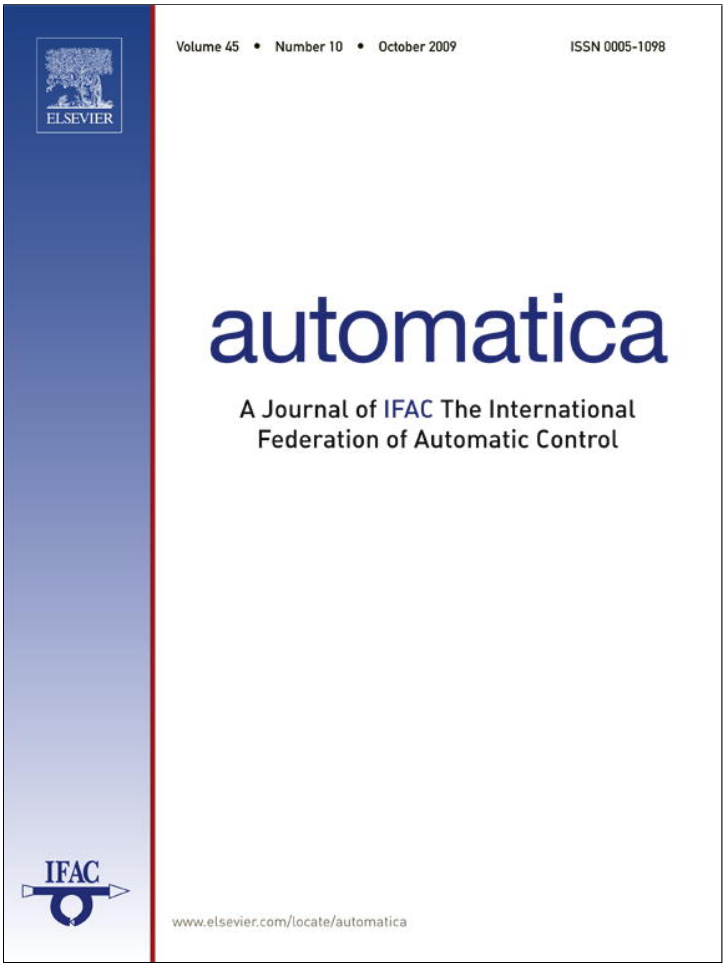
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Brief paper

An integrated perturbation analysis and Sequential Quadratic Programming approach for Model Predictive Control[☆]

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ABSTRACT

Computationally efficient algorithms are critical in making Model Predictive Control (MPC) applicable to broader classes of systems with fast dynamics and limited computational resources. In this paper, we propose an integrated formulation of Perturbation Analysis and Sequential Quadratic Programming (InPA-SQP) to address the constrained optimal control problems. The proposed algorithm combines the complementary features of perturbation analysis and SQP in a single unified framework, thereby leading to improved computational efficiency and convergence property. A numerical example is reported to illustrate the proposed method and its computational effectiveness.

1. Introduction

Model Predictive Control (MPC) is a promising control technique with conspicuous advantages, such as the capability to deal with constraints (Mayne, Rawlings, & Scokaert, 2000) and hybrid systems. It has found wide acceptance in industry, especially in the chemical and process industry (Qin & Badgwell, 1997). Since a constrained optimization problem has to be solved repeatedly on-line for each control update, the feasibility and effectiveness of the MPC is often limited to systems with slow dynamics and adequate computational resources. Improvements in computational algorithms of MPC can expand the range of applicability of MPC to systems with fast dynamics and limited computing power.

Several methodologies and algorithms have been proposed in the literature to reduce the on-line computational effort. For example, computational time reduction can be achieved via model reduction techniques (Dufour, Toure, Blanc, & Laurent, 2003; Hovd, Lee, & Morari, 1993), or by using regional linear approximation of nonlinear models (Garcia, 1984; Zheng, 1997) and exploiting off-line polyhedral partitioning of the state space and storing the obtained solution for piecewise affine linear systems (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Johansen, Petersen, & Slupphaug, 2002). In addition, the computational time can be lowered by reducing the number of constraints through approximation of constraints by the well-known geometric object like an ellipsoid or a polytope (VanAntwerp & Braatz, 2000). An alternative approach is to construct a functional approximation of the MPC law off-line (Canale, Fagiano, & Milanese, 2009).

