k.*

To prove the claim, we first suppose that $|h_k| = 0$. Then, since h_k has type 1, we know $h_k \in G_0 \setminus J_1$. Then we take $\ell = 1$, and apply Lemma 6.7 to find the required set K and elements j_k. Otherwise, suppose that $|h_k| \geq 1$, and write out a normal form for h_k:

$$h_k = f^{\alpha_{k,1}}g_{k,1} \cdots f^{\alpha_{k,n}}g_{k,n}.$$

We consider the inverse word

$$g_{k,n}^{-1} f^{-\alpha_{k,n}} \cdots g_{k,1}^{-1} f^{-\alpha_{k,1}}.$$

By Lemma 5.6, the sub-word $g_{k,n}h_k^{-1} = f^{-\alpha_{k,n}} \cdots g_{k,1}^{-1} f^{-\alpha_{k,1}}$ is a normal form, which must have length at least 1. Up to subsequence, for every k this normal form is type ℓ for some fixed $\ell = \pm 1$, meaning that $g_{k,n}h_k^{-1}B_\ell \subset B_1$ if $-\alpha_{k,n} > 0$ and $g_{k,n}h_k^{-1}B_\ell \subset B_{-1}$ if $-\alpha_{k,n} < 0$. After extracting another subsequence we can assume one of these conditions holds for every k.

In the case where $-\alpha_{k,n} < 0$ for every k, we can use Lemma 6.7 to find elements $j_k \in J_1$ and a compact $K \subset A \cup B_{-1}$ so that $j_k g_{k,n}^{-1} B_{-1} \subset K$ for every k. Then, we know that for every k, we have

$$j_k h_k^{-1} B_\ell = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_\ell \subset j_k g_{k,n}^{-1} B_{-1} \subset K.$$

On the other hand, if $-\alpha_{k,n} > 0$, then since h_k has type 1 we know that $g_{k,n} \in G_0 \setminus J_1$. Then again by Lemma 6.7 we can find a compact $K \subset A \cup B_{-1}$ and $j_k \in J_1$ so that $j_k g_{k,n}^{-1} B_1 \subset K$. Thus, we have

$$j_k h_k^{-1} B_\ell = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_\ell \subset j_k g_{k,n}^{-1} B_1 \subset K.$$

We have shown the claim above, so now consider the sequence $(h_k j_k^{-1})$. Since all the translates $h_k B_1$ are distinct, the group elements h_k lie in infinitely many left J_1-cosets, hence so do the group elements $h_k j_k^{-1}$. In particular, the sequence $h_k j_k^{-1}$ is divergent in G, so we can extract a convergence subsequence and assume that there are points $z_+, z_- \in M$ so that $(h_k j_k^{-1} y)$ converges to z_+ whenever $y \neq z_-$. Equivalently, $(j_k h_k^{-1} y)$ converges to z_- whenever $y \neq z_+$.

Proposition 5.3 tells us that the set B_1 is infinite, so in particular there must be some $y \in B_1 \setminus \{z_+\}$. Then, since $j_k h_k^{-1} B_1 \subset K$ we conclude that $z_- \in K$. Finally, since B_1 is a compact set in the complement of K, we see that $(h_k j_k^{-1} B_1) = (h_k B_1)$ must converge to $\{z_+\}$. □

6.3. **Geometrical finiteness of the extension.** We now prove that (G_0 geometrically finite) $\implies$ (G geometrically finite). This gives one of the directions of Theorem B part (iv).

As in the proof of the analogous direction of Theorem A, the key for this direction of theorem is to show that limit points in $\Lambda(G) \setminus \Lambda(G_0)$ can be "coded" by sequences of (i, j)-forms in G. The precise statement is:

Proposition 6.9 (HNN coding for G-limit points)**.** *Suppose that either G or G_0 is geometrically finite, and let $x \in B_1 \cup B_{-1}$ be a point in $\Lambda(G) \setminus G(\Lambda(G_0))$. Then for fixed ℓ, there is a sequence of type-ℓ forms (h_k) in G so that $|h_k| \to \infty$, each h_k is a prefix of h_{k+1}, and $x \in h_k B_\ell$ for every k.*

