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Brief paper

Stabilization bound of singularly perturbed systems subject to actuator saturation[☆]

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ABSTRACT

This paper considers the stabilization bound problem for singularly perturbed systems (SPSs) subject to actuator saturation. A state feedback stabilization controller design method is proposed and a basin of attraction depending on the singular perturbation parameter is constructed, which facilitates the formulation of the convex optimization problem for maximizing the basin of attraction of SPSs. Finally, examples are given to show the advantages and effectiveness of the obtained results.

1. Introduction

Singularly perturbed systems (SPSs), with a small singular perturbation parameter ε determining the degree of separation between the "slow" and "fast" modes of the systems, have been one of the major research subjects of control science due to their widespread applications. The stability bound problem for SPSs, which is referred to as the problem of determining the stability bound ε_0 such that the system is stable for all $\varepsilon \in (0, \varepsilon_0)$ or $(0, \varepsilon_0]$, is a fundamental problem and has attracted much attention (Abed, 1985; Cao & Schwartz, 2004; Feng, 1988; Saydy, 1996; Sen & Datta, 1993). Frequency- and time-domain methods were proposed in Cao and Schwartz (2004) and Feng (1988) to provide the largest stability bound for SPSs. The stabilization bound problem aiming at designing controllers to enlarge the stability bound of SPSs has also been considered (Chiou, Kung, & Li, 1999; Li & Li, 1992; Liu, Paskota, Sreeram, & Teo, 1997).

Actuator saturation is a common phenomenon in practical systems and thus intensive research efforts have been devoted to control systems subject to actuator saturation. The problem of global/semi-global stabilization is one of the most interesting topics and has been discussed in great depth (Cao, Lin, & Ward, 2002; Hu, Teel, & Zaccarian, 2006; Lin & Saberi, 1993). Since global stabilization cannot be achieved for open-loop unstable systems in the presence of actuator saturation, local results have to be developed. In this context, a key issue is to estimate the domain of stability for the closed-loop system (estimation of the basin of attraction). Most of the results on this topic are based on characterizing the basin of attraction by Lyapunov functions, by which the design parameters can be incorporated into optimization problems to maximize the basin of attraction for the closed-loop systems (Cao et al., 2002; Hu et al., 2006). However, the associated Lyapunov function for SPSs is usually ε-dependent, which leads to difficulties in generalizing the approaches for normal systems to SPSs.

Recently, the problems of analysis and synthesis for SPSs subject to actuator saturation have attracted more attention. Applying the routine methods for normal systems to SPSs usually leads to ill-conditioned numerical problems (Kokotovic, Khalil, & O'Reilly, 1986). The conventional approaches to avoiding the ill-conditioned problem are based on decomposing the original SPSs into fast and slow subsystems. Liu (2001) proposed a controller design method for SPSs subject to actuator saturation under the assumption that the fast dynamics is stable. Garcia and Tarbouriech (2003) designed

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a composite stabilizing controller and estimated the basin of attraction of SPSs by solving a convex optimization problem. In Xin, Gan, Huang, and Wang (2010); Xin, Wu, Gan, and Qin (2008), some methods to estimate the basin of attraction of SPSs were proposed by introducing the so-called reduced-order adjoint systems. These methods are all based on the decomposition of the original systems, which leads to difficulties for analyzing stability bound of the systems. An alternative approach that is independent of system decomposition was proposed in Lizarraga, Tarbouriech, and Garcia (2005) to avoid the possible ill-conditioned numerical problems. However, the proposed results did not consider the stability bound either. All in all, the problems of stabilization bound and optimization of the basin of attraction of SPSs subject to actuator saturation are still open.

This paper will consider the stabilization bound problem for SPSs subject to actuator saturation. The objective is to propose a state feedback controller design method to achieve a given stabilization bound of the closed-loop system. First, by a Lyapunov function that gives full consideration of the singular perturbation structure, a state feedback controller is designed such that the closed-loop system is asymptotically stable and a basin of attraction is constructed. Then, an optimization problem is formulated to enlarge the basin of attraction of the closed-loop system. Finally, two examples are given to show the effectiveness of the obtained results. The main contributions of the paper are as follows: (1) the proposed method implicitly employs the singular perturbation structure of the SPSs rather than depends on decomposing the original systems into reduced-order subsystems, which provides convenience for stability bound analysis and synthesis of SPSs subject to actuator saturation; (2) a given stabilization bound is one of the design objectives; (3) a novel basin of attraction is constructed, which facilitates the formulation of a well-conditioned convex optimization problem for maximizing the basin of attraction of SPSs.