On the other hand, approximating the solution of the MPC optimization problem using a pre-computed nominal optimal solution can also reduce the on-line MPC computational requirements. If the current state is sufficiently close to the initial condition associated with the nominal solution, the optimal solution corresponding to the current state can be represented as the perturbed solution using neighboring extremal approximation. The nominal solution can be pre-computed off-line for different regions, or it can be computed on-line between two sample instances, using state predictions (Gholami, Gordon, & Rabbath, 2005; Milam, Franz, Hauser, & Murray, 2005). The real-time approximation of the perturbed solution for continuous-time systems has been considered in several papers (Kugelmann & Pesch, 1990; Malanowski & Maurer, 1996). For discrete-time systems, the neighboring extremal method (Bryson & Ho, 1975) can be used to calculate the closed-form perturbed solution for the cases with no constraints. However, the authors of this paper are not aware of analogous results for discrete-time systems in the presence of constraints, except for (Büskens & Maurer, 2001), which converts the optimal control problem into a nonlinear programming problem.

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In this paper, we introduce a method, and thereafter we refer to as the Integrated Perturbation Analysis and Sequential Quadratic Programming (InPA-SQP) approach, for the MPC implementation. It synergistically combines the solutions derived using perturbation analysis and SQP to solve the optimization problem with initial state perturbation and input/state constraints. The development of this InPA-SQP method is accomplished in three steps.

First, we consider the problem of solving the perturbed optimal control problem for small perturbations that do not change the active constraints set. For cases involving large perturbations in the initial condition, an algorithm similar to active set method is proposed, which calculates intermediate initial states and their corresponding approximations of optimal perturbed solution to handle the change in the activity status of the constraints.

The intermediate initial states are considered as new "nominal solutions" at each successive application of the perturbation analysis. Since the error can accumulate in the iterative process, in the third step a special formulation of SQP with active set method being used to modify the solution to compensate the error and achieve optimality. We thus integrate the perturbation analysis and SQP into a single unified framework which provides accurate and fast calculation of the optimal solution. This unification is dependent on a special formulation of the SQP algorithm (presented in Section 4) that allows us to represent the solution of the SQP problem by a formula similar to that of the perturbation analysis, thereby facilitating their seamless integration.

The proposed method adopts a different approach than those of Borbow, Park, and Sideris (2006), Ferreau, Bock, and Diehl (2006) and Zavala, Laird, and Biegler (2006) in the sense that it is based on indirect (variational) approach using first-order necessary conditions for optimality, rather than direct approaches (Allgöwer, Findeisen, & Biegler, 2007). Introducing a suitable switching structure, active inequality constraints can be handled and the perturbed solution is calculated using the closed form formulation in our proposed algorithm.

To illustrate the application of InPA-SQP in solving the optimal control problem associated with the Model Predictive Control (MPC), a ship steering problem with control constraints is considered. A reduction in computing time, as compared to the SQP, is demonstrated for this example without any loss in performance.

2. Perturbation analysis for discrete-time optimal control problem subject to constraints

In this section, we develop the perturbation analysis for a discrete-time optimal control problem with fixed ending time and no terminal constraint. Consider the problem of minimizing a cost function,

$$J[u] = \sum_{k=0}^{N-1} L(x(k), u(k)) + \Phi(x(N)), \tag{1}$$

over all control sequences $u : [0, N-1] \to \mathbb{R}^m$ and all state vector sequences $x : [0, N] \to \mathbb{R}^n$ subject to the following constraints:

$$x(k+1) = f(x(k), u(k)), \quad f : \mathbb{R}^{n+m} \to \mathbb{R}^n; \tag{2}$$

$$x(0) = x_0, \quad x_0 \in \mathbb{R}^n; \tag{3}$$

$$C(x(k), u(k)) \le 0, \quad C : \mathbb{R}^{n+m} \to \mathbb{R}^l. \tag{4}$$

It is assumed that functions $L : \mathbb{R}^{n+m} \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$, $f(x(\cdot), u(\cdot))$, $C(x(\cdot), u(\cdot))$ are twice continuously differentiable with respect to x and u.$C_u(k)$ is the partial derivative of $C(x(k), u(k))$ with respect to u at the time instant k. $C^a(x(k), u(k))$ is a vector consisting of those elements of the vector $C(x(k), u(k))$ whose corresponding inequality constraints are active. That is, $C^a(x(k), u(k))$ = empty vector if no inequality constraint is active at the time instant k and $C^a(x(k), u(k)) \in \mathbb{R}^{l'}$, if l' (out of l) constraints are active. Moreover, it is assumed that if the constraint is active, $C_u^a(x(k), u(k)) \neq 0$.

Remark 1. If $C_u^a(x(k), u(k)) = 0$ or approaches zero, the algorithm described in the sequel can be modified as shown in Ghaemi, Sun, and Kolmanovsky (2008), which replaces C_u^a by a full row rank matrix derived using the constraint back-propagation approach.

For simplicity, we treat the case $l = 1$ in this paper. The results can be generalized to the case $l > 1$ without much difficulty but the notations become more cumbersome.

