

Math 668 Fall 2026 Problem Set 2
Due Thursday October 15

Hand in at most 4 problems.

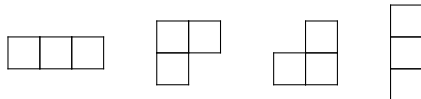
1. Show that $H^*(\text{Gr}(k, n))$ has a presentation given by $\mathbb{Z}[h_1, h_2, \dots, h_{n-k}]/I$ where $I = (e_{k+1}, e_{k+2}, \dots, e_n)$ is the ideal generated by elementary symmetric functions (which can be expressed as polynomials in homogeneous symmetric functions).
2. Prove that $\lambda \preceq \mu$ in dominance order if and only if $\mu' \preceq \lambda'$ in dominance order.
3. Recall that we defined $H(t) = \prod_i 1/(1 - x_i t)$. Expand $\log H(t)$ as a power series whose coefficients are symmetric functions. Use this to express h_n in terms of the power sum symmetric functions.
4. Compute $\omega(p_n)$, where $\omega : \Lambda \rightarrow \Lambda$ is the algebra involution of symmetric functions from class.
5. Prove that the power sum symmetric functions are algebraically independent and generate Λ over the rational numbers.
6. Prove, using lattice paths, the dual Jacobi-Trudi formula expressing s_λ as a determinant of the elementary symmetric functions e_k . We did the version with h_k in class.
7. Prove the following statements for the alternant a_α mentioned in class:
 - (a) $a_\delta = \prod_{i < j} (x_i - x_j)$
 - (b) a_α is divisible by a_δ
 - (c) The set $\{a_\alpha\}$ where $\alpha_1 > \alpha_2 > \dots > \alpha_k$ forms a basis for the space of alternating polynomials in k variables.

8. Prove that

$$s_\lambda(1, q, q^2, \dots, q^{m-1}) = x^a \prod_{i < j} \frac{x^{\lambda_i - \lambda_j + j - i} - 1}{x^{j-i} - 1}$$

where $a = \sum_i (i-1)\lambda_i$.

9. Let $\delta = (n-1, n-2, \dots, 1)$. Prove that $s_\delta(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j)$.
10. Prove that s_δ is a polynomial in the odd power sums.
11. Find a product formula for the power series $\sum_\lambda s_\lambda$. The summation is over all partitions.
12. Let λ be a partition of n with length r . The forgotten symmetric functions are defined as $f_\lambda = (-1)^{n-r} \omega(m_\lambda)$. Show that the coefficient of m_μ in f_λ is equal to the number of distinct permutations $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of $(\lambda_1, \dots, \lambda_r)$ such that the set $\{\alpha_1 + \dots + \alpha_i : 1 \leq i \leq r\}$ contains the set $\{\mu_1 + \dots + \mu_j : 1 \leq j \leq \ell(\mu)\}$.
13. A ribbon $R = \lambda/\mu$ is an edge connected skew-shape that does not contain any 2×2 squares. For example, the 3-ribbons are



The p -core of a partition λ is the Young diagram obtained by p -ribbons from λ (from the outside, keeping the shape a Young diagram at each step) until this is no longer possible. Show that the p -core of a partition does not depend on the order that these ribbons are removed.

14. Let λ/μ be a skew shape. A *domino (horizontal) strip* is a tiling of λ/μ by dominos (or 2-ribbons) $D_1, D_2, \dots, D_r$ such that for each i , we have that every box of D_{i+1} is to the right of every box of D_i , and $\mu \cup D_1 \cup \dots \cup D_i$ is a Young diagram. A *domino tableau* T of shape λ is a tiling of λ by dominos and a filling of each domino by $1, 2, \dots$ such that for each i , the union of all dominos labeled i is a domino strip. The *spin* of a domino tableau T is equal to the number of vertical dominos in it.

Define

$$G_\lambda = \sum_T q^{\text{spin}(T)} \mathbf{x}^T$$

the summation over all domino tableaux of shape λ , and $\mathbf{x}^T = x_1^{\#1-s} x_2^{\#2-s} \dots$ is the usual weight. Prove that G_λ is a symmetric function in $\Lambda \otimes \mathbb{Z}[q]$.

15. A *reverse plane partition* is a filling F of a Young diagram λ with numbers $1, 2, 3, \dots$ which is weakly increasing along both rows and columns. Define the weight of a reverse plane partition F by

$$\text{wt}(F) = (\alpha_1, \alpha_2, \dots), \quad \alpha_i = \# \text{columns containing an } i.$$

Define $g_\lambda = \sum_F x^{\text{wt}(F)}$, where the summation is over all reverse plane partitions of shape λ . Prove that g_λ is a symmetric function. (Bonus: prove that g_λ is Schur-positive.)

16. Let G be a simple undirected graph with vertex set $V(G)$. A proper coloring is a function $\kappa : V(G) \rightarrow \{1, 2, \dots\}$ such that adjacent vertices receive different colors. The *chromatic symmetric function* is defined as

$$X_G(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where the summation is over all proper colorings of G . Observe that X_G is a symmetric function.

(a) Compute X_G for the complete graph, the path, and the cycle.

(b) For a subset $A \subseteq E(G)$ of edges, let $\lambda(A)$ denote the partition with parts equal to the sizes of the connected components of the induced subgraph of G keeping the edges in A . Show that

$$X_G = \sum_{A \subseteq E(G)} (-1)^{|A|} p_{\lambda(A)}$$

where $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell}$.