Notation. The superscript T stands for matrix transposition and the notation M^{-T} denotes the transpose of the inverse matrix of M. $\star$ denotes the block induced by symmetry. For a matrix M, $M_{(i)}$ denotes the ith row of M.

2. Problem formulation

Consider the following system

$$E(\varepsilon)\dot{x}(t) = Ax(t) + B\text{sat}(u(t)), \tag{1}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^n$ is the state, $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $u \in R^m$ is the control input, $E(\varepsilon) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix} \in R^{n\times n}$, $A \in R^{n\times n}$, and $B \in R^{n\times m}$ are constant matrices. $\text{sat}(\cdot)$ is a componentwise saturation map $R^m \mapsto R^m$ defined as:

$$\text{sat}(u_i(t)) = \text{sign}(u_i(t))\min\{1, |u_i(t)|\}, \quad i = 1, 2, \ldots, m. \tag{2}$$

The following state feedback controller is considered to stabilize system (1),

$$u(t) = K(\varepsilon)x(t). \tag{3}$$

Then, we have the following closed-loop system

$$E(\varepsilon)\dot{x}(t) = Ax(t) + B\text{sat}(K(\varepsilon)x(t)). \tag{4}$$

The problem under consideration is formulated as follows:

Problem 1. Given a stabilization bound $\varepsilon_0 > 0$, determine feedback gain matrix $K(\varepsilon)$ and a region $\Omega(\varepsilon) \subseteq R^n$, as large as possible, such that for any initial condition $x_0 \in \Omega(\varepsilon)$, the closed-loop system (4) is asymptotically stable for any $\varepsilon \in (0, \varepsilon_0]$.

Remark 1. In many SPSs, the singular perturbation parameter ε can be measured. In these cases, ε is available for the related synthesis problems, which has attracted much attention (Assawinchaichote & Nguang, 2006; Yang & Zhang, 2009). In Problem 1, under the assumption that the singular perturbation parameter ε is available, the stabilization bound and basin of attraction of system (4) are considered simultaneously.

There are basically two methods handling saturation nonlinearity, namely, the sector bound approach (Hindi & Boyd, 1998) and the convex hull approach (Hu & Lin, 2001). The latter is less conservative than the the former (Hu, Lin, & Chen, 2002). Thus the convex hull approach is used in this paper. We now recall some standard notations and preliminary lemmas.

For a given $F \in R^{m\times n}$, define $\mathcal{L}(F) = \{x \in R^n : |F_{(i)}x| \le 1, i \in [1, m]\}$. Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{D}$. Suppose that these elements of $\mathcal{D}$ are labeled as D_i, $i \in [1, 2^m]$. Denote $D_i^- = I - D_i$. Clearly, $D_i^- \in \mathcal{D}$ if $D_i \in \mathcal{D}$.

The following lemmas will be used in the sequel.

Lemma 1 (Hu & Lin, 2001)**.** *Let $F, H \in R^{m\times n}$. Then, for any $x \in \mathcal{L}(H)$, it holds that $\text{sat}(Fx) \in \text{co}\{D_iFx + D_i^-Hx, i \in [1, 2^m]\}$, where co stands for the convex hull.*

Lemma 2 (Yang & Zhang, 2009)**.** *For a positive scalar ε_0 and symmetric matrices S_1, S_2 and S_3 of appropriate dimensions, if*

$$S_1 \ge 0, \tag{5}$$

$$S_1 + \varepsilon_0 S_2 > 0, \tag{6}$$

$$S_1 + \varepsilon_0 S_2 + \varepsilon_0^2 S_3 > 0, \tag{7}$$

hold, then

$$S_1 + \varepsilon S_2 + \varepsilon^2 S_3 > 0, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{8}$$

Lemma 3 (Yang & Zhang, 2009)**.** *If there exist matrices Z_i ($i = 1, 2, \ldots, 5$) with $Z_i = Z_i^T$ ($i = 1, 2, 3, 4$) satisfying*

$$Z_1 > 0, \tag{9}$$

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \varepsilon_0 Z_5^T \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 \end{bmatrix} > 0, \tag{10}$$

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \varepsilon_0 Z_5^T \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 + \varepsilon_0^2 Z_4 \end{bmatrix} > 0, \tag{11}$$

then

$$E(\varepsilon)Z(\varepsilon) = Z^T(\varepsilon)E(\varepsilon) > 0, \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{12}$$

where $Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}$.

3. Main results

In this section, a state feedback controller is designed and then a convex optimization problem is formulated to enlarge the basin of attraction of the closed-loop system.