We can think of this proposition as a less explicit version of Proposition 4.7 in the amalgamated free product case. The construction in this case is slightly more involved, and we need a little more information about the location of certain points in $\Lambda(G)$. So, we start by showing the following:

Lemma 6.10. *Suppose that either G or G_0 is geometrically finite. Then the only limit points of G in $\partial B_{\pm 1}$ are limit points of $J_{\pm 1}$. That is, $\Lambda(G) \cap \partial B_{\pm 1} = \Lambda(J_{\pm 1})$.*

Proof. We will show that the intersection $\Lambda(G) \cap (\partial B_1 \cup \partial B_{-1})$ is a subset of $\Lambda(G_0)$; then we will be done by condition (3) in Definition B. So, let $x \in \Lambda(G) \cap \partial B_1$. We can find a sequence (g_k) in G so that $g_k z \to x$ for all but perhaps a single $z \in M$. Now, if $g_k \in G_0$ for infinitely many k, the conclusion immediately follows. So we may assume $|g_k| \geq 1$ for every k.

Up to subsequence, the g_k are all (i, j)-forms for fixed $i \in \{0, \pm 1\}$ and $j \in \{\pm 1\}$. If $i = -1$, then $g_k B_j \subset \mathrm{Int}(B_{-1})$. But for some $z \in B_j$, the sequence $(g_k z)$ converges to $x \in \partial B_1$. So, we know that either $i = 0$ or $i = 1$.

If $i = 0$, then by Lemma 5.4, for some $\ell = \pm 1$ and some $h_k \in G_0 \setminus J_i$, we have $g_k B_j \subset h_k B_\ell \subset A$. There must be infinitely many distinct translates $h_k B_\ell$, since otherwise each $g_k z$ would lie in a fixed compact subset of A, and $(g_k z)$ could not converge to $x \in B_1$. So, the left cosets $h_k J_\ell$ are all distinct. Then by Lemma 6.8, the sequence of sets $(h_k B_\ell)$ converges to a singleton, which must be x. But since $h_k \in G_0$ this again implies that $x \in \Lambda(G_0)$.

Finally, we consider the case $i = 1$. We write out a normal form for g_k:

$$g_k = f^{\alpha_{k,1}} g_{k,1} \cdots f^{\alpha_{k,n}} g_{k,n}.$$

First observe that if $\alpha_{k,1} > 1$ for infinitely many k, then the word

$$g'_k = f^{-1} g_k = f^{\alpha_{k,1} - 1} g_{k,1} \cdots f^{\alpha_{k,n}} g_{k,n}$$

is still a $(1, j)$-form, which means that $f^{-1} g_k B_j \subset B_1$ for infinitely many k. But then for infinitely many k, the point $g_k z$ lies in the compact subset $f B_1 \subset \mathrm{Int}(B_1)$, which is impossible if $g_k z \to x \in \partial B_1$.

We conclude that after extraction, we have $\alpha_{k,1} = 1$ for every k. After further extraction, we can assume that one of the the three conditions below holds for every k:

(a) The length of g'_k is zero;
(b) g'_k is a $(0, j)$-form;
(c) g'_k is not a normal form, hence $g_{k,1} \in J_1$ and $\alpha_{k,2} > 0$.

If either (a) or (b) holds, we can use the first two cases of this proof to see that that $f^{-1} x$ lies in $\Lambda(G_0) \cap \partial B_{-1} = \Lambda(J_{-1})$, and thus $x \in \Lambda(J_1)$. And, (c) cannot occur: if $\alpha_{k,2} > 0$, then the word $f^{\alpha_{k,2}} g_{k,2} \cdots f^{\alpha_{k,n}} g_{k,n}$ is a $(1, j)$-form, which means $g'_k z \in g_{k,1} B_1$, and if $g_{k,1} \in J_1$ then $g_k z = f g'_k z \in f B_1$ and again $(g_k z)$ cannot converge to $x \in \partial B_1$. ☐

We now set about proving Proposition 6.9. As in the analogous situation in the amalgamated free product case, we follow Maskit's strategy by defining certain "ping-pong" sets in M. Let $T_{0,i} = G_0(B_i)$, and $T_0 = T_{0,1} \cup T_{0,-1} = G_0(B_1 \cup B_{-1})$, the union of all G_0 translates of B_1 and B_{-1}.