The augmented cost, obtained by adjoining the constraints (2)–(4), is:

$$\bar{J}[u] = \Phi(x(N)) + \sum_{k=0}^{N-1} (H(k) - \lambda(k+1)x(k+1)) \tag{5}$$

where:

$$H(k) = L(x(k), u(k)) + \lambda(k+1)^{\mathrm{T}} f(x(k), u(k)) + \mu(k)^{\mathrm{T}} C(x(k), u(k)), \tag{6}$$

$\mu(k)$, $\lambda(k+1)$ are the Lagrange multipliers associated with (4) and (2) at the time instant k respectively.

Let $x(k), u(k)$, $k \in [0, N]$ be the state and control corresponding to the optimal solution with the initial condition $x(0)$, hereafter referred to as the nominal solution. If there is a perturbation $\delta x(0)$ in the initial state $x(0)$, which does not change the set and the time instants at which constraints are active, then the following theorem gives the neighboring extremal solution to the optimal control problem. By neighboring extremal solution, we refer to the state and control sequences which satisfy the first order necessary conditions for optimality when the initial state is perturbed. In what follows, the subscripts u and x stand for the partial derivatives of a function with respect to u and x, respectively.

Theorem 2. *The neighboring extremal approximation of the solution to the optimization problem, defined by the cost function (1) and constraints (2), (4) and initial state $x(0) = x_0 + \delta x(0)$, is $x(k) + \delta x(k)$ and $u(k) + \delta u(k)$, $k \in [0, N]$, provided*

$$Z_{uu}(k) \succ 0 \quad \text{for } k \in [0, N] \tag{7}$$

for the nominal solution. Where[1]*:*

$$\delta u(k) = K^*(k)\delta x(k), \tag{8}$$

$$K^*(k) = -[I\ 0]K_0(k) \begin{bmatrix} Z_{ux}(k) \\ C_x^a(k) \end{bmatrix}, \tag{9}$$

and

$$K_0(k) = \begin{bmatrix} Z_{uu}(k) & C_u^{a\mathrm{T}}(k) \\ C_u^a(k) & 0 \end{bmatrix}^{-1} \tag{10}$$

$$\begin{aligned} Z_{uu}(k) &= H_{uu}(k) + f_u^{\mathrm{T}}(k)S(k+1)f_u(k), \\ Z_{ux}(k) &= Z_{xu}(k)^{\mathrm{T}} = H_{ux}(k) + f_u^{\mathrm{T}}(k)S(k+1)f_x(k), \\ Z_{xx}(k) &= H_{xx}(k) + f_x^{\mathrm{T}}(k)S(k+1)f_x(k), \end{aligned} \tag{11}$$

and $C^a(k) = C(x(k), u(k))$ if the constraint is active and it is empty if the constraint is not active at the time instant k. Moreover, the zero matrix appearing in Eqs. (9) and (10) is empty if the constraint is not

[1] Following the notation used in Bryson and Ho (1975), H_{uu}, H_{ux}, H_{xx}, Φ_{xx}, etc., denote the partial derivative with respect to x and/or u. Namely, $H_{uu} = \frac{\partial^2 H}{\partial u^2}$, $H_{ux} = \frac{\partial}{\partial x}(\frac{\partial H}{\partial u}^{\mathrm{T}})$, etc. The variables Z_{uu}, Z_{ux}, Z_{xx} are the exception, and they are defined by (11).

active at the time instant k. $S(k)$ in Eq. (11) is given by:

$$S(k) = Z_{xx}(k) - [Z_{xu}(k)\; C_x^{a\mathrm{T}}(i)]K_0(k)\begin{bmatrix} Z_{ux}(k) \\ C_x^a(k) \end{bmatrix}, \quad k = 0, \ldots, N-1 \qquad (12)$$
$$S(N) = \Phi_{xx}(N).$$

Remark 3. Theorem 2 allows one to calculate a first-order approximation to the optimal solution in the form of $x(k) + \delta x(k)$ and $u(k) + \delta u(k)$, without solving an optimization problem. $\delta x(k)$ and $\delta u(k)$ are calculated using the above recursive updates.

Proof. Given space limitations, we only provide a sketch of the proof. Consider the second order expansion of the augmented cost function $\bar{J}$ around the nominal solution. Note that the first order variation satisfies $\delta\bar{J} = 0$. The neighboring extremal solution minimizes the following cost:

$$\delta^2\bar{J} = 1/2\delta x(N)^{\mathrm{T}}\Phi_{xx}(N)\delta x(N) + 1/2\sum_{k=0}^{N-1}\begin{bmatrix}\delta x(k)\\ \delta u(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}H_{xx}(k) & H_{xu}(k)\\ H_{ux}(k) & H_{uu}(k)\end{bmatrix}\begin{bmatrix}\delta x(k)\\ \delta u(k)\end{bmatrix}, \qquad (13)$$

subject to the linearized constraints:

$$\delta x(k+1) = f_x(k)\delta x(k) + f_u(k)\delta u(k), \qquad (14)$$
$$\delta x(0) = \delta x_0, \qquad (15)$$
$$C_x^a(k)\delta x(k) + C_u^a(k)\delta u(k) = 0\,. \qquad (16)$$