3.1. Controller design

Theorem 1. *Given a scalar $\varepsilon_0 > 0$, if there exist matrices $Y \in R^{m\times n}$, $G_1 \in R^{m\times n_1}$, $G_2 \in R^{m\times n_2}$, $Z_1 \in R^{n_1\times n_1}$, $Z_2 \in R^{n_2\times n_2}$, $Z_3 \in R^{n_1\times n_1}$,*

$Z_4 \in R^{n_2\times n_2}, Z_5 \in R^{n_2\times n_1}$ with $Z_i = Z_i^T$ $(i = 1, 2, 3, 4)$, such that LMIs (9)–(11) and

$$Z^T(0)A^T + AZ(0) + (D_iY + D_i^- G(0))^T B^T + B(D_iY + D_i^- G(0)) < 0, \quad i \in [1, 2^m], \tag{13}$$

$$Z^T(\varepsilon_0)A^T + AZ(\varepsilon_0) + (D_iY + D_i^- G(\varepsilon_0))^T B^T + B(D_iY + D_i^- G(\varepsilon_0)) < 0, \quad i \in [1, 2^m], \tag{14}$$

$$\begin{bmatrix} Z_1 & \star \\ G_{1(i)} & 1 \end{bmatrix} \geq 0, \quad i \in [1, m], \tag{15}$$

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \star & \star \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 & \star \\ G_{1(i)} & \varepsilon_0 G_{2(i)} & 1 \end{bmatrix} \geq 0, \quad i \in [1, m], \tag{16}$$

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \star & \star \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 + \varepsilon_0^2 Z_4 & \star \\ G_{1(i)} & \varepsilon_0 G_{2(i)} & 1 \end{bmatrix} \geq 0, \quad i \in [1, m], \tag{17}$$

hold, where $Z(\varepsilon) = \begin{bmatrix} Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\ Z_5 & Z_2 + \varepsilon Z_4 \end{bmatrix}$, $G(\varepsilon) = \begin{bmatrix} G_1 & \varepsilon G_2 \end{bmatrix}$.

Then the controller (3) with $K(\varepsilon) = YZ^{-1}(\varepsilon)$, $Z(\varepsilon) = U_1 + \varepsilon U_2$ stabilizes the system (4) for any $\varepsilon \in (0, \varepsilon_0]$. And the ellipsoid $\Omega(\varepsilon) = \{x | x^T Z^{-T}(\varepsilon)E(\varepsilon)x \leq 1\}$ is a basin of attraction of the closed-loop system.

Proof. From Lemma 2, LMIs (13) and (14) imply

$$Z^T(\varepsilon)A^T + AZ(\varepsilon) + (D_iY + D_i^- G(\varepsilon))^T B^T + B(D_iY + D_i^- G(\varepsilon)) < 0, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{18}$$

Pre- and post-multiplying (18) by $Z^{-T}(\varepsilon)$ and its transpose, respectively, we have

$$A^T Z^{-1}(\varepsilon) + Z^{-T}(\varepsilon)(D_iY + D_i^- G(\varepsilon))^T B^T Z^{-1}(\varepsilon) + Z^{-T}(\varepsilon)A + Z^{-T}(\varepsilon)B(D_iY + D_i^- G(\varepsilon))Z^{-1}(\varepsilon) < 0, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{19}$$

Letting $K(\varepsilon) = YZ^{-1}(\varepsilon)$, $P(\varepsilon) = Z^{-1}(\varepsilon)$, we have

$$\begin{aligned} \Pi_i \triangleq\ & A^T P(\varepsilon) + P^T(\varepsilon)A \\ & + (D_iK(\varepsilon) + D_i^- G(\varepsilon)P(\varepsilon))^T B^T P(\varepsilon) \\ & + P^T(\varepsilon)B(D_iK(\varepsilon) + D_i^- G(\varepsilon)P(\varepsilon)) < 0, \\ & \forall \varepsilon \in (0, \varepsilon_0], i \in [1, 2^m]. \end{aligned} \tag{20}$$

Using Lemma 2 again, LMIs (15)–(17) imply that

$$\begin{bmatrix} Z^T(\varepsilon)E(\varepsilon) & \star \\ G_{(i)}(\varepsilon) & 1 \end{bmatrix} \geq 0, \quad \forall i \in [1, m], \forall \varepsilon \in (0, \varepsilon_0], \tag{21}$$

which is equivalent to

$$\begin{bmatrix} E^{-1}(\varepsilon)Z^T(\varepsilon) & \star \\ G_{(i)}(\varepsilon)E^{-1}(\varepsilon) & 1 \end{bmatrix} \geq 0, \quad \forall i \in [1, m], \forall \varepsilon \in (0, \varepsilon_0]. \tag{22}$$