More generally, let

$$T_{m,-1} = \bigcup g B_{-1},$$

where the union is taken over length-m normal forms g of type -1. Similarly, let

$$T_{m,1} = \bigcup g B_1,$$

where the union is taken over length-m normal forms of type 1. Let

$$T_m = T_{m,1} \cup T_{m,-1}.$$

Lemma 5.9 implies that the sets T_m are decreasing: for any length-m normal form g with type i, we can use the lemma to find a length-$(m-1)$ normal form g' with type j, and $g_0 \in G_0$ so that $g B_i = g' f^j g_0 B_i \subset g' B_j$. So, we can now consider the set

$$T = \bigcap_{m=0}^{\infty} T_m.$$

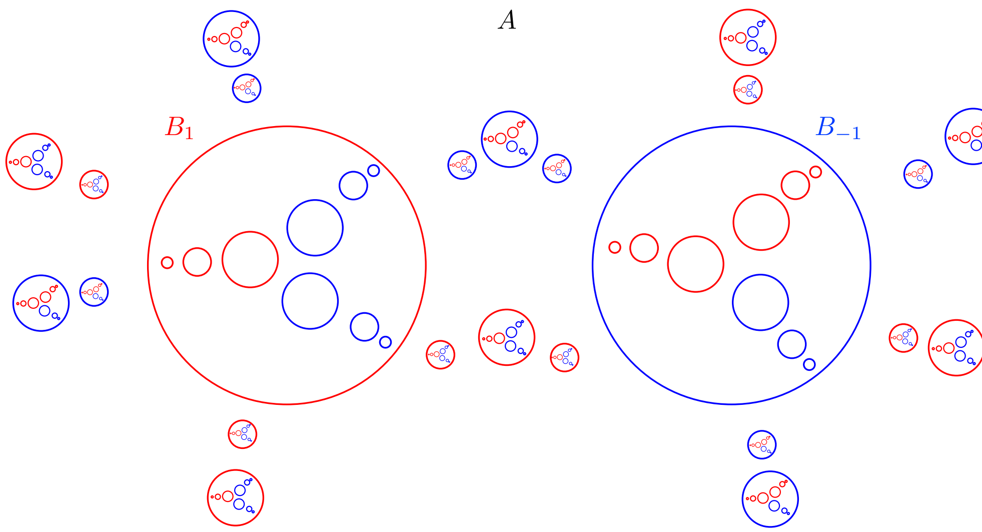

FIGURE 6.1. Part of the sets T_0 and T_1.

The proof of Proposition 6.9 mainly involves showing that $\Lambda(G) \setminus G(\Lambda(G_0)) \subset T$. Then, we construct the desired sequence using the definition for T. The first step is:

Lemma 6.11. *Suppose either G or G_0 is geometrically finite. Then $\Lambda(G) \setminus \Lambda(G_0) \subset T_0$.*

Proof. This argument is similar to the proof of Lemma 6.10. Suppose $y \in \Lambda(G)$ does not lie in T_0 (that is, $y \in A_0$). We must show $y \in \Lambda(G_0)$. Since $y \in \Lambda(G)$, we can find a sequence (g_k) in G so that $g_k w \to y$ for all but a single $w \in M$. If $g_k \in G_0$ for infinitely many k we will have $y \in \Lambda(G_0)$ as desired, so now suppose that $|g_k| \geq 1$ for infinitely many k. After extracting a subsequence we can assume that each g_k is an (i,j)-form for i, j fixed. Since B_j is an infinite set, we can fix some $w \in B_j$ so that $(g_k w)$ converges to y.

By definition, we know that $g_k w \in T_n$, so in particular $g_k w \in T_0$ for every k. Then, we can extract a further subsequence so that $g_k w \in G_0(B_i)$ for fixed i and write $g_k w = g'_k z_k$ for $g'_k \in G_0$ and $z_k \in B_i$.