The Karush–Kuhn–Tucker (KKT) conditions lead to:

$$\delta\lambda(k) = \tilde{H}_{\delta x(k)}(k), \quad k \in [1\; N-1], \qquad (17)$$
$$\tilde{H}_{\delta u(k)}(k) = 0, \quad k \in [0\; N-1], \qquad (18)$$
$$\delta\lambda^{\mathrm{T}}(N) = \delta x(N)^{\mathrm{T}}\Phi_{xx}(N). \qquad (19)$$

where $\tilde{H}$ is the Hamiltonian function corresponding to the optimization problem (13) subject to constraints (14)–(16). These conditions, after algebraic manipulations, result in

$$\begin{bmatrix}\delta u(k)\\ \delta\mu(k)\end{bmatrix} = -K_0(k)\begin{bmatrix}Z_{ux}(k)\\ C_x^a(k)\end{bmatrix}. \qquad (20)$$

Moreover, it can be proved that if $Z_{uu}(k)$ is positive definite, the optimization problem (13)–(16) is convex and there exists a unique perturbed solution. ◇

Remark 4. When the algorithm is used in the MPC context, as will be elaborated on later, repeated feasibility of the optimization problem (1)–(4) has to be assumed. This feasibility issue is in no way unique to our approach and method. If feasibility becomes an issue, any method to guarantee feasibility, including treating constraints as soft, extending the horizon or appropriately defining the terminal set, can be used to assure feasibility.

3. Augmented perturbation analysis for handling large perturbations

Theorem 2 is derived under the assumption that the set of active and inactive constraints remain unchanged after perturbation. To deal with the initial state variation that is large enough to change activity status of constraints, we can analyze the perturbed Lagrange multipliers associated with the inequality constraints and the perturbed value of $C(x(k), u(k))$ to determine the status of constraint activity after perturbation. The following proposition provides the relation between the initial condition perturbation and the Lagrange multipliers that will allow us to predict the constraint activity change.

Proposition 5. *If the initial condition $x(0)$ is perturbed by $\delta x(0)$, the optimal Lagrange multiplier perturbation $\delta\mu(k)$ at the time instant k when the constraint is active can be approximated as follows:*

$$\delta\mu(k) = -[0\; I]K_0(k)\begin{bmatrix}Z_{ux}(k)\\ C_x^a(k)\end{bmatrix}\Upsilon(k)\delta x(0),$$
$$\Upsilon(k) = \prod_{i=0}^{k-1} M(i), \qquad (21)$$
$$M(i) := f_x(i) + f_u(i)K^*(i), \quad i = 0, \ldots, k-1.$$

In addition, if the constraint is not active, then the constraint perturbation can be expressed as

$$\delta C(x(k), u(k)) = (C_x(k) + C_u(k)K^*(k))\Upsilon(k)\delta x(0), \qquad (22)$$

with $K_0(\cdot)$ and $K^*(\cdot)$ being defined as in Theorem 2.

Proof. By combining Eqs. (20), (8) and (14), the expression (21) can be derived. (22) follows directly by taking partial derivatives of $C(\cdot,\cdot)$ and noting that $\delta x(k) = \Upsilon(k)\delta x(0)$. ◇

Note that the perturbed optimal Lagrange multiplier $\mu^1(k)$ is:

$$\mu^{(1)}(k) = \mu(k) + \delta\mu(k), \quad k \in \mathbb{K}^a \qquad (23)$$

where $\mu(\cdot)$ is the nominal Lagrange multiplier and $\delta\mu(\cdot)$ is calculated from (21). If $\mu^{(1)}(k) \geq 0$, one can conclude that the constraint will remain active at the time k for the perturbed solution. Otherwise, it may become inactive because the Lagrange multiplier must always be greater than or equal to zero. Similarly, using Eq. (22), the value of the constraint function corresponding to the perturbed optimal solution is:

$$C(x^{(1)}(k), u^{(1)}(k)) = C(x(k), u(k)) + \delta C(x(k), u(k)), \qquad (24)$$

where $x^{(1)}(k)$ and $u^{(1)}(k)$ are the following linear approximations of the optimal solution

$$x^{(1)}(k) = x(k) + \delta x(k), \qquad u^{(1)}(k) = u(k) + \delta u(k). \qquad (25)$$

If $C(x^{(1)}(k), u^{(1)}(k)) < 0$, the constraint remains inactive. Otherwise, it will become active.

