

Math 668 Fall 2026 Problem Set 3
Due November 5

Hand in at most 4 problems.

1. Complete the proof in class showing that the curves $C(I, J)$ for $|I \cap J| = k-1$ are exactly the T -invariant curves in $\text{Gr}(k, n)$.
2. Let $Z \subseteq X$ be a T -invariant subvariety. Let $\iota : \{p\} \hookrightarrow X$ denote the inclusion of a T -fixed point, and assume that $p \notin Z$. Give a careful proof that

$$\iota^*[Z]^T = 0 \in H_T^*(\{p\}).$$

3. Compute $(\xi_{2,1})^2$ in $H_T^*(\text{Gr}(3, 6))$. Compute $s_{2,1}(x_1, x_2, x_3|y)^2$ and compare.
4. Let ω be a symplectic form on $\mathbb{C}^{2n}$. That is, ω is a nondegenerate skew-symmetric bilinear form. Let $LG(2n) \subset \text{Gr}(n, 2n)$ be the Lagrangian Grassmannian, consisting of all the isotropic n -dimensional subspaces $V \subset \mathbb{C}^{2n}$. (A subspace V is isotropic if the restriction of ω to V is 0.) Let $T = (\mathbb{C}^*)^n \subset \text{GL}(2n)$ be a n -dimensional torus preserving ω . Compute the GKM graph for T acting on $LG(2n)$. (You may assume that the GKM theorem applies in this case.)
5. By piggybacking on the known result from ordinary cohomology, prove the following statement. Suppose X is a T -variety and $Y, Z \subseteq X$ are T -subvarieties. If Y, Z intersect properly and transversally, then $[Y]^T \cdot [Z]^T = \sum_W [W]^T$, where the sum is over the irreducible components W of $Y \cap Z$.
6. Let $M_{k,n}^\circ$ denote the space of full rank $k \times n$ complex matrices. Prove that $\pi_i(M_{k,n}^\circ) = 0$ for $0 \leq i \leq 2(n-k)$. (As an intermediate step, you may want to prove that the space of lower rank matrices inside $M_{k,n}$ has codimension $\geq (n-k-1)$.) Use this to identify $H_{\text{GL}(k)}^*(\text{pt})$.
7. Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ by $t \cdot (x, y) = (t^2x, t^3y)$. Show that $E = \{(x, y) \mid x^3 - y^2 = 0\}$ is a one-dimensional orbit closure with a unique fixed point. So in general it is possible for orbit closures to have singularities.
 Now suppose that we have a T -variety X that is smooth, projective, and equivariantly formal. Show that these conditions forbid singular one-dimensional T -orbit closures.
8. Find a formula for the one-row Schubert classes $\xi_{(r)} \in H_T^*(\text{Gr}(k, n))$ as functions on $\binom{[n]}{k}$.
9. Find an equivariant Pieri formula, giving the expansion of $\xi_{(r)}\xi_\lambda$ in terms of ξ_μ .
10. Consider the Grassmannian $\text{Gr}(1, n)$, also known as projective space $\mathbb{P}^{n-1}$. Explicitly write down the GKM ring, the equivariant Schubert classes, and write down a closed formula for all equivariant Littlewood-Richardson coefficients.
11. Let $w \in S_n$ and λ a partition fitting in a $k \times (n-k)$ rectangle. Compute the equivariant cohomology class $[w \cdot X_\lambda]^T$, as a function on $\binom{[n]}{k}$.
12. The following characterization of the equivariant LR-coefficients $c_{\lambda\mu}^\nu$ was mentioned in class. Write $D = \xi_{(1)}$, the equivariant class of the Schubert divisor. Let $a_{\lambda\mu}^\nu \in S$ satisfy:
 - (a) For all λ , we have $a_{\lambda\lambda}^\lambda = \xi_\lambda(\lambda)$,
 - (b) For all λ, μ , we have $(D(\lambda) - D(\mu))a_{\lambda\mu}^\lambda = \sum_\nu a_{\lambda\nu}^\lambda$ where the summation is over ν such that ν/μ is a single box,
 - (c) For all λ, μ, ν , we have $(D(\nu) - D(\lambda))a_{\lambda\mu}^\nu = \sum_{\lambda'} a_{\lambda'\mu}^\nu - \sum_{\nu'} a_{\lambda\mu}^{\nu'}$ where the summation is over λ' (resp. ν') such that λ'/λ (resp. ν/ν') is a single box.

Show that $a_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu$.

13. In class we gave two monomial formulae for the super Schur functions $s_\lambda[x|y]$ (defined as the limit of factorial Schur functions). One is from expanding the summation $s_\lambda[x|y] = \sum_{\mu \subseteq \lambda} s_\mu(x)s_{\lambda'/\mu'}(y)$. The other is $s_\lambda[x|y] = \sum_T (x|y)^T$. Give a direct combinatorial proof that the two formulae are equal.

14. Let $S = H_T^*(\text{pt}) = \mathbb{Z}[y_1, \dots, y_n]$. Prove that the equivariant cohomology of the Grassmannian has a presentation of the following form:

$$H_T^*(\text{Gr}(k, n)) \simeq S[e_1(x|y), e_2(x|y), \dots, e_k(x|y)] / (h_{n-k+1}(x|y), \dots, h_n(x|y)).$$

In particular, $H_T^*(\text{Gr}(k, n))$ is generated by algebraically independent elements $e_1(x|y), e_2(x|y), \dots, e_k(x|y)$. The ideal generators $h_r(x|y)$ are to be interpreted as polynomials in $e_1(x|y), \dots, e_k(x|y)$ with coefficients in S .

15. Let $\text{Gr}(k, n)$ and $\text{Gr}(n - k, n)$ be dual Grassmannians (cf. Pset 2). Let $c_{\lambda\mu}^\nu$ and $d_{\lambda\mu}^\nu$ denote the equivariant LR coefficients for the two Grassmannians. Show that

$$d_{\lambda'\mu'}^{\nu'}(\mathbf{y}') = c_{\lambda\mu}^\nu(\mathbf{y})$$

where $y'_i = -y_{n+1-i}$.

16. There is a pushforward map $\rho_* : H_T^*(\text{Gr}(k, n)) \rightarrow H_T^*(\text{pt})$. It can be computed by the following *integration formula* ([AF, Corollary 5.2.4]):

$$\rho_*(\xi) = \sum_{\lambda} \frac{\xi(\lambda)}{c_d^T(T_{\lambda}\text{Gr}(k, n))}$$

where $c_d^T(T_{\lambda}\text{Gr}(k, n)) = \prod \chi$, the product over the weights of the T -representation $T_{\lambda}\text{Gr}(k, n)$, viewing λ as a T -fixed point. Prove *equivariant Poincare duality*:

$$\rho_*(\sigma_{\lambda} \cdot \tilde{\sigma}_{\mu}) = \delta_{\lambda, \mu^c}.$$

where $\tilde{\sigma}_{\mu} = [\tilde{X}_{\mu}]^T$ is the equivariant fundamental class of the opposite Schubert variety.