As $y \notin T_0$, there must be infinitely many distinct translates $g'_k B_i$, because otherwise every $g'_k z_k$ would lie in a fixed compact subset of T_0. Thus there are infinitely many distinct cosets $g'_k J_i$, and Lemma 6.8 tells us that after extraction, $(g'_k B_i)$ must converge to a singleton. Since $g'_k z_k \to y$, it follows that this singleton is y. Hence for any choice of $z \in B_i$ we have $g'_k z \to y$, so $y \in \Lambda(G_0)$. □

Proof of Proposition 6.9. We first prove that $\Lambda(G) \setminus G(\Lambda(G_0)) \subset T$. So, fix $z \in \Lambda(G)$, and suppose $z \notin T$. We will show $z \in G(\Lambda(G_0))$.

If $z \in \Lambda(G_0)$ we are done, hence by Lemma 6.11 we can assume $z \in T_0$. Then we can find $m > 0$ so that $z \in T_{m-1} \setminus T_m$ since these sets are decreasing. Without loss of generality, we have $z \in gB_{-1}$ for $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ a normal form with length $m-1$ and type -1. If $g^{-1}z \in \partial B_{-1}$, then since $\Lambda(G)$ is G-invariant we have $g^{-1}z \in \Lambda(G) \cap \partial B_1 = \Lambda(J_{-1})$ by Lemma 6.10 and we are done. So suppose $g^{-1}z \in \mathrm{Int}(B_{-1})$.

Since $z \notin T_m$, we have $fg^{-1}z \notin B_{-1}$ since gf^{-1} has length m. Also, $fg^{-1}z \notin B_1$ since f does not map any points of $\mathrm{Int}(B_{-1})$ into B_1. It follows that $fg^{-1}z \notin B_1 \cup B_2$, but also, $fg^{-1}z$ cannot be in a translate of B_1 nor B_2. Indeed, if $fg^{-1}z = hy$ for $y \in B_i$ and $h \in G_0$, then $h \in G_0 \setminus J_i$ since $hy \notin B_i$ and B_i is J_i-invariant. Hence $z = gf^{-1}hy \in T_m$, a contradiction. Hence $fg^{-1}z \notin T_0$, and so by Lemma 6.11 we have $fg^{-1}z \in \Lambda(G_0)$ and $z \in G(\Lambda(G_0))$ as desired.

We have now shown that $\Lambda(G) \setminus G(\Lambda(G_0)) \subset T$, so consider $z \in T$. We will construct our sequence (h_k) of normal forms inductively. We know that $z \in T_1$, so we can find some normal form h_1 with type i_1 so that $z \in h_1 B_{i_1}$. Now, assume that we have constructed a normal form h_k of type i so that $z \in h_k B_i$. Since $z \in T_{k+1}$, we can find a normal form h'_{k+1} with length $k+1$ and type ℓ so that $x \in h'_{k+1}B_\ell$. Then, by Lemma 5.9, there is a type-i' form h'_k with length k and $g_0 \in G_0$ so that $h'_{k+1} = h'_k f^{i'} g_0$ and $h'_{k+1}B_\ell \subset h'_k B_{i'}$. Then $h'_k B_{i'}$ has nonempty intersection with $h_k B_i$, so by Lemma 5.10 we have $i = i'$ and $h_k j = h'_k$ for $j \in J_i$. We can write $jf^i = f^i j'$ for $j' \in J_{-i}$. Then since h_k has type i, h_k is a prefix of the type-ℓ form $h_{k+1} = h_k f^i j' g_0$. This form is equivalent in G to the type-ℓ form h'_{k+1}, hence $z \in h_{k+1}B_\ell$.

Finally, by taking a subsequence, we can assume that each h_k is a form of type ℓ for ℓ fixed, and we are done. $\square$

As for amalgamated free products, we can use the coding given by Proposition 6.9 to construct conical limit sequences for points in $\Lambda(G) \setminus G(\Lambda(G_0))$:

Lemma 6.12. *If G_0 is geometrically finite, then every point of $\Lambda(G)\setminus G(\Lambda(G_0))$ is a conical limit point for G.*

Proof. Let $x \in \Lambda(G)\setminus G(\Lambda(G_0))$. Proposition 6.9 says that for i fixed, we can find a sequence (h_k) of ping-pong forms of type i, with $|h_k| \to \infty$, so that each h_k is a prefix of h_{k+1}, and $x \in h_k B_i$ for all k. Possibly after relabeling we may assume $i = 1$.