Using Proposition 5, we propose the following algorithm to approximate the optimal solution when initial state perturbations change the status of the constraints activity:

1- Set $i = 0$, $\delta x^{(0)}(0) = \delta x(0)$ and $x^0(0) = x(0)$;

2- If constraint is active at the time instant k, compute α_{ik} as:

$$\alpha_{ik} = \frac{\mu^{(i)}(k)}{\delta\mu^{(i)}(k)}. \qquad (26)$$

If it is inactive at k, compute α_{ik} as

$$\alpha_{ik} = -\frac{C(x^{(i)}(k), u^{(i)}(k))}{\delta C(x^{(i)}(k), u^{(i)}(k))}, \qquad (27)$$

where $\delta C(x^{(i)}, u^{(i)})$ is calculated by (22), all the matrices used in (26) should be evaluated at $x^{(i)}(k)$ and $u^{(i)}(k)$, using (21) and (23) at the ith iteration. Then find the smallest $\alpha_{ik} \in [0, 1]$ such that the perturbation $\alpha_{ik}\delta x^{(i)}(0)$ changes the status of the constraint at least at one instant, namely:

$$\alpha_i = \min_k\{\alpha_{ik}, k = 0, \ldots, N-1,\; 0 \leq \alpha_{ik} \leq 1\}.$$

If, for all $k \in [0 : N]$, $\alpha_{ik} < 0$ or $\alpha_{ik} > 1$, set $\alpha_i = 1$.

3- Compute an approximation to the perturbed optimal solution $\delta x^{(i)}(\cdot)$, $\delta u^{(i)}(\cdot)$ for the intermediate perturbation $\min\{\alpha_i, 1\}\,\delta x^{(i)}(0)$ and initial condition $x^{(i)}(0)$ using the perturbation analysis developed in Section 2.

4- If $\alpha_i = 1$, terminate. Otherwise:

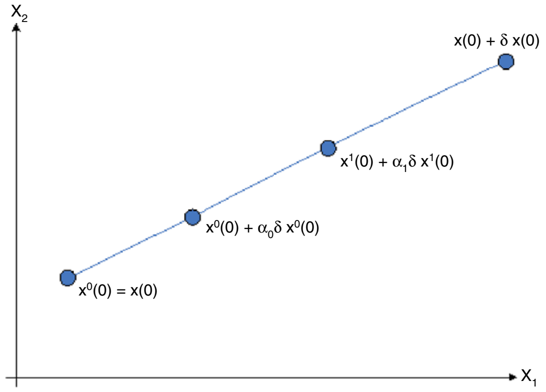

Fig. 1. Intermediate initial states which handle the large perturbation.

- If $\alpha_i = 0$, change the activity status of the corresponding constraint accordingly. That is, if α_i corresponds an active (inactive) constraint, set the constraint inactive (active). Go to step 2.
- If $\alpha_i < 1$ set

  $\delta x^{(i+1)}(0) = (1 - \alpha_i)\delta x^{(i)}(0)$,
  $x^{(i+1)}(0) = x^{(i)}(0) + \alpha_i \delta x^{(i)}(0)$,
  $i = i + 1$.

  Go to step 2.

Fig. 1 illustrates the procedure, where the intermediate points are highlighted. Note that the intermediate perturbed initial states lie on the line connecting $x(0)$ to $x(0) + \delta x(0)$. These intermediate states are nominal states at which Theorem 2 can be repeatedly applied to derive approximations to the optimal solution. Therefore the perturbed optimal control solution corresponding to a large perturbation $\delta x(0)$ can be approximated by augmenting the nominal solution as:

$$u(k) + \sum_i \delta u^i(k).$$

4. Sequential quadratic optimal control based on active set method

In this section, we formulate the Sequential Quadratic Programming (SQP) method for the optimization problem (1) as a prelude to introducing the InPA-SQP approach in the next section. This formulation is different from the one of Glad and Johnson (1984) in the sense that it reduces to a recursive Riccati equation instead of solving a system of recursive linear equations. When there is no constraint, this formulation becomes the one proposed in McRaynolds and Bryson (1965). Therefore the SQP method presented in this section can be considered as an extended version of McRaynolds and Bryson (1965) to the cases with input-state constraints.

