II, Sec. 6 section 1 and 2]). Hence, $P_\xi = P_0 + O(\xi)$ where P_0 is such that $P_0^\dagger \sigma_x^{lk} P_0 = \sigma_z^{lk}$ is real and diagonal. $\sigma_z^{lk} = |l\rangle\langle l| - |k\rangle\langle k|$. We find $\forall\tau$: $\Sigma_x^{lk}(\tau) = \sigma_x^{lk} + O(\xi)$, where $O(\xi)$ is a first-order term in ξ and a bounded function of τ.

ACKNOWLEDGMENT

The authors thank M. Mirrahimi and A. Goun for discussions.

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Robust Control of Constrained Linear Systems With Bounded Disturbances

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***Abstract*—This technical note develops a novel robust control algorithm for linear systems subject to additive and bounded disturbances. The approach is based on the constraint tightening method. While this problem can be tackled using existing robust model predictive control techniques, the proposed method has an advantage in that it is computationally efficient and avoids the need to solve repeatedly an online optimization problem, while the optimization problem solved at initialization is a simple linear programming problem. The algorithm elaborated in this technical note guarantees convergence to a minimal disturbance invariant set, and the terminal predicted state constraint set is allowed to be larger than the minimal disturbance invariant set. As an illustration, the developed algorithm is applied to constrained roll control of a ship operating in a wave field. Simulation results show that the proposed approach reduces the ship roll motion while the input and dynamic stall constraints are satisfied.**

***Index Terms*—Model predictive control (MPC).**

I. INTRODUCTION

In this technical note, we consider a control problem for constrained discrete-time linear systems that are subject to bounded additive disturbances. Our goal is to provide a control method that enforces specified state and input constraints in the presence of disturbances and steers state trajectories to a given target set.

This problem has been studied employing invariant set methods (see [1], [2] and references therein) and optimization based control strategies such as model predictive control (MPC) [3]. In the MPC literature, one approach relies on sufficient contractivity of the open-loop system [4]. MPC strategies in which a deterministic control sequence is optimized, may result in a small domain of attraction hence another approach has been proposed in which optimization is performed over feedback policies [5]. However, optimization over arbitrary feedback policies, in the presence of constraints, may be especially difficult. As an alternative, affine feedback policies were employed where the state feedback gain(s) are calculated off-line and optimization was performed over constant offset terms [6]–[8].

Many robust MPC schemes are based on tightening the constraints (on states and controls) over the prediction horizon. This method was proposed initially in [9] as well as in [6], [10]–[12]. Based on constrained tightening approach, a robust MPC for nonlinear systems subject to bounded disturbances has been introduced in [13], that guarantees convergence to an ellipsoidal disturbance invariant set. An alter-

Manuscript received January 09, 2011; revised June 22, 2011; accepted March 14, 2012. Date of publication April 19, 2012; date of current version September 21, 2012. This work is supported in part by the National Science FoundationNSF ECS-0501284 and ONR N00014-05-1-0537. Recommended by Associate Editor T. Zhou.

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Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2012.2192362

native approach for handling constraints in the presence of bounded disturbances and parametric uncertainty is the technique of quadratic boundedness [15]. For linear systems with model uncertainties where state-space matrices lie in a polytope, [14] provides an off-line MPC approach.

In this technical note, we introduce a robust control method for linear discrete-time systems subject to mixed input-state constraints. The proposed scheme, which is also based on the constraint tightening approach, has several special features.

First, it guarantees convergence to a robust invariant set in N steps, N being the length of prediction horizon. Namely, after N steps, the state is in the maximal robust invariant set[1] and the proposed controller is a linear state feedback. We note that in the presence of disturbances, asymptotic convergence to an invariant set is achievable using MPC for linear systems [11] and nonlinear systems [13]. However, to the authors' knowledge, convergence in N steps to the maximal robust invariant set in the presence of persistent additive disturbances has not been previously reported for MPC methods.

Second, our method does not involve repeated online optimization to determine the control action and it does not use a performance cost/index. Our approach is similar to MPC in a broader sense that we update a sequence of control actions over a receding horizon and we guarantee that constraints are enforced. But it is different from on-line MPC, in that the control action is determined by simple linear operations on known data at each sampling time and therefore it is computationally effective. Our approach is also different from off-line or explicit MPC because we do not compute the control as a solution to a finite horizon optimal control problem for a specified cost function but solve a lower complexity linear programming problem online at the initialization thus preserving online reconfiguration ability for changes in problem data. While constraints are enforced by both robust MPC and with our approach, the receding horizon cost optimization in robust MPC can provide additional flexibility to improve transient response characteristics of the system.