Pre- and post multiplying (22) by $\mathrm{diag}([E^{-1}(\varepsilon)Z^T(\varepsilon)]^{-1}, 1)$ and its transpose, respectively, we have $\begin{bmatrix} E(\varepsilon)Z^{-1}(\varepsilon) & \star \\ G_{(i)}(\varepsilon)Z^{-1}(\varepsilon) & 1 \end{bmatrix} \geq 0, \forall i \in [1, m], \forall \varepsilon \in (0, \varepsilon_0]$, which implies $E(\varepsilon)Z^{-1}(\varepsilon) \geq Z^{-T}(\varepsilon)G_{(i)}^T(\varepsilon)G_{(i)}(\varepsilon)Z^{-1}(\varepsilon)$, $\forall \varepsilon \in (0, \varepsilon_0]$. Then for any $x \in \Omega(\varepsilon)$, it holds that $x^T Z^{-T}(\varepsilon)G_{(i)}^T(\varepsilon)G_{(i)}(\varepsilon)Z^{-1}(\varepsilon)x \leq 1, \forall i \in [1, m], \forall \varepsilon \in (0, \varepsilon_0]$, which implies that $\Omega(\varepsilon) \subseteq \mathcal{L}(G(\varepsilon)Z^{-1}(\varepsilon))$, that is, $\Omega(\varepsilon) \subseteq \mathcal{L}(G(\varepsilon)P(\varepsilon))$. As a result, by Lemma 1, for any $x \in \Omega(\varepsilon)$, we have

$$Ax + B\mathrm{sat}(K(\varepsilon)x) \in \mathrm{co}\{Ax + B(D_iK(\varepsilon) + D_i^- G(\varepsilon)P(\varepsilon))x, i \in [1, 2^m]\}. \tag{23}$$

By Lemma 3, LMIs (9)–(11) guarantee that (12) holds, which implies

$$E(\varepsilon)P(\varepsilon) = P^T(\varepsilon)E(\varepsilon) > 0, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{24}$$

Define an ε-dependent Lyapunov function

$$V(x) = x^T E(\varepsilon)P(\varepsilon)x. \tag{25}$$

Computing the derivative of $V(x)$ along the trajectories of system (4) and taking into account (20) and (23), we have

$$\begin{aligned} \dot{V}|_{(4)} &= 2(E(\varepsilon)\dot{x})^T P(\varepsilon)x \\ &= 2(Ax + B\mathrm{sat}(K(\varepsilon)x))^T P(\varepsilon)x \\ &\leq \max_{i\in[1,2^m]} 2(Ax + B(D_iK(\varepsilon) + D_i^- G(\varepsilon)P(\varepsilon))x)^T \times P(\varepsilon)x \\ &= \max_{i\in[1,2^m]} x^T \Pi_i x \\ &< 0, \quad \forall \varepsilon \in (0, \varepsilon_0], \forall x \in \Omega(\varepsilon), x \neq 0. \end{aligned} \tag{26}$$

Therefore, the closed-loop system is asymptotically stable for any $x_0 \in \Omega(\varepsilon)$ and any $\varepsilon \in (0, \varepsilon_0]$. And the ellipsoid $\Omega(\varepsilon)$ is a basin of attraction of the closed-loop system. □

Remark 2. LMIs (9) and (10) indicate that $Z_1 > 0$ and $Z_2 > 0$. As a result, the matrix $Z(0) = \begin{bmatrix} Z_1 & 0 \\ Z_5 & Z_2 \end{bmatrix}$ is nonsingular. In addition, the proof of Theorem 1 has shown that $Z(\varepsilon)$ is nonsingular for all $\varepsilon \in (0, \varepsilon_0]$. Then $K(\varepsilon) = YZ^{-1}(\varepsilon)$ is well-defined for all $\varepsilon \in (0, \varepsilon_0]$ and robust with respect to ε. Since $\lim_{\varepsilon\to 0^+} K(\varepsilon) = YZ^{-1}(0)$, if ε_0 is sufficiently small, controller (3) can be reduced to an ε-independent one.

Remark 3. The problem under consideration assumes that the upper bound ε_0 for the singular perturbation parameter is given according to prior information. In fact, the best estimate for the stabilization bound ε_0 can be obtained by a one dimensional search algorithm.

Remark 4. The basin of attraction is usually described by the associated Lyapunov function, which admits an easy way to formulate convex optimization problems. There have been some mature methods for normal systems to optimize the basin of attraction (Boyd, Ghaoui, Feron, & Balakrishnan, 1994; Hu et al., 2002, 2006). This paper constructs an ε-dependent basin of attraction for SPSs for the first time, which gives full consideration to the singular perturbation structure of SPSs and is quite different from that for normal systems. Such a construction facilitates the formulation of well-conditioned convex optimization problems for maximizing the basin of attraction of the closed-loop system.