We start with a feasible initial guess of $u(k)$, $x(k)$, $\lambda(k)$, and $\mu(k)$ such that they satisfy Eqs. (2)–(4) and the following equations

$$\lambda(k) = H_{x(k)}(k), \quad k = 1, \dots, N-1, \qquad \lambda(N) = \Phi_x(N). \tag{28}$$

Note that since the initial guess is not an optimal solution, it may not satisfy the optimality condition

$$H_{u(k)}(k) = 0. \tag{29}$$

The inequality constraints, when active, are treated as the equality constraints during the active set iteration. The corrections $\delta u(k)$ and $\delta x(k)$ are obtained as the solution of the following equality constrained quadratic programming problem (QP) for the linearized system

$$\begin{aligned} &\min_{\delta u(\cdot),\delta x(\cdot)} \quad \Delta\bar{J} \\ &\text{subject to: } \delta x(k+1) = f_x(k)\delta x(k) + f_u(k)\delta u(k), \\ &\qquad \delta x(0) = 0, \\ &\qquad C_x^a(k)\delta x(k) + C_u^a(k)\delta u(k) = 0, \end{aligned} \tag{30}$$

where

$$\Delta\bar{J} = \sum_{k=0}^{N-1} H_{u(k)}^{\mathrm{T}}(k)\delta u(k) + 1/2\delta x(N)^{\mathrm{T}}\Phi_{xx}(N)\delta x(N) + 1/2\sum_{k=0}^{N-1}\begin{bmatrix}\delta x(k)\\ \delta u(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}H_{xx}(k) & H_{xu}(k)\\ H_{ux}(k) & H_{uu}(k)\end{bmatrix}\begin{bmatrix}\delta x(k)\\ \delta u(k)\end{bmatrix}. \tag{31}$$

Theorem 6. *Let $u(k)$, $x(k)$ and $\lambda(k)$ be the control, state and co-state, respectively, that satisfy (2)–(4) and (28) and $\mu(k)$ be the Lagrange multiplier associated with the inequality constraint. In addition, assume that $Z_{uu}(k)$, defined in (11), is positive definite. Then the solution of the QP with equality constraint (30) is given by*

$$\begin{aligned} &\delta u(k) = -[I\ 0]K_0(k)\begin{bmatrix} Z_{ux}(k)\delta x(k) + f_u^{\mathrm{T}}(k)T(k+1) + H_u(k) \\ C_x^a(k)\delta x(k)\end{bmatrix} \\ &\delta x(k+1) = f_x(k)\delta x(k) + f_u(k)\delta u(k) \\ &\delta x(0) = 0 \end{aligned} \tag{32}$$

where $K_0(k)$, $Z_{uu}(k)$, $Z_{ux}(k)$ and $Z_{xx}(k)$ are defined in (10) and (11). Moreover, the matrices $S(\cdot)$ and $T(\cdot)$ are calculated using the following backward recursive equations

$$\begin{aligned} &S(N) = \Phi_{xx}(N), \\ &S(k) = Z_{xx}(k) - [Z_{xu}(k)\ C_x^{a\mathrm{T}}(k)]K_0(k)\begin{bmatrix} Z_{ux}(k) \\ C_x^a(k)\end{bmatrix}, \\ &T(N) = 0, \\ &T(k) = f_x^{\mathrm{T}}(k)T(k+1) \\ &\qquad - [Z_{xu}(k)\ C_x^{a\mathrm{T}}(k)]K_0(k)\begin{bmatrix} f_u^{\mathrm{T}}(k)T(k+1) + H_u(k) \\ 0\end{bmatrix}. \end{aligned} \tag{33}$$

Using Theorem 6, the active set method, which is introduced in Flecher (1981), may be implemented as follows:

First, we find the minimum value of $0 < \alpha \le 1$ such that there exists a time instant k at which $C(x(k), u(k)) < 0$ and

$$C(x(k) + \alpha\delta x(k), u(k) + \alpha\delta u(k)) = 0 \tag{34}$$

where $\delta u(k)$ and $\delta x(k)$ are calculated using Eq. (32). If there exists such α that satisfies the above condition, then the corresponding time instant at which the inactive constraint becomes active is added to active constraints and the equality constraint problem (30) is solved at the next iteration with the initial solution $x(k) + \alpha\delta x(k)$ and $u(k) + \alpha\delta u(k)$.

Otherwise the sign of Lagrange multipliers $\mu(k)$, calculated using Eq. (20), is examined. If all the Lagrange multipliers $\mu(k)$ are nonnegative, then the necessary optimality conditions are satisfied. If, in addition, $Z_{uu}(k) \succ 0$ then we have reached a local optimal solution. In the other case, one inequality constraint with negative multiplier is deleted from the set of active constraints.

5. InPA-SQP approach

In this section we introduce a method that unifies the perturbation analysis and SQP to achieve fast and accurate method for calculation of the perturbed optimal solution, when the

Table 1
Constant parameters of ship model.

Parameter	Value	Unit
a	1.084	1/min
b	0.62	min/rad^2
c	3.553	1/min^2
r_1	−0.0375	nm/rad
r_3	0	Nmmin^2/rad^3
f	0.86	1/min
W	0.067	nm/rad^2

nominal optimal solution is given. This algorithm combines the computational advantages of the perturbation analysis approach and the capability of the SQP to enforce the condition $H_u(k) = 0$.