Third, our control algorithm provides a constraint feasible solution at each time instant (in the form of control and state sequences) by linearly combining previous state and control sequences. Thus the recursive feasibility and constraint enforcement are guaranteed.

Fourth, the minimal invariant set corresponding to the off-line calculated state feedback is an attractor, i.e., all trajectories will converge to this set. As we will demonstrate, convergence to the target set is guaranteed as long as the target set contains the minimal invariant set. Moreover, our method requires no explicit knowledge of the minimal invariant set.

It should be noted that it is the specifics of our algorithm and the combination of all the features that define the novelty of the approach described in this technical note.

This technical note goes significantly beyond our previous conference paper [16]. Specifically, in our present technical note we establish the convergence to robust invariant set in a finite number of steps. Our previous result (Theorem 3.2 in [16]) only established the asymptotic convergence, as opposed to finite-time convergence, and only its formulation but not the actual proof was included in [16]. Our treatment of the application example (ship roll stabilization) in Section IV is also expanded compared to [16] and we include the comparison with the robust MPC.

II. Problem Statement

Consider a class of linear, time-invariant, discrete-time systems described by

$$x^+ = Ax + Bu + w,\ x(k) \in \mathbb{R}^n,\ u(k) \in \mathbb{R}^m,\ w(k) \in \mathbb{R}^n \quad (1)$$

[1]The maximal robust invariant set is the largest set in which the linear feedback law satisfies constraints in the presence of disturbances.

where x, u and w are, respectively, the state, control and disturbance vectors; x^+ denotes the successor state of x, and $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. We assume that the disturbance w belongs to a polytope W, the control and state are subject to hard constraints, i.e.

$$(u, x) \in \Omega \subset \mathbb{U} \times \mathbb{X} \text{ and } w \in W \quad (2)$$

where $\mathbb{U}$ and W are (convex, compact) polytopes, containing the origin in their interior, and $\mathbb{X}$ is a (convex) closed polyhedron. Finally, a target set $\mathbb{X}_t$ is given by

$$\mathbb{X}_t = \{x \in \mathbb{R}^n | Yx \leq q\},\ Y \in \mathbb{R}^{r\times n},\ q \in \mathbb{R}^r. \quad (3)$$

We assume that $\mathbb{X}_t$ is bounded (so it is a polytope) and $0 \in int(\mathbb{X}_t)$. The control objective is to find u that steers the state into the target set $\mathbb{X}_t$. Moreover, we assume the existence of a feedback gain matrix $K \in \mathbb{R}^{m\times n}$ such that $A_K = A + BK$ is an exponentially stable matrix and the minimal robust invariant set[2] F_K for the system $x^+ = A_K x + w$, defined in [17], satisfies

$$F_K \subseteq \mathbb{X}_t. \quad (4)$$

In the sequel, we use the Pontryagin difference [17] which, for two sets S and T, is defined as $S \sim T = \{x | x + t \in S,\ \forall\, t \in T\}$.

III. Robust Control Algorithm

For any initial state $x \in \mathbb{X}$, the following control sequence:

$$\mathbf{u}^*(x) := \{u_0^*(x),\ u_1^*(x), \ldots, u_{N-1}^*(x)\}$$

and associated state sequence

$$\mathbf{x}^*(x) := \{x_0^*(x),\ x_1^*(x), \ldots, x_N^*(x)\}$$

are feasible if they satisfy the following set of constraints $C(x)$:

$$\begin{aligned} x_0^*(x) &= x, \\ x_{i+1}^*(x) &= Ax_i^*(x) + Bu_i^*(x),\ i = 0, \ldots, N-1 \quad (5) \\ \Omega_0 &= \Omega, \\ \Omega_{i+1} &= \Omega_i \sim [\,K^T \quad I\,]^T A_K^i W,\ i = 0, \ldots, N-1, \\ (u_i^*(x), x_i^*(x)) &\in \Omega_i,\ i = 0, \ldots, N-1 \quad (6) \\ x_N^*(x) &\in \mathbb{X}_f \quad (7) \end{aligned}$$

where $\mathbb{X}_f$ is a robust invariant set for the system

$$x^+ = A_K x + w \quad (8)$$

with $w \in A_K^N W$, i.e., $A_K \mathbb{X}_f + A_K^N W \subset \mathbb{X}_f$ which satisfies the constraint

$$[\,K^T \quad I\,]^T \mathbb{X}_f \subset \Omega_N. \quad (9)$$

Let us assume that for an initial state $x(0)$, $\mathbf{u}^*(x(0))$ and $\mathbf{x}^*(x(0))$ are feasible control and state sequences. Computing these initial sequences involves finding a point inside the polyhedron defined by (5)–(7). It can be shown that the point can be determined by solving a linear programming problem, which can be further decomposed into a sequence of N programming problems of dimension m [21]. Now we propose the following algorithm, where at each time instant k, the feasible control sequence $\mathbf{u}^*(x(k))$ is constructed using the feasible