3.2. Optimization of the basin of attraction

With all the feasible solutions satisfying the LMI conditions of Theorem 1, we are interested in obtaining the largest basin of attraction of the closed-loop system. There are generally two approaches to obtain the largest basin of attraction of the closed-loop system. One is proposed in Boyd et al. (1994), where the size of a set is measured by its volume. The other takes the shape of a set into consideration (Hu et al., 2002, 2006).

Here, we choose the first method and the second one can be obtained in a similar way. Maximizing the volume of the basin of attraction $\Omega(\varepsilon)$ can be reduced to the following optimization problem

$$\begin{aligned} & \min_{S,M,Y,U_1,U_2} \lambda \\ & \text{s.t. (9)–(11) and (13)–(17)}, \\ & \lambda > 0 \quad \text{and} \quad Z^{-T}(\varepsilon)E(\varepsilon) \leq \lambda I. \end{aligned} \tag{27}$$

It can be seen that $Z^{-T}(\varepsilon)E(\varepsilon) \leq \lambda I$ with $\lambda > 0$ is equivalent to

$$\begin{bmatrix} Z^T(\varepsilon)E(\varepsilon) & E(\varepsilon) \\ E(\varepsilon) & \lambda I \end{bmatrix} \geq 0. \quad (28)$$

By Lemma 2, inequality (28) is guaranteed by

$$\begin{bmatrix} Z_1 & \star & \star & \star \\ 0 & 0 & \star & \star \\ I & 0 & \lambda I & \star \\ 0 & 0 & 0 & \lambda I \end{bmatrix} \geq 0, \quad (29)$$

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \star & \star & \star \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 & \star & \star \\ I & 0 & \lambda I & \star \\ 0 & \varepsilon_0 I & 0 & \lambda I \end{bmatrix} \geq 0, \quad (30)$$

and

$$\begin{bmatrix} Z_1 + \varepsilon_0 Z_3 & \star & \star & \star \\ \varepsilon_0 Z_5 & \varepsilon_0 Z_2 + \varepsilon_0^2 Z_4 & \star & \star \\ I & 0 & \lambda I & \star \\ 0 & \varepsilon_0 I & 0 & \lambda I \end{bmatrix} \geq 0. \quad (31)$$

It is easy to see that inequality (29) is equivalent to

$$\begin{bmatrix} Z_1 & \star \\ I & \lambda I \end{bmatrix} \geq 0. \quad (32)$$

Then the optimization problem (27) can be reformulated as the following convex optimization problem

$$\min_{S,M,Y,U_1,U_2} \lambda$$

s.t. (9)–(11) and (13)–(17), (32), (30) and (31). (33)

Remark 5. Analysis and synthesis problems for SPSs subject to actuator saturation have been considered by several researchers. Compared with the existing methods in Garcia and Tarbouriech (2003), Liu (2001), Lizarraga et al. (2005), and Xin et al. (2010, 2008), the advantages of the proposed method are as follows: (1) achieving a given stabilization bound is one of the design objectives; (2) a well-conditioned convex optimization algorithm is proposed to maximize the basin of attraction of the closed-loop system.

4. Examples

This section will illustrate various features of the proposed methods and show their advantages over the existing results.

Example 1. To show the advantages of the proposed methods over the existing results, consider system (1) with

$$E(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad A = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using the method in Lizarraga et al. (2005), we have the following controller

$$u = \begin{bmatrix} -16.3461 & 0.3371 \end{bmatrix} x. \quad (34)$$

Based on the results of Lizarraga et al. (2005), the system is stabilized by (34) if the perturbation parameter ε is small enough. The obtained basin of attraction of the closed-loop system is shown in Fig. 1 (see the solid ellipsoid). When $\varepsilon = 0.02$, the trajectory starting from $[-0.1 \;\; 3.9]^T$ remains in the ellipsoid and converges to the equilibrium point of the system (see the dotted line in Fig. 1). However, when $\varepsilon = 0.06$, the trajectory starting from $[-0.1 \;\; 3.9]^T$ diverges to infinity (see the dashed line in Fig. 1). Similar problem may also arise if the existing methods in Garcia and Tarbouriech (2003), Liu (2001), and Xin et al. (2010, 2008) are used because all of them can not produce an estimate of the stability bound of the closed-loop system. The newly developed results in this paper can overcome this problem.