We write h_k in a normal form:

$$f^{\alpha_1} g_1 \cdots f^{\alpha_{n_k}} g_{n_k}.$$

If $\alpha_1 = 0$, then $\alpha_2 \neq 0$, in which case $h'_k = f^{\alpha_2} g_2 \cdots f^{\alpha_{n_k}} g_{n_k}$ is a ping-pong form of type 1 such that $h'_k B_1$ contains $x' = g_1^{-1}x$. Since x' is a conical limit point if and only if x is, if necessary we can replace x with x' and h_k with h'_k, and assume that $\alpha_1 \neq 0$. That is, h_k is an (ℓ, j)-form for $\ell \neq 0$, so $h_k B_1 \subset B_\ell$.

Further, since $x \in h_1 B_1$, by replacing x with $h_1^{-1}x$ and h_k with $h_1^{-1}h_k$, we can assume that also $x \in B_1$, hence $h_k B_1 \cap B_1 \neq \varnothing$. Since $h_k B_1 \subset B_\ell$ we have $\ell = 1$, meaning $\alpha_1 > 0$.

Now, consider the sequence of sets

$$(h_k^{-1} B_{-1}) = (g_{n_k}^{-1} f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1} B_{-1}).$$

By Lemma 5.6, the word $f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1}$ is a normal form; since $\alpha_1 > 0$ it is a form of type -1, implying that $f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1} B_{-1}$ is a subset of B_1 if $\alpha_{n_k} < 0$ and a subset of B_{-1} if $\alpha_{n_k} > 0$. And, since h_k is a form of type 1, we know that either $g_{n_k} \in G_0 \setminus J_1$ or $\alpha_{n_k} > 0$.

To prove that x is conical, we want to apply Lemma 2.10, which means we need to produce distinct elements g_k, a set Y with at least two points, and disjoint compact sets K_1 and K_2 so that $g_k x \in K_2$ and $g_k Y \subset K_1$. Let $K \subset A \cup B_{-1} = M \setminus B_1$ be the compact from Lemma 6.7, and take $Y = B_{-1}, K_1 = K$ and $K_2 = B_1$. We know Y contains at least two points from Proposition 5.3, so we just need to produce the sequence (g_k) by modifying h_k^{-1}.

For each fixed k, we already have $h_k^{-1}x \in B_1$ as desired. If $\alpha_{n_k} > 0$, then $h_k^{-1}B_{-1} \subset g_{n_k}^{-1}B_{-1}$. From the definition of K, we can find $j_k \in J_1$ so that $j_k g_{n_k}^{-1}B_{-1} \subset K$, hence $j_k h_k^{-1}B_{-1} \subset K$.

On the other hand, if $\alpha_{n_k} < 0$, then we necessarily have $g_{n_k} \in G_0 \setminus J_1$, and $h_k^{-1}B_{-1} \subset g_{n_k}^{-1}B_1 \subset A$. Again using the definition of K, we can find $j_k \in J_1$ so that $j_k g_{n_k}^{-1}B_1 \subset K$, hence $j_k h_k^{-1}B_{-1} \subset K$.

In either of these cases, we have $j_k h_k^{-1}B_{-1} \subset K$ and $j_k h_k^{-1}x \in j_k B_1 = B_1$, which means we can take $g_k = j_k h_k^{-1}$ to complete the proof. □

We next consider parabolic points.

Lemma 6.13. *Suppose that G_0 is geometrically finite. If $p \in \Lambda(G_0)$ is a parabolic point for the action of G_0 on $\Lambda(G_0)$, then p is a bounded parabolic point for the action of G on $\Lambda(G)$.*

Proof. Let $p \in \Lambda(G_0)$ be a parabolic point for G, and let P be the stabilizer of p in G. Since p is a bounded parabolic point, and P contains the stabilizer of p in G_0, we know that there is a compact $\widehat{K} \subset \Lambda(G_0) \setminus \{p\}$ so that $P(\widehat{K}) = \Lambda(G_0) \setminus \{p\}$. We want to find a compact $K \subset \Lambda(G) \setminus \{p\}$ so that $P(K) = \Lambda(G) \setminus \{p\}$.