[2]The robust invariant set F_K for the system $x^+ = A_K x + w$ is minimal if for all closed robust invariant sets X such that $A_K X + W \subset X$, it follows that $F_K \subset X$[17].

control and state sequences $\mathbf{u}^*(x(k-1))$ and $\mathbf{x}^*(x(k-1))$, where $x(k)$ is the observed state at the time instant k:

$$\begin{aligned} u_i^*(x(k)) &= u_{i+1}^*(x(k-1)) \\ &\quad + K(x_i^*(x(k)) - x_{i+1}^*(x(k-1))), \\ &\quad \text{for } i = 0,\dots,N-2, \\ u_{N-1}^*(x(k)) &= Kx_{N-1}^*(x(k)); \end{aligned} \tag{10}$$

$$\begin{aligned} x_0^*(x(k)) &= x(k), \\ x_{i+1}^*(x(k)) &= Ax_i^*(x(k)) + Bu_i^*(x(k)), \\ &\quad i = 0,\dots,N-1. \end{aligned} \tag{11}$$

At each time instant, the first element of the feasible control sequence is applied as the control signal, thus we form the robust control law as

$$u(k) = \kappa_N^*(x(k)) := u_0^*(x(k)). \tag{12}$$

The proposed algorithm is summarized as follows:

1) At time instant $k = 0$, find feasible control and state sequences $\mathbf{u}^*(x(0))$ and $\mathbf{x}^*(x(0))$ that satisfy the set of constraints $C(x(0))$, at time instant $k = 0$.
2) At time instant $k > 0$, given control and state sequences $\mathbf{u}^*(x(k-1))$ and $\mathbf{x}^*(x(k-1))$, observe the state $x(k)$ at time instant k and calculate control and state sequences $\mathbf{u}^*(x(k))$ and $\mathbf{x}^*(x(k))$ according to (10) and (11) as follows:
   a) Set $i = 0$ and $x_i^*(x(k)) = x(k)$.
   b) Set
   $$u_i^*(x(k)) = u_{i+1}^*(x(k-1)) + K(x_i^*(x(k)) - x_{i+1}^*(x(k-1)))$$
   if $i \neq N-1$. Otherwise, set $u_i^*(x(k)) = Kx_i^*(x(k))$.
   c) Set $x_{i+1}^*(x(k)) = Ax_i^*(x(k)) + Bu_i^*(x(k))$.
   d) If $i = N-1$, terminate. Otherwise, set $i = i+1$ and go to 2b.
3) Apply the first element of the control sequence $\mathbf{u}^*(x(k))$, i.e., $u_0^*(x(k))$.

Theorem 3.1: Suppose the set of constraints $C(x(0))$ is satisfied with the feasible control, $\mathbf{u}^*(x(0))$, and state, $\mathbf{x}^*(x(0))$, sequences. Then the state and input trajectories of the system (1) with the control law defined by (12) satisfy the input and state constraints (2). Furthermore, the set of constraints $C(x(k))$ is satisfied by the control and state sequences $\mathbf{u}^*(x(k))$ and $\mathbf{x}^*(x(k))$, defined by (10) and (11), for all $k > 0$.

Proof: See [18]. ■

Remark 3.1: Note that since more than one feasible solution exists that satisfies the constraint $C(x(0))$, the initial feasible control and state sequences can be chosen such that they optimize a secondary objective subject to the constraint $C(x(0))$.

Remark 3.2: According to [16], $x_{N-1}^*(x^+) - x_N^*(x) \in A_K^{N-1}W$. Since $x_N^*(x^+) = A_K x_{N-1}^*(x^+)$, $x_N^*(x^+) \in \{A_K x_N^*(x)\} + A_K^N W$. This is achieved if the set $\mathbb{X}_f$ is robust invariant with respect to the disturbance set $A_K^N W$. Note that $A_K^N W \rightarrow \{0\}$ as $N \rightarrow \infty$. Therefore, for long length of prediction horizon N, $\mathbb{X}_f$ needs to be robust invariant with respect to a smaller disturbance set.

