**Math 668 Fall 2026 Problem Set 1**
**Due Tuesday September 29**

Hand in no more than four of the problems. (Solve as many as you like!). The problems vary in difficulty a lot, and some may require knowledge of algebraic geometry that we have not covered in class.

(1) Show that the Plücker embedding $p : \mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$ is injective.

(2) Consider the Plücker relations
$$\Delta_{i_1\cdots i_k}\Delta_{j_1\cdots j_k} - \sum \Delta_{i'_1\cdots i'_k}\Delta_{j'_1\cdots j'_k}$$
summed over all pairs obtained by interchanging a fixed set of $d$ of the subscripts $j_1,\ldots,j_k$ with $d$ of the subscripts $i_1,\ldots,i_k$, with order maintained.

Show that any collection of complex numbers $\Delta_{i_1\cdots i_k}$ satisfying the Plücker relations lies in the image of $p$.

(3) Prove or find a counterexample to the following statement: the Plücker ideal is generated by Plücker relations for $d = 1$.

(4) The Grassmannian $\mathrm{Gr}(2,4)$ is a 4-dimensional subvariety of $\mathbb{P}^5$ in the Plücker embedding. Write down the single relation that cuts out $\mathrm{Gr}(2,4)$, and check directly that it defines a smooth projective subvariety. (For example, use the Jacobian criterion for smoothness.)

Calculate the ideal of the Schubert variety $X = X_{(1)}(E_\bullet)$. Use this to show that $X$ is not smooth.

(5) Determine exactly when $X_\lambda(E_\bullet) \subset \mathrm{Gr}(k,n)$ is a smooth variety.

(6) Compute the ideal of the Schubert variety $X_\lambda(E_\bullet)$ inside the Grassmannian $\mathrm{Gr}(k,n)$.

(7) The group $\mathrm{GL}(n)$ acts on the polynomial ring $\mathbb{C}[\Delta_I \mid I \in \binom{[n]}{k}]$ as follows. For $g \in \mathrm{GL}(n)$, $J \in \binom{[n]}{k}$, and $V$ a $k \times n$ matrix we have
$$\Delta_J(V \cdot g) = \text{some polynomial } p(\Delta_*) \text{ in } \Delta_J(V) \text{ with coefficients depending on } g.$$
Show that the polynomial $p$ does not depend on any choices, and declare that $g \cdot \Delta_J = p(\Delta_*)$. Prove that the Plücker ideal is preserved by the action of $\mathrm{GL}(n)$.

(8) Check directly that the triple intersection of Schubert varieties we used to compute the Pieri rule is a transverse intersection.

(9) Call two flags $E_\bullet$ and $F_\bullet$ *opposing* if
$$\dim(E_i \cap F_j) = \begin{cases} i+j-n & \text{if } i+j-n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
Show that if $E_\bullet$ and $F_\bullet$ are opposing then $X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$ is a transverse intersection.

(10) Suppose that $E_\bullet$ and $F_\bullet$ are opposing flags. Show that $X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$ is either empty, or an irreducible subvariety.

(11) Let $V \in \mathrm{Gr}(k,n)$. Describe the tangent space $T_V(\mathrm{Gr}(k,n))$ as a subspace of $T_V(\mathbb{P}^{\binom{n}{k}-1})$ under the Plücker embedding.

(12) Prove that the codimension one Schubert variety $X_{(1)}(F_\bullet)$ for any flag $F_\bullet$ is the transverse intersection of $\mathrm{Gr}(k,n)$ with some hyperplane $H \subset \mathbb{P}^{\binom{n}{k}-1}$. (By hyperplane I mean the set of points satisfying a single non-trivial linear equation $\sum_I a_I \Delta_I = 0$.)

(13) Let $X \subset \mathbb{P}^d$ be a projective variety of dimension $m$. The *degree* of $X$ (in this embedding) is the number of points in the intersection $X \cap L$ where $L$ is a generic linear hyperspace of dimension $d-m$. Show that the degree of $\mathrm{Gr}(k,n)$ in the Plücker embedding is equal to the number of standard Young tableaux of shape $(n-k)^k$ i.e. a rectangle of size $k \times (n-k)$. (You can look up "standard Young tableau" on Wikipedia.)

(14) Let $W^*$ denote the dual vector space to $W$, and suppose that $\dim(W) = n$. Show that the Grassmannians $\mathrm{Gr}(k,W)$ and $\mathrm{Gr}(n-k,W^*)$ are isomorphic under the map
$$V \mapsto \ker(W^* \to V^*).$$
Show that this isomorphism takes a Schubert variety $X_\lambda \subset \mathrm{Gr}(k,W)$ to the Schubert variety $X_{\lambda'} \subset \mathrm{Gr}(k,W^*)$, where $\lambda'$ is the conjugate of $\lambda$.