Let $x(\cdot)$, $u(\cdot)$ be the nominal solution for $x(0) = x_0$. As the first step to calculate the optimal solution corresponding to the first intermediate initial state $x^1(0) = x_0 + \delta x^1(0)$, we assume that the perturbation $\delta x^1(0)$ is small enough so that the activity status of the constraints does not change over the time interval $[0, N - 1]$. In this case, because $x(\cdot)$ and $u(\cdot)$ satisfy an approximation to the optimality conditions, we can use Theorem 2 to calculate the optimal solution corresponding to $x(0) + \delta x^1(0)$.

When moving to the next intermediate state, the neighboring optimal solution for $x^{(1)}(\cdot)$ can be taken as initial estimate for nominal optimal solution and one can perform SQP iterations using the algorithm described in Section 4. Therefore, for the intermediate point, one can first solve the optimization problem (30) using the SQP solution (32), then approximate the solution for the next intermediate state based on the perturbation analysis.

Instead of applying the above two-step method, we introduce the unifying approach which exploits the SQP formulation to modify the large perturbation analysis. Comparing the two optimization problems (13) and (30), we note that the solution to the problem (30) is identical to that of (13) if we set $H_u(k) = 0$ and $\delta x(0) = \delta x_0$ in (32) and (33). Based on the above observation, we propose the following formulation which merges the two optimization steps into one:

$$\begin{aligned} \delta u(k) &= -[I\ 0]K_0(k)\begin{bmatrix} Z_{ux}(k)\delta x(k) + f_u^{\mathrm{T}}(k)T(k+1) + H_u(k) \\ C_x^a(k)\delta x(k) \end{bmatrix}, \\ \delta x(k+1) &= f_x(k)\delta x(k) + f_u(k)\delta u(k), \\ \delta x(0) &= \delta x_0. \end{aligned} \tag{35}$$

The $\delta u(\cdot)$ calculated using (35) will not only correct for the initial state perturbation, but also move $x(\cdot)$ and $u(\cdot)$ in the direction suggested by SQP to enforce $H_u(k) = 0$ condition. With the perturbation analysis approach of Section 3, modified using (35), a more computationally efficient InPA-SQP algorithm is obtained as compared to solving SQP for initial condition $x(0)$ from scratch.

Remark 7. If we consider the equality $x(0) - x_0 = 0$ as an equality constraint with $\lambda(0)$ being the associated Lagrange variable calculated using (28), then we can reformulate the augmented cost function by adding the term $\lambda(0)(x(0) - x_0)$. If the SQP method is applied to the reformulated problem, the equality constraint $\delta x(0) = x_0 - x(0)$ appears in the Eq. (32) and forms the Eq. (35). Therefore, the InPA-SQP is the result of applying SQP on the reformulated problem and consequently benefits from the convergence properties that SQP provides.

6. InPA-SQP for MPC implementation: An example

The InPA-SQP approach can be employed to reduce the computational time of solving the optimal control problem associated with MPC, compared to the conventional SQP-based approach.

According to the MPC strategy, at the time instant k, the state of the system, $x(k)$, is observed, the optimization problem (1)–(4) is solved, and the first element of the optimal control sequence is implemented as the control signal. At the time instant $k + 1$, the state $x(k + 1)$ is observed and the same optimal control problem is solved. Note that by the time instant $k + 1$, the solution to the MPC optimization problem with initial state $x(k)$ is available, which can be exploited to improve the efficiency of optimization. Defining

$$dx(k) := x(k+1) - x(k) \tag{36}$$

the solution of the MPC optimization problem for initial state $x(k+1)$ can be approximated using the solution for initial state $x(k)$ and the InPA-SQP method, $dx(k)$.

As an example to illustrate the application of the proposed algorithm, we consider a problem of steering a ship from an initial position to a desired target position. The following ship model, taken out from Casado, Fernandez, and Iglesis (2001), is used for numerical simulation:

$$\begin{aligned} \dot{x}_1 &= x_5\cos(x_3) - (r_1x_4 + r_3x_4^3)\sin(x_3), \\ \dot{x}_2 &= x_5\sin(x_3) + (r_1x_4 + r_3x_4^3)\cos(x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -ax_4 - bx_4^3 + cu_r, \\ \dot{x}_5 &= -fx_5 - Wx_4^2 + u_t, \end{aligned} \tag{37}$$

where x_1 and x_2 are the ship position (in nautical miles (nm)) in the $X_1 - X_2$ plane, x_3 is the heading angle (in radians (rad)), x_4 is the yaw rate (rad/min), and x_5 is the forward velocity (nm/min). The two control inputs are: the rudder angle u_r (rad), and the propeller's thrust u_t (nm/min^2).