We now proceed to establish that the algorithm (12) will in fact converge in N steps to a robust invariant set for system (1).

Let $\mathbb{X}_f^i$, $i = 0,\dots,N$ be defined as

$$\mathbb{X}_f^{i+1} = \mathbb{X}_f^i + A_k^{N-i-1}W,\ i = 0,\dots,N,\ \mathbb{X}_f^0 := \mathbb{X}_f. \tag{13}$$

Lemma 3.1: The sets $\mathbb{X}_f^i$, $i = 0,\dots,N$, are robust positive invariant with respect to the system $x^+ = A_k x + A_k^{N-i}w$, $w \in W$. Moreover, $\begin{bmatrix} K \\ I \end{bmatrix} \mathbb{X}_f^i \subset \Omega_{N-i}$.

Proof: Proceeding by induction, the statement of the lemma is true for $\mathbb{X}_f^0$, by definition. Now assume the statement holds for $\mathbb{X}_f^i$. Then

$$\begin{aligned} A_k\mathbb{X}_f^{i+1} + A_k^{N-i-1}W &\subset A_k\mathbb{X}_f^i \\ &\quad + A_k^{N-i}W + A_k^{N-i-1}W \subset \mathbb{X}_f^i + A_k^{N-i-1}W \subset \mathbb{X}_f^{i+1}. \end{aligned} \tag{14}$$

Therefore, $\mathbb{X}_f^{i+1}$ is a robust positive invariant set for the system $x^+ = A_k x + A_k^{N-i-1}w$, $w \in W$. The first relation in (14) comes from (13) while the second relation follows from induction hypothesis. For the second part of statement of lemma, again we proceed by induction. The statement holds true for $\mathbb{X}_f^0$. Assuming it is true for $\mathbb{X}_f^i$, from (13) we have

$$\begin{aligned} \begin{bmatrix} K \\ I \end{bmatrix} \mathbb{X}_f^{i+1} &= \begin{bmatrix} K \\ I \end{bmatrix} \mathbb{X}_f^i + \begin{bmatrix} K \\ I \end{bmatrix} A_k^{N-i-1}W \\ &\subset \Omega_{N-i} + \begin{bmatrix} K \\ I \end{bmatrix} A_k^{N-i-1}W \subset \Omega_{N-i-1} \end{aligned} \tag{15}$$

where the last relation follows from (6). ■

Theorem 3.2: Suppose $x(0)$ is the initial state of the system (1) with the control law defined by (12), and $x(\cdot)$ and $u(\cdot)$ denote the state and control trajectories, respectively. Then $x(N) \in \mathbb{X}_f^N$ and $\kappa_N^*(x(N)) = Kx(N)$.

Proof: First we prove that

$$x_i^*(x(N-i)) \in \mathbb{X}_f^{N-i},\ i = 0,\dots,N. \tag{16}$$

Proceeding by induction, (7) implies $x_N^*(x(0)) \in \mathbb{X}_f^0$. Therefore, (16) holds for $i = N$. Now assume $x_j^*(x(N-j)) \in \mathbb{X}_f^{N-j}$, $j = i,\dots,N$. Considering the state evolution (11) and control update (10), we have

$$\begin{aligned} x_{i+1}^*(x^+) &= Ax_i^*(x^+) + Bu_i^*(x^+) \\ &= Ax_i^*(x^+) + Bu_{i+1}^*(x) + BK(x_i^*(x^+) - x_{i+1}^*(x)) \\ &= A_K(x_i^*(x^+) - x_{i+1}^*(x)) + x_{i+2}^*(x), \\ &\quad i = 0,\dots,N-2 \end{aligned} \tag{17}$$

where the last equality is achieved by adding and subtracting $A_K x_{i+1}^*(x)$ and using (11). From (11), we have $x_0^*(x^+) - x_1^*(x) = x^+ - Ax - Bu_0^*(x) = w_0 \in W$, and using (17) it can be easily shown that $x_i^*(x^+) - x_{i+1}^*(x) = A_K^i w_0 \in A_K^i W$, $i = 0,\dots,N-1$. Hence

$$x_{i-1}^*(x(N-i+1)) \in x_i^*(x(N-i)) + A_K^{i-1}W.$$

According to the hypothesis of induction, $x_{i-1}^*(x(N-i+1)) \in \mathbb{X}_f^{N-i} + A_K^{i-1}W$. From (13) we have $x_{i-1}^*(x(N-i+1)) \in \mathbb{X}_f^{N-i+1}$, and induction is complete. Hence, (16) holds for $i = 0,\dots,N$ and therefore, $x(N) = x_0^*(x(N)) \in \mathbb{X}_f^N$. According to (10) if $u_i^*(x(N-i-1)) = Kx_i^*(x(N-i-1))$ then $u_{i-1}^*(x(N-i)) = Kx_{i-1}^*(x(N-i))$. According to (10), $u_{N-1}^*(x(1)) = Kx_{N-1}^*(x(1))$. Hence, by induction, we conclude $\kappa_N^*(x(N)) = u_0^*(x(N)) = Kx_0^*(x(N)) = Kx(N)$. ■