As in the proof of Lemma 4.11, our strategy is to show that $\Lambda(G) \setminus \{p\}$ can be decomposed into several pieces, such that each piece is either far away from p to begin with, or can be pushed uniformly far away from p using either the boundedness of p in $\Lambda(G_0)$ or an application of Proposition 2.18. We consider two cases:

Case 1: $p \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$. In this case, Lemma 6.11 tells us that each point in $\Lambda(G) \setminus \{p\}$ lies in the union $\Lambda(G_0) \cup T_0$. We can further decompose T_0 by intersecting it with $(B_1 \cup B_{-1})$ and its complement A, meaning we decompose $\Lambda(G) \setminus \{p\}$ into three pieces lying in

$$L_1 = \Lambda(G_0), \quad L_2 = (B_1 \cup B_{-1}), \quad L_3 = T_0 \cap A.$$

For each L_i, we need to find a compact set $K_i \subset M \setminus \{p\}$ so that if $y \in \Lambda(G) \cap L_i$, then we can find $h \in P$ so that $hy \in K_i$. Then we can take $K = K_1 \cup K_2 \cup K_3$.

We know we can take $K_1 = \widehat{K}$ from the boundedness of p in $\Lambda(G_0)$, and from part (3) of Definition B we know that $B_1 \cup B_{-1}$ is already a compact subset of $M \setminus \{p\}$. So, we just need to find the compact set K_3.

We apply Proposition 2.18, taking $H = G_0$, $J_1 = P$, $U_1 = M \setminus \{p\}$, $J_2 = J_{\pm 1}$, and $U_2 = M \setminus B_{\pm 1}$, to see that there are sets $K_+, K_- \subset M \setminus \{p\}$ such that for any $g \in G_0 \setminus J_{\pm 1}$, we can find $h \in P$ so that $hgB_{\pm 1} \subset K_\pm$. To justify the application of the proposition, we need to check that for every $g \in G_0 \setminus J_{\pm 1}$, we have $gB_{\pm 1} \subset M \setminus \{p\}$, but this follows from part (iii) of Proposition 6.4. Then, we take $K_3 = K_+ \cup K_-$.

Now, if $y \in T_0 \cap A$, then by definition we know that $y \in (G_0 \setminus J_1)(B_1) \cup (G_0 \setminus J_{-1})(B_{-1})$. But then by definition of $K_\pm$ we know that we can find $h \in P$ so that $hy \in K_+ \cup K_- = K_3$ and we are done.

Case 2: $p \in G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$. Since G acts by homeomorphisms it suffices to consider $p \in \Lambda(J_1) \cup \Lambda(J_{-1})$. Without loss of generality take $p \in \Lambda(J_1)$. As in the previous case, we decompose $\Lambda(G) \setminus \{p\}$ into several different pieces, by writing M as the union

$$M = fB_1 \cup fA \cup \partial B_1 \cup A \cup B_{-1}.$$

Since $p \in \partial B_1$, the sets $fB_1 \subset \mathrm{Int}(B_1)$ and B_{-1} are compact sets in the complement of p.

So, we only need to consider the three pieces of $\Lambda(G) \setminus \{p\}$ contained in the three sets

$$\partial B_1, \quad A, \quad fA.$$

We can further decompose these pieces by intersecting each of them with the sets $\Lambda(G_0)$, $f\Lambda(G_0)$ and their complements in M. By Lemma 6.10, we know that $\partial B_1 \cap \Lambda(G) \subset \Lambda(G_0)$. Also, from Lemma 6.11, we know that $\Lambda(G) \setminus \Lambda(G_0)$ lies in T_0, which means we now only need to consider the pieces of $\Lambda(G) \setminus \{p\}$ contained in the four sets

$$L_1 = \Lambda(G_0), \quad L_2 = T_0 \cap A, \quad L_3 = f\Lambda(G_0), \quad L_4 = f(T_0 \cap A).$$

We want to find compact sets $K_1, K_2, K_3, K_4 \subset M \setminus \{p\}$ so that for each $y \in L_i \cap (\Lambda(G) \setminus \{p\})$, we can find $h \in P$ so that $hy \in K_i$.

We already know that we can take $K_1 = \widehat{K}$, and to find K_2, we can use the exact same construction we used for K_3 in Case 1. To justify the application of Proposition 2.18 in this situtation, we again need to check that for any $g \in G_0 \setminus J_{\pm 1}$, we have $gB_{\pm 1} \subset M \setminus \{p\}$. This time, the desired inclusion follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) and the fact that $p \in B_1$.