The model parameters are summarized in Table 1. With these parameters, the ship has a maximum speed of 0.25 nm/min = 15 knots for a maximum thrust of 0.215 nm/min^2. For maximal rudder angle of 35°, the stationary rate of turn is 1°/s.

The discrete-time model of ship dynamics is derived using Euler approximation with sampling period $T = 0.1$ s. The target position is described by a circle with a radius 0.1 (nm) around the origin. To minimize the energy consumption during the maneuvering, we define

$$\begin{aligned} L(x(k), u(k)) &= 0.1u_r(k)^2 + 10u_t(k)^{3/2} \\ \Phi(x) &= 2000(x_1^2 + x_2^2) \end{aligned} \tag{38}$$

and with $N = 140$ as the length of horizon. The resulting MPC optimization problem is

$$\begin{aligned} &\min_{u(\cdot),x(\cdot)} \sum_{k=0}^{139}(0.1u_r(k)^2 + 10u_t(k)^{3/2}) \\ &\quad + 2000(x_1(140)^2 + x_2(140)^2) \end{aligned} \tag{39}$$

subject to constraints:

$$\begin{aligned} &0.02 \le u_r(k) \le 0.61 \text{ rad} \\ &-0.215 \le u_t(k) \le 0.215 \text{ nm/min}^2. \end{aligned} \tag{40}$$

The total number of optimization variables, including states and inputs, for the length of horizon $N = 140$, is 978, which is substantial from computational point of view. The initial optimal solution for $k = 0$ is calculated off-line using SQP algorithm.

For a fair comparison, the formulation proposed in Section 4 is implemented for SQP, which is equivalent to the InPA-SQP in the absence of initial state perturbation. Simulations are performed on a computer with Intel(R) CPU @ 1.83 GHz and computation time is measured using CPU time and controller code is implemented in Matlab.

Fig. 2 shows the ship trajectory in the $X_1 - X_2$ plane and the propeller's thrust using both SQP and InPA-SQP for initial condition of $x(0) = [3, 0, \pi/3, 0, 0.25]$. The two solutions overlap in Fig. 2 as they are nearly identical. The computational time of the two methods are compared in Fig. 3. The InPA-SQP results in almost 280% reduction in the average computational time when compared with the SQP.

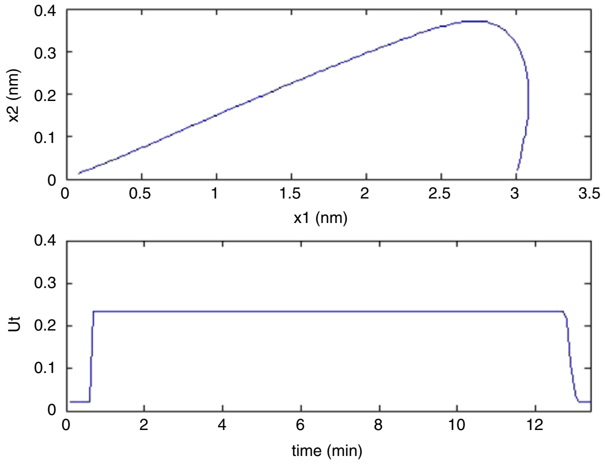

Fig. 2. Implementing MPC using SQP with active set method and InPA-SQP approach on ship.

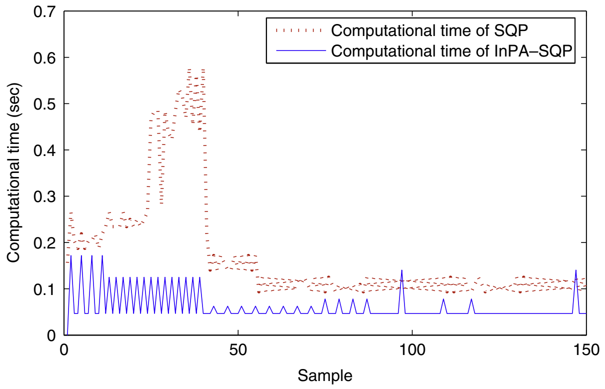

Fig. 3. Computational time of SQP with active set method and InPA-SQP approach for ship steering problem.

7. Conclusion

In this paper we proposed a numerical optimization algorithm, referred to as Integrated Perturbation Analysis and Sequential Quadratic Programming (InPA-SQP), for treating discrete-time optimal control problems for nonlinear systems with pointwise-in-time constraints. This method, based on combining complementary features of Perturbation Analysis and Sequential Quadratic Programming, is primarily intended for MPC applications, where the requirements to compute the optimal solution in real-time can present a very tight constraint. A numerical example has been reported which confirmed the improvements in the computing speed of InPA-SQP versus SQP.

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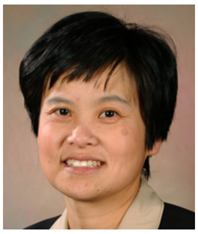

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