Theorem 3.2, shows that the proposed controller steers state trajectories to a robust invariant set where states inside the set and corresponding linear state feedback satisfy the constraints. Moreover, convergence occurs after N steps. This, in turn, guarantees state convergence to a minimal invariant set expressed as [17]

$$F_K = \sum_{i=0}^{\infty} A_K^i W. \tag{18}$$

The following theorem states convergence property of the proposed algorithm:

Theorem 3.3: If for an initial state $x(0)$, there exist control and state sequences satisfying the set of constraints $C(x(0))$, then the set F_K is robustly attractive (all trajectories converge to F_K despite disturbances) for the controlled uncertain system

$$x^+ = Ax + B\kappa_N^*(x) + w \tag{19}$$

where $w \in W$. Furthermore, the region of attraction is $R = \{x \in \mathbb{R}^n \mid C(x) \text{ is feasible}\}$.

Proof: Here we provide a sketch of the proof. Assuming for an initial state $x(0)$, there exist control and state sequences satisfying the set of constraints $C(x(0))$, then according to Theorem 3.1 the set of constraints $C(x(k))$ is feasible for all $k > 0$. Therefore, the control law (12) satisfies constraints for all instants of the time. According to Theorem 3.2, after N time steps $x(N) \in \mathbb{X}_f^N$ where $\mathbb{X}_f^N$ is robust positive invariant set for the system $x^+ = A_k x + w,\ w \in W$, according to Lemma 3.1. Based on the structure of the controller in (10) and (11), $\kappa_N^*(x(k)) = Kx(k)$ for $k > N$. This linear feedback controller that is the result of the proposed control strategy for $k > N$ satisfies constraints because $\begin{bmatrix} K \\ I \end{bmatrix} \mathbb{X}_f^N \subset \Omega$, according to Lemma 3.1. The control law $u(k) = Kx(k)$ reduces the system to $x^+ = A_k x + w,\ w \in W$. Therefore, all trajectories converge to F_K despite disturbances. ■

Remark 3.3: According to (4), $F_K \subset \mathbb{X}_t$ and F_K is robustly attractive for system (19). Hence, trajectories converge to the target set $\mathbb{X}_t$. If F_K is contained in the interior of the target set, then convergence to the target set occurs in finite time.

Remark 3.4: The important feature of the proposed method is that, after N steps, the state is guaranteed to be in a robust invariant set $\mathbb{X}_f^N$ for system (19) with $w \in W$, and our controller reduced to a linear state feedback. This property is not available in the usual MPC based methods. Once the state is inside a robust invariant set, the linear feedback guarantees convergence to F_K. Moreover, no repeated optimization or minimal robust invariant set approximation is involved, while in MPC based methods [6], [11] attraction to F_K is achieved by repeatedly solving online an optimization problem.

Remark 3.5: The proposed robust control method may be viewed as based on tightening constraints, at each time instance over the prediction horizon, by $A_K^i W$. In this respect, the proposed scheme is similar to [4], [12]. However, the advantage of the proposed method is that it does not require the terminal constraint set $\mathbb{X}_f$ to be a subset of the desired target set $\mathbb{X}_t$. In fact, the target set $\mathbb{X}_t$ is only required to contain the minimal robust invariant set F_K, i.e., $F_K \subset \mathbb{X}_t$, in order to be attractive.

Remark 3.6: Note that the control law (12) represents the first element of the feasible control sequence that is updated by laws (10) and (11). Even though only the first element of the control sequence is used, it is necessary to perform the updates (10) and (11) to be able to calculate the control for future sample time instants.

Remark 3.7: In [18], we showed that the approach of this technical note can provide a larger region of attraction compared to the method proposed in [11].

IV. Control of Ship Fin Stabilizer

In this section, we illustrate an application of the proposed algorithm to a ship roll control problem. We consider the fin stabilizer control as a robust control problem, where the linear dynamics of the system are affected by a bounded additive disturbance, and we employ the robust control algorithm proposed in Section III, which does not require repeated on-line solution of an optimization problem.