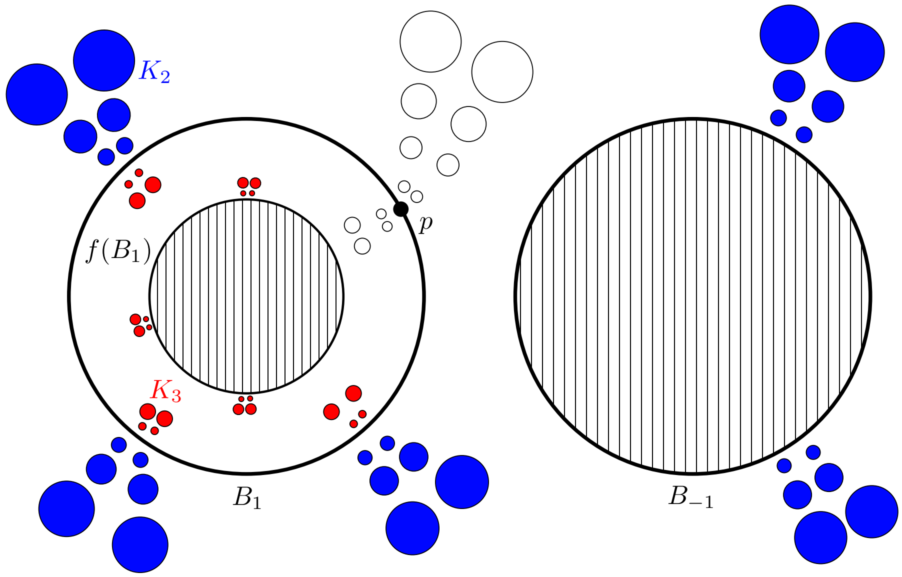

FIGURE 6.2. Illustration for Case 2 of Lemma 6.13. The sets B_{-1} and $f(B_1)$ are already compact subsets of $M \setminus \{p\}$, so we need to divide the rest of $\Lambda(G)$ into pieces. The sets K_1 and K_4 (not pictured) lie in $\Lambda(G_0) \setminus \{p\}$ and $f(\Lambda(G_0)) \setminus \{p\}$.

Finally, to find K_3 and K_4, we just apply the same exact arguments to the parabolic point $f^{-1}p \in \Lambda(J_{-1})$ and its stabilizer $f^{-1}Pf$, to obtain a pair of compact sets $K_3', K_4' \subset M \setminus \{f^{-1}p\}$ such that for any $z \in (\Lambda(G) \setminus \{f^{-1}p\}) \cap (L_1 \cup L_2)$, we can find $h \in P$ so that $f^{-1}hfz \in K_3' \cup K_4'$. We can take $K_3 = fK_3'$ and $K_4 = fK_4'$ (see Figure 6.2). Then if $y \in (\Lambda(G) \setminus \{p\}) \cap (L_3 \cup L_4) = (\Lambda(G) \setminus \{p\}) \cap (fL_1 \cup fL_2)$, we have $y = fz$ for $z \in (\Lambda(G) \setminus \{f^{-1}p\}) \cap (L_1 \cup L_2)$, and we can find $h \in P$ so that $f^{-1}hfz \in f^{-1}K_3 \cup f^{-1}K_4$, hence $hy \in K_3 \cup K_4$. $\square$

Finally, we complete the proof of this direction of Theorem B part (iv):

Proposition 6.14. *If G_0 is geometrically finite, then G is geometrically finite.*

Proof. We must show that any $x \in \Lambda(G)$ is a conical limit point or bounded parabolic point. By Lemma 6.12, we may assume $x \in G(\Lambda(G_0))$. Since G acts by homeomorphisms, in fact we can assume that $x \in \Lambda(G_0)$. Since G_0 is geometrically finite, x is either a conical limit point or bounded parabolic point for the action of G_0 on $\Lambda(G_0)$. In the former case, x is also a conical limit point for G acting on $\Lambda(G)$, and in the latter case x is a bounded parabolic point for G acting on $\Lambda(G)$ by Lemma 6.13. Hence G is geometrically finite. □