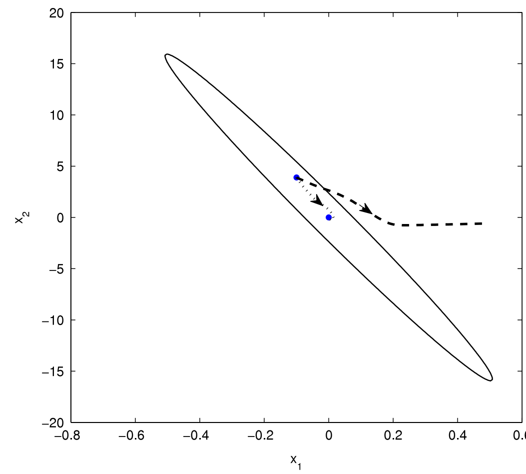

Fig. 1. The basin of attraction of the system under the control of (34) and trajectories starting from $[-0.1 \;\; 3.9]^T$ for $\varepsilon = 0.02$ (the dotted line) and $\varepsilon = 0.06$ (the dashed line), respectively.

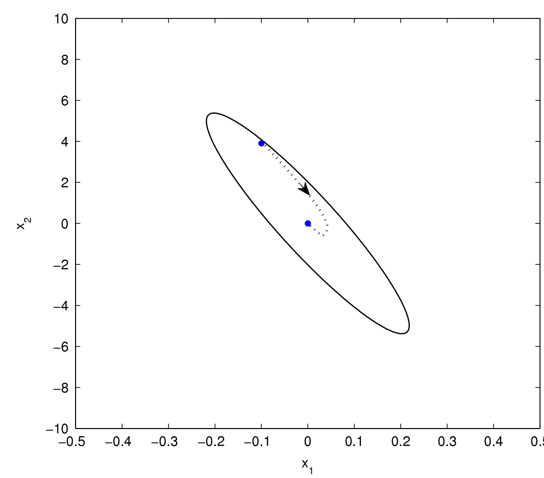

Fig. 2. The basin of attraction of the system with $\varepsilon = 0.06$ under the control of (35) and the converging trajectory starting from $[-0.1 \;\; 3.9]^T$.

Set $\varepsilon_0 = 0.1$. Solving the LMIs in Theorem 1, we have $Z_1 = 0.0071$, $Z_2 = 1.4691$, $Z_3 = 0.8137$, $Z_4 = -0.3860$, $Z_5 = -1.0908$, $G_1 = -0.0824$, $G_2 = 0.1714$, and $Y = \begin{bmatrix} -2.2496 & -6.2612 \end{bmatrix}$. The resulted controller is as follows

$$u = \begin{bmatrix} -2.2496 & -6.2612 \end{bmatrix} Z^{-1}(\varepsilon)x, \quad (35)$$

where $Z(\varepsilon) = \begin{bmatrix} 0.0071 + 0.8137\varepsilon & -1.0908\varepsilon \\ -1.0908 & 1.4691 - 0.3860\varepsilon \end{bmatrix}$.

By Theorem 1, the controller (35) stabilizes the system for any $\varepsilon \in (0, 0.1]$. And the ellipsoid $\Omega(\varepsilon) = \{x | x^T Z^{-1}(\varepsilon)E(\varepsilon)x \leq 1\}$ is a basin of attraction of the closed-loop system. For the case $\varepsilon = 0.06$, the basin of attraction of the system under the control of (35) and the trajectory starting from $x_0 = [-0.1\; 3.9]^T$ are shown in Fig. 2. It can be seen that the trajectory starting from $x_0 = [-0.1\; 3.9]^T \in \Omega$ remains in Ω and converges to the equilibrium point of the system.

Example 2. This example will demonstrate how the proposed method is applied to an inverted pendulum system controlled by a DC motor via a gear train. The model, which was first established

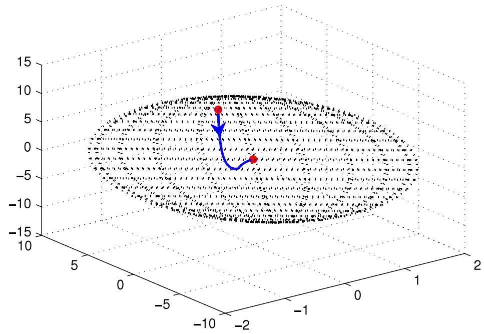

Fig. 3. The basin of attraction Ω and the converging trajectory starting from $x_0 = [-0.4\ \ 1.2\ \ 9]^T \in \Omega$.