The following linear equations describe the roll motion expressed in a frame fixed at the center of gravity of the ship [19]:

$$\dot{\phi} = p,\ I_{\phi\phi}\dot{p} + Dp + G\phi = \tau_c + \tau_w \tag{20}$$

where ϕ is the roll angle, p is the roll rate, τ_c is the control moment produced by the fins, and τ_w is the wave excitation moment. Moreover, $I_{\phi\phi}$ is the total roll inertia about the axis along the ship longitudinal direction, D is the equivalent linear damping (which accounts for potential and viscous effects), and G is the linear roll restoring coefficient.

For a ship fin stabilizer, the effective angle of attack can be calculated by $\alpha_e = -\alpha_{pu} - \alpha_m$, where α_m is the mechanical angle of the fin (control input) and α_{pu} is the flow angle induced by the combination of forward speed, U, and roll rate, p. It is calculated as follows:

$$\alpha_{pu} = arctan\left(\frac{r_f p}{U}\right) \approx \frac{r_f}{U} p. \tag{21}$$

We consider two set of constraints:

- Input constraint which reflects saturation of the mechanical angle of the fin

$$|\alpha_m| \le \alpha_{sat}. \tag{22}$$

- Input-state constraint that is aimed at preventing dynamic stall

$$|\alpha_e| = |\frac{r_f}{U} p + \alpha_m| \le \alpha_{stall}. \tag{23}$$

To proceed with the controller design and performance evaluation of the proposed system, we use the vessel model introduced in [20] where parameters for (20)–(23) are given. The vessel travels at $U = 15$ kts forward speed with a magnitude constraint for the mechanical angle of the fin of 0.436 rad, and a magnitude constraint for the angle of attack of 0.41 rad. Moreover, the coefficients in (20), (23) are: $I_{\phi\phi} = 3.4263 \times 10^6\ \mathrm{kgm^2/rad}$, $D = 0.5 \times 10^6\ \mathrm{kgm^2/(rad/sec)}$, $G = 3.57 \times 10^9\ \mathrm{Nm/rad}$, $r_f = 4.22$ m. A discrete-time model of (20), with sampling period T_s=0.1 sec, is

$$x(k+1) = A_d x(k) + B_d u(k) + B_w \tau_w(k) \tag{24}$$

where $x = [\phi\ p]^T$, $u = \alpha_m$ and

$$A_d = \begin{bmatrix} 0.99 & 0.095 \\ -0.08 & 0.90 \end{bmatrix}, \quad B_d = \begin{bmatrix} -0.007 \\ -0.142 \end{bmatrix},$$
$$B_w = \begin{bmatrix} 0.004 \\ 0.095 \end{bmatrix}. \tag{25}$$

Assuming $|\tau_w| \le 0.2 I_{\phi\phi}$, according to the general formulation (1), the disturbance set W is

$$W = \{B_w w, |w| \le .2 I_{\phi\phi}\}.$$

The feedback gain $K = [-6.31 - 3.66]$ is designed using LQR technique with weight $R = 10$ for control input and the weight $Q = diag[10\ 2]$ for the states. With the designed feedback gain K, the corresponding minimal invariant set is a subset of the target set

$$\mathbb{X}_t = \{(\phi\ p) | \phi \in [-0.02\ 0.02],\ p \in [-0.06\ 0.06]\}$$

for the disturbance set W. Considering the constraints (22) and (23), the sets $\Omega_i,\ i = 1, \ldots, N$ are defined as follows:

$$\Omega_i = \left\{ (u, x) \left| \begin{aligned} &|c_u u + c_x x| \le \alpha_{stall} - \sum_{j=0}^{i-1} h_{A_k^j W}(c_x + c_u K) \\ &|u| \le \alpha_{sat} - \sum_{j=0}^{i-1} h_{A_k^j W}(K) \end{aligned} \right. \right\} \tag{26}$$

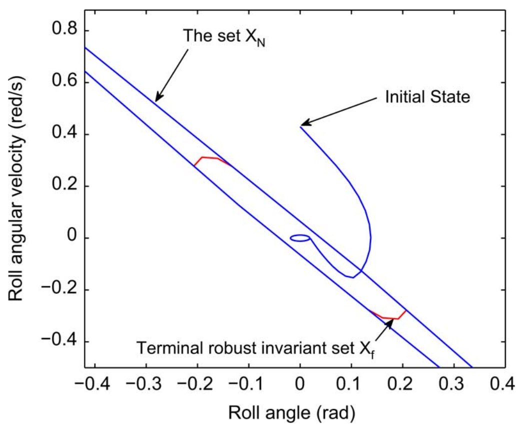


Fig. 1. Trajectory of the system with initial condition $[\phi \;\; p] = [0 \text{ rad}, \; 0.45 \text{ rad/sec}]$.