6.4. **Geometrical finiteness of G_0.** Finally, we prove the other direction of Theorem B part (iv), and show that if G is geometrically finite, then so is G_0. As for the amalgamated free product case, the first step is the following:

Lemma 6.15. *Assume that G is geometrically finite. Let $x \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, and suppose that $h_k \in G$ is a conical limit sequence for x. Then, after extracting a subsequence, we can find some $h \in G$ so that $h_k \in hG_0$ for every k.*

Proof. By Proposition 6.4 part (iii), we know $x \in A_0 \subset A$. As x is a conical limit point, we can find a conical limit sequence (h_k) for x, so that for distinct points $a, b \in M$, we have $h_k x \to a$ and $h_k z \to b$ for any $z \in M \setminus \{x\}$.

If $h_k \in G_0$ for infinitely many k then we are done, so we may assume $|h_k| \geq 1$ for every k. Suppose we can write $h_k = h'_k f g_k$ where $|h'_k| = |h_k| - 1$ (the case where $h_k = h_k f^{-1} g_k$ is similar). We note that $g_k x \in A_0 \subset A$ still since A_0 is G_0-invariant.

Consider the sequence $(h_k g_k^{-1}) = (h'_k f)$. We know that $f g_k x \in fA \subset B_1$, so $h_k x$ lies in $h'_k B_1$ for every k. If the h'_k are all in distinct left J_1-cosets in G, the sequence $(h'_k B_1)$ converges to a singleton by Lemma 6.8. The limit of $(h_k x)$ is contained in this singleton, so the singleton is $\{a\}$. On the other hand, since A is infinite, the set $g_k^{-1}A$ is also infinite, so there is at least one point z in $g_k^{-1}A \setminus \{x\}$. But then $h_k z \in h_k g_k^{-1} A$, so we have

$$h_k z \in h'_k f g_k g_k^{-1} A \subset h'_k f A \subset h'_k B_1.$$

This means that $(h_k z)$ converges to a, which contradicts the fact that (h_k) is a conical limit sequence for x.

So, after taking a subsequence, we must have $h'_k \in h' J_1$ for some fixed $h' \in G$. Then for every k, we have $h_k \in h' J_1 f g_k = h' f J_{-1} g_k \subset h' f G_0$, and we are done. □

Proposition 6.16. *If G is geometrically finite, then G_0 is geometrically finite.*

Proof. We must show that any $x \in \Lambda(G_0)$ is a conical limit point or bounded parabolic point for the G_0-action. Since G is geometrically finite, x is either a conical limit point or bounded parabolic point for the G-action. In the former case, by Lemma 6.15 we conclude that there is a conical limit sequence of the form (hg_k) for x, where $h \in G$ and $g_k \in G_0$. Then (g_k) is a conical limit sequence for x in G_0 and we are done.

In the latter case, let $P < G$ be the stabilizer of x, a parabolic subgroup of G. We claim that in fact P is a subgroup of G_0. If $x \in \Lambda(J_1) \cup \Lambda(J_{-1})$, then x lies in either ∂B_1 or ∂B_{-1}, and then this follows from Lemma 6.5. And, if $x = gy$ for $y \in \Lambda(J_1) \cup \Lambda(J_{-1})$ and $g \in G_0$, then the stabilizer of x lies in $gG_0 g^{-1} = G_0$. Finally, if $x \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, then part (iii) of Proposition 6.4 says that $x \in A_0$, and Lemma 5.4 implies that no element of G with positive length can fix a point in A_0.

Now, since x is a bounded parabolic point for the G-action on $\Lambda(G)$, local compactness of $\Lambda(G) \setminus \{x\}$ implies that there is some compact $K \subset \Lambda(G)$ so that $P(K) = \Lambda(G) \setminus \{x\}$. We let $K_0 = K \cap \Lambda(G_0)$, which is a compact in $\Lambda(G_0) \setminus \{x\}$.

Using G_0-invariance (and hence P-invariance) of $\Lambda(G_0)$, we now have that

$$P(K_0) = P(K \cap \Lambda(G_0)) = P(K) \cap \Lambda(G_0) = \Lambda(G_0) \setminus \{x\}$$

as desired. □

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Department of Mathematics, University of Virginia, Charlottesville, VA 22902, USA
Email address: at3kk@virginia.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: tjwei@umich.edu