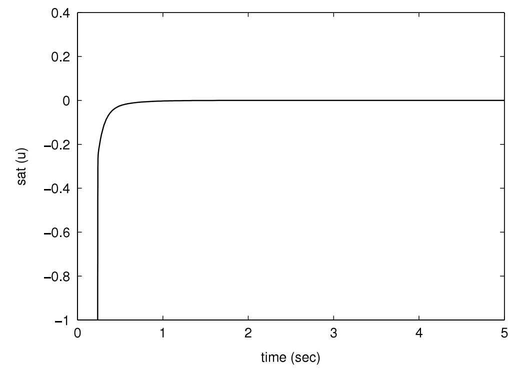

Fig. 4. The control input.

in Zak and Maccarley (1986), is described by

$$\begin{cases}\dot{x}_1(t) = x_2(t),\\ \dot{x}_2(t) = \dfrac{g}{l}\sin x_1(t) + \dfrac{NK_m}{ml^2}x_3(t),\\ L_a\dot{x}_3(t) = -K_bNx_2(t) - R_ax_3(t) + u(t),\end{cases} \tag{36}$$

where $x_1(t) = \theta_p(t)$ denotes the the angle (rad) of the pendulum from the vertical upward, $x_2(t) = \dot{\theta}_p(t)$, $x_3(t) = I_a(t)$ denotes the current of the motor, $u(t)$ is the control input voltage, K_m is the motor torque constant, K_b is the back emf constant, N is the gear ratio, and L_a is the inductance which is usually a small positive constant. The parameters for the plant are as follows: $g = 9.8$ m/s^2$N = 10$, $l = 1$ m, $m = 1$ kg, $K_m = 0.1$ Nm/A, $K_b = 0.1$ Vs/rad, $R_a = 1\ \Omega$ and $L_a = 0.05$ H and the input voltage is required to satisfy $|u| \le 1$. Note that L_a represents the singular perturbation parameter of the system. Substituting the parameters into (36), we have

$$\begin{cases}\dot{x}_1(t) = x_2(t),\\ \dot{x}_2(t) = 9.8\sin x_1(t) + x_3(t),\\ \varepsilon\dot{x}_3(t) = -x_2(t) - x_3(t) + u,\end{cases} \tag{37}$$

where $\varepsilon = L_a$.

The equilibrium point of system (37), that is, $x_e = [0\ 0\ 0]^T$ corresponds to the upright rest position of the inverted pendulum. We will design a controller to balance the pendulum around its upright rest position.

The linearized system of (37) can be transformed into the form of (1) with

$$E(\varepsilon) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \varepsilon\end{bmatrix}, \qquad A = \begin{bmatrix}0 & 1 & 0\\ 9.8 & 0 & 1\\ 0 & -1 & -1\end{bmatrix},$$

$$B = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

Solving the optimization problem (33) with $\varepsilon_0 = 0.1$, we have $Z_1 = \begin{bmatrix}2.8775 & -10.3632\\ -10.3632 & 37.4051\end{bmatrix}$, $Z_2 = 6.7152$, $Z_3 = \begin{bmatrix}6.1643 & -17.0102\\ -17.0102 & 47.5824\end{bmatrix}$, $Z_4 = -0.5595$, $Z_5 = \begin{bmatrix}-3.8177 & 8.0017\end{bmatrix}$, $G_1 = \begin{bmatrix}-0.0176 & -0.2207\end{bmatrix}$, $G_2 = 0.0826$, $Y = \begin{bmatrix}-15.2684 & 42.5480 & -66.0277\end{bmatrix}$, $\lambda = 177.1625$.

Taking into account $\varepsilon = 0.05$, we have the controller gain matrix $K = \begin{bmatrix}-2.9201 & -0.8130 & -0.0446\end{bmatrix} \times 10^3$and the basin of attraction of the closed-loop system $\Omega = \{x \in R^3 | x^TPx \le 1\}$, where $P = \begin{bmatrix}153.7649 & 42.9750 & 1.8181\\ 42.9750 & 12.0363 & 0.5066\\ 1.8181 & 0.5066 & 0.0291\end{bmatrix}$.

The basin of attraction of the closed-loop system and the trajectory starting from $x_0 = [-0.4\ \ 1.2\ \ 9]^T$ are shown in Fig. 3 and the control input is shown in Fig. 4. It can be seen from Fig. 3 that the trajectory starting from $x_0 = [-0.4\ \ 1.2\ \ 9]^T \in \Omega$ remains in Ω and converges to the equilibrium point of system (37), that is, $x_e = [0\ 0\ 0]^T$.