where

$$c_u = 1, \quad c_x = [0 \; \frac{r_f}{U}] \tag{27}$$

and for a set $S \subset \mathbb{R}^n$, $h_S(\cdot)$ denotes its support function, see e.g., [17]. Consequently, $h_{A_k^j W}$ stands for the support function for the set $A_k^j W$. The value of $N = 10$ was chosen to provide large domain of attraction. Moreover, for this example, the set $\mathbb{X}_f$ in the robust control algorithm is chosen as the maximal robust invariant set. The set $\mathbb{X}_f$ is contained in the following set as shown in Fig. 1

$$\mathbb{X}_N := \left\{ x \left| \begin{array}{l} |(c_u K + c_x)x| \le \alpha_{stall} - \sum_{j=0}^{N-1} h_{A_k^j W}(c_x + c_u.K) \\ |Kx| \le \alpha_{sat} - \sum_{j=0}^{N-1} h_{A_k^j W}(K) \end{array} \right. \right\}. \tag{28}$$

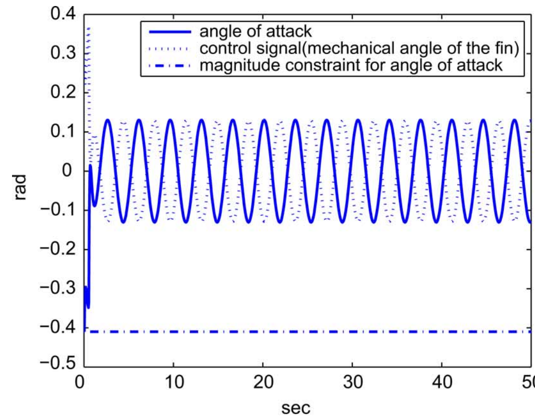
Given the sets $\mathbb{X}_f$ and Ω_i, the control law for the fin stabilizer is determined according to (10) and (11). The simulation of the closed loop was performed for a given sinusoidal wave torque profile with period of 7 sec and magnitude of $0.2 I_{\phi\phi}$.

Fig. 1 shows the trajectory of the system with initial condition $[\phi \;\; p] = [0 \text{ rad } 0.45 \text{ rad/sec}]$. It can be seen that in the presence of sinusoidal wave disturbance, the ship roll motion is stabilized around the origin within a minimal invariant set characterized by the matrix A_K and the set W, while saturation constraints as well as the constraint on the angle of attack α_e are satisfied. Fig. 2(a) shows the angle of the fin and the angle of attack.

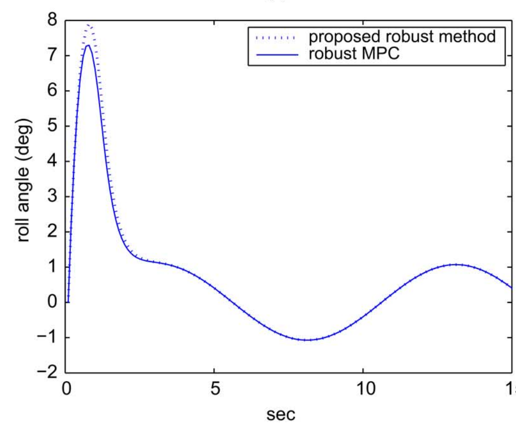
For comparison, we also applied a robust MPC method to the roll control problem. The set of tightened constraints (26) and (28) was used along with the cost function

$$\sum_{i=0}^{9} R u(i)^2 + x(i)^T Q x(i) + x(10)^T S_f x(10)$$

where S_f is the solution of the associated discrete-time Riccati equation for the infinite horizon problem. Fig. 2(b) shows the roll angle when the approach of this technical note and robust MPC are employed. Referring to Fig. 2(b), similar performance can be observed while no repeated on-line optimization was employed for our method. For the initialization of the proposed algorithm, an LP problem was solved at $k = 0$ to provide a feasible trajectory. No optimization problem is solved for $k > 0$. As one can observe from these figures, the constraints are satisfied. The MPC is implemented using the same set of constraints as the robust controller; thereby providing the same region of attraction.



(a)



(b)

Fig. 2. (a) Angle of attack and fin angle of the system with initial condition $[\phi \; p] = [0 \text{ rad}, \; 0.45 \text{ rad/sec}]$. Angle of attack is constrained to ± 0.41 rad and fin is constrained to ± 0.436 rad (b) Roll angle of the system with initial condition $[\phi \; p] = [0 \text{ rad}, \; 0.45 \text{ rad/sec}]$.

The average computational time for the MPC method is 4.1 ms and the maximum computational time is 22 ms. However, for the proposed method, the maximum computational time is 0.015 ms and the average is 0.00194 ms. We observe that there is two order of magnitude difference in computational time.

V. Conclusion

This technical note developed a robust control approach for a class of constrained linear systems subject to mixed state and input constraints with bounded disturbances. The novel feature of our robust controller is that its control action is a linear combination of known data at each sampling time instant and therefore it is highly computationally effective. The proposed controller guarantees convergence to a robust invariant set for a system with reduced disturbance set in a finite number of steps. Inside the robust invariant set, the proposed algorithm is equivalent to a linear state feedback. Moreover, it guarantees convergence of state trajectory to a minimal invariant set of the desired system while explicit specification or approximation of such set is not required. The method was employed in this technical note for control of roll motion of a surface ship, to prevent the dynamic stall and enforce fin saturation constraints. Convergence to a desired target set in the presence of disturbances such as those caused by sea waves has been demonstrated. Simulation results were presented to show the effectiveness of the proposed method. Other constraints, e.g., rate limits, can be handled similarly.

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On Achieving Size-Independent Stability Margin of Vehicular Lattice Formations With Distributed Control

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***Abstract*—We study the stability margin of a vehicular formation with distributed control, in which the control at each vehicle only depends on the information from its neighbors in an information graph. We consider a D-dimensional lattice as information graph, of which the 1-D platoon is a special case. The stability margin is measured by the real part of the least stable eigenvalue of the closed-loop state matrix, which quantifies the rate of decay of initial errors. In [1], it was shown that with symmetric control, in which two neighbors put equal weight on information received from each other, the stability margin of a 1-D vehicular platoon decays to 0 as $O(1/N^2)$, where N is the number of vehicles. Moreover, a perturbation analysis was used to show that with vanishingly small amount of asymmetry in the control gains, the stability margin scaling can be improved to $O(1/N)$. In this technical note, we show that, with judicious choice of non-vanishing asymmetry in control, the stability margin of the closed loop can be bounded away from zero uniformly in N. Asymmetry in control gains thus makes the control architecture highly scalable. The results are also generalized to D-dimensional lattice information graphs that were studied in [2], and the correspondingly stronger conclusions than those derived in [2] are obtained. In addition, we show that the size-independent stability margin can be achieved with relative position and *relative velocity* (RPRV) feedback as well as relative position and *absolute velocity* (RPAV) feedback, while the analysis in [1], [2] was only for the RPAV case.**

***Index Terms*—Asymmetric control, automated platoon, distributed control, multiagent system, stability margin.**

I. Introduction

We study cooperative control of a large vehicular formation with distributed control. The vehicles are modeled as double integrators, and the control action at each vehicle is computed based on information from its neighbors, where the neighbor relationship is characterized by a lattice information graph. The control objective is to make the vehicular formation track a constant-velocity type desired trajectory while maintaining prespecified constant separation among neighbors. The desired trajectory of the entire vehicular formation is given in terms of trajectories of a set of fictitious reference vehicles.

The problem of distributed control for multiagent coordination is relevant to many applications such as automated highway system, collective behavior of bird flocks and animal swarms, and formation flying of unmanned aerial and ground vehicles for surveillance, reconnaissance and rescue, etc. [3]–[8]. A typical issue faced in distributed control is that as the number of agents increases, the performance (stability margin and sensitivity to external disturbances) of the closed loop degrades. Several recent papers have studied the scaling of performance of vehicle formations as a function of the number of vehicles. The [1], [2] have studied the scaling of the stability margin of D-dimensional lattice formations. The stability margin is defined as the absolute value of the real part of the least stable eigenvalue of the closed loop. The stability margin characterizes the rate at which initial errors decay. The

Manuscript received February 09, 2011; revised September 28, 2011; accepted February 05, 2012. Date of publication March 15, 2012; date of current version September 21, 2012. This work was supported by the National Science Foundation by Grant CNS-0931885 and ECCS-0925534. Recommended by Associate Editor M. Egerstedt.

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Digital Object Identifier 10.1109/TAC.2012.2191179