5. Conclusion

In this paper, we have considered the stabilization bound problem for singularly perturbed systems subject to actuator saturation. We first proposed a state feedback stabilization controller design method and constructed an ε-dependent basin of attraction, by which a convex optimization algorithm was formulated to maximize the basin of attraction of the closed-loop system. The results presented in this paper generalize the existing ones for normal systems. Finally, examples were given to demonstrate the utility of the proposed methods and the contributions of the results.

References

Abed, E. H. (1985). A new parameter estimate in singular perturbations. *Systems & Control Letters*, *6*(3), 193–198.

Assawinchaichote, W., & Nguang, S. K. (2006). Fuzzy H_∞ output feedback control design for singularly perturbed systems with pole placement constraints: an LMI approach. *IEEE Transactions on Fuzzy Systems*, *14*(3), 361–371.

Boyd, S., Ghaoui, L. E., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. Philadelphia, PA: SIAM.

Cao, Y. Y., Lin, Z., & Ward, D. G. (2002). An antiwindup approach to enlarging domain of attraction for linear systems subject to actuator saturation. *IEEE Transactions on Automatic Control*, *47*(1), 140–145.

Cao, L., & Schwartz, H. M. (2004). Complementary results on the stability bounds of singularly perturbed systems. *IEEE Transactions on Automatic Control*, *49*(11), 2017–2021.

Chiou, J. S., Kung, F. C., & Li, T. H. S. (1999). An infinite ε-bound stabilization design for a class of singularly perturbed systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *46*(12), 1507–1510.

Feng, W. (1988). Characterization and computation for the bound ε^* in linear time-invariant singularly perturbed systems. *Systems & Control Letters*, *11*(3), 195–202.

Garcia, G., & Tarbouriech, S. (2003). Control of singularly perturbed systems by bounded control. In *2003 America control conference* (pp. 4482–4487).

Hindi, H., & Boyd, S. (1998). Analysis of linear systems with saturating using convex optimization. In *the 37th IEEE conference on decision and control* (pp. 903–908).

Hu, T., & Lin, Z. (2001). *Control systems with actuator saturation: analysis and design*. Boston, MA: Birkhäuser.

Hu, T., Lin, Z., & Chen, B. M. (2002). *Automatica*, *38*(2), 351–359.

Hu, T., Teel, A. R., & Zaccarian, L. (2006). Stability and performance for saturated systems via quadratic and non-quadratic Lyapunov functions. *IEEE Transactions on Automatic Control*, *51*(11), 1770–1786.

Kokotovic, P. V., Khalil, H. K., & O'Reilly, J. (1986). *Singular perturbation methods in control: analysis and design*. New York: Academic.

Li, T. H. S., & Li, J. H. (1992). Stabilization bound of discrete two-time-scale systems. *Systems & Control Letters*, *18*(6), 479–489.

Lin, Z., & Saberi, A. (1993). Semi-global exponential stabilization of linear systems subject to input saturation via linear feedbacks. *Systems & Control Letters*, *21*(1), 225–239.

Liu, P. L. (2001). Stabilization of singularly perturbed multiple-time-delay systems with a saturating actuator. *International Journal of Systems Science*, *32*(8), 1041–1045.

Liu, W. Q., Paskota, M., Sreeram, V., & Teo, K. L. (1997). Improvement on stability bounds for singularly perturbed systems via state feedback. *International Journal of Systems Science*, *28*(6), 571–578.

Lizarraga, I., Tarbouriech, S., & Garcia, G. (2005). Control of singularly perturbed systems under actuator saturation. In *16th IFAC world congress: vol. 16* (pp. 243–248).

Saydy, L. (1996). New stability/performance results for singularly perturbed systems. *Automatica*, *32*(6), 807–818.

Sen, S., & Datta, K. B. (1993). Stability bounds of singularly perturbed systems. *IEEE Transactions on Automatic Control*, *38*(2), 302–304.

Xin, H., Gan, D., Huang, M., & Wang, K. (2010). Estimating the stability region of singular perturbation power systems with saturation nonlinearities: an linear matrix inequality based method. *IET Control Theory & Applications*, *4*(3), 351–361.

Xin, H., Wu, D., Gan, D., & Qin, J. (2008). A method for estimating the stability region of singular perturbation systems with saturation nonlinearities. *Aata Autamatica Sinica*, *34*(12), 1549–1555.

Yang, C., & Zhang, Q. (2009). Multi-objective control for *T–S* fuzzy singularly perturbed systems. *IEEE Transactions on Fuzzy Systems*, *17*(1), 104–115.

Zak, S. H., & Maccarley, C. A. (1986). State-feedback control of non-linear systems. *International Journal of Control*, *43*(5), 1497–1514.

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