Simultaneous input and parameter estimation with input observers and set-membership parameter bounding: theory and an automotive application

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SUMMARY

The paper addresses an on-line, simultaneous input and parameter estimation problem for a first-order system affected by measurement noise. This problem is motivated by practical applications in the area of engine control. Our approach combines an input observer for the unknown input with a set-membership algorithm to estimate the parameter. The set-membership algorithm takes advantage of *a priori* available information such as (i) known bounds on the unknown input, measurement noise and time rate of change of the unknown input; (ii) the form of the input observer in which the unknown parameter affects only the observer output; and (iii) the input observer error bounds for the case when the parameter is known exactly. The asymptotic properties of the algorithm as the observer gain increases are delineated. It is shown that for accurate estimation the unknown input needs to approach the known bounds a sufficient number of times (these time instants need not be known). Powertrain control applications are discussed and a simulation example based on application to engine control is reported. A generalization of the basic ideas to higher order systems is also elaborated. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: input observers; set-membership parameter estimation; adaptive control; engine control

1. INTRODUCTION

In this paper, we consider a problem of simultaneous input and parameter estimation. Although generalizations to higher order systems will be touched upon later in the paper, for the most part our developments concern a first-order system,

$$\dot{x} = \theta^{\mathrm{T}} z(t) + u(t) \tag{1}$$

In (1), x is a scalar state variable, u is a scalar input variable, θ is an m-vector of constant parameters, and $z(t)$ is an m-vector. Our objective is, by measuring $x(t)$ and from the assumption

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that $z(t)$ is known at each time t (but not prior to time t), to simultaneously estimate the unknown input $u(t)$ and the unknown constant parameter vector, θ.

This problem (even in the first-order case) arises quite frequently in applications. For instance, in mechanical systems first-order dependencies exist between translational velocity and forces, angular rate of rotation and torques, pressure or mass in a volume and inlet/outlet flows, etc. In these situations, the practical need to estimate and adapt models of unmeasured forces, torques or flows can rather naturally lead to the problem formulation which we treat here. We will touch upon several concrete examples of this kind, which arise in the automotive applications, later in the paper.

In the subsequent treatment of this problem, we assume that the signal $x(t)$ can be measured but its measurements are contaminated by noise, $\zeta(t)$, so that

$$y(t) = x(t) + \zeta(t), \quad \sup_{t \geqslant 0} |\zeta(t)| \leqslant \mu \tag{2}$$

where μ is a known upper bound on the noise.¶

It is clear that without additional assumptions θ and u are not uniquely identifiable from the observed time-histories of x and z. Indeed, if θ and u are such that $\dot{x}(t) = \theta^T z(t) + u(t)$, then with $\tilde{u}(t) = u(t) + \frac{1}{2}\theta^T z(t)$, $\tilde{\theta} = \frac{1}{2}\theta$ it is also true that $\dot{x}(t) = \tilde{\theta}^T z(t) + \tilde{u}(t)$. As we will show in this paper, if the bounds on the unknown input, u, and its time rate of change are known and are tight at some time instants (which do not need to be known), then θ and u can be accurately estimated. Specifically, let $\underline{u}(t)$ and $\bar{u}(t)$ denote, respectively, the lower and upper bounds on $u(t)$ that are *known* at each time t (but may not be known prior to time t), and let h denote the *a priori* known bound on $|\dot{u}|$, i.e.

$$\underline{u}(t) \leqslant u(t) \leqslant \bar{u}(t), \quad t \geqslant 0 \tag{3}$$

$$\sup_{t \geqslant 0} |\dot{u}(t)| \leqslant h \tag{4}$$

Even if bounds (3), (4) are known, additional conditions on their tightness at some time instants will be needed. Indeed, if $\underline{u}(t) < u(t) < \bar{u}(t)$, $t \geqslant 0$, $\sup_{t \geqslant 0} |\dot{u}(t)| < h$ and $\dot{x} = \theta^T z(t) + u(t)$ then for $\tilde{u}(t) = u(t) + \varepsilon z(t)$, $\tilde{\theta} = \theta - \varepsilon$, where $|\varepsilon|$ is sufficiently small, it also follows that $\dot{x}(t) = \tilde{\theta}^T z(t) + \tilde{u}(t)$, $\underline{u}(t) \leqslant \tilde{u}(t) \leqslant \bar{u}(t)$, $t \geqslant 0$, and $\sup_{t \geqslant 0} |\dot{\tilde{u}}(t)| \leqslant h$. Specific bound tightness conditions that guarantee accurate input and parameter estimation will be developed in this paper.

Bound tightness conditions are typically used in bounded error identification problems (see e.g. Reference [1]) to assure parameter convergence. Note, however, that system (1), (2) is not in the form treated in Reference [1]. Earlier conference papers [2, 3] have described our approach and have included some of the results of this paper.

An estimation algorithm should also take an advantage of any *a priori* known bounds on the unknown parameter. Specifically, we assume that at $t = 0$ it is known that

$$\theta \in \Theta^0 \tag{5}$$

where Θ^0 is a specified *compact* set in $\mathbf{R}^m$.

¶The functions z and ζ are assumed to be Lebesgue-measurable and bounded.

The paper is organized as follows. In Section 2, we review the estimation procedure for u based on the input observer proposed in Reference [4] for the case when θ is known. In Section 3, we combine this input observer with a set-membership [5–7] algorithm to estimate θ on-line. The set-membership algorithm takes advantage of bounds (3), (4), (5) and of the input observer error bound if the parameter were accurately known. A special structure of the input observer, with the parameter affecting only the observer output equation, facilitates the application of the set-membership procedure. In Section 4, we summarize the asymptotic properties of our input and parameter estimation algorithm, as the observer gain increases and the measurement noise magnitude decreases. The computational implementation of the set-membership algorithm updates is briefly discussed in Section 5. In Section 6, we discuss the applications of the results to internal combustion engines.

For the special case of $m = 1$, our approach and the relevant results have been previously reported in the conference article [2]. While our approach is closest to Reference [1], the problem can be also treated with other techniques, including References [8–11]. Reference [12] applies these techniques to a vehicle adaptive control problem in which vehicle mass and the time-varying road grade are simultaneously estimated.

Note that a higher dimensional case, $\dot{X} = Z^{\mathrm{T}}(t)\theta + u(t)$, where $X(t)$ is an n-vector, $u(t)$ is an n-vector and $Z(t)$ is an $m \times n$ matrix can be viewed as a set of n systems of form (1); the bounds for θ on the basis of each of these first-order systems can be intersected to obtain tighter bounds for θ. Thus this higher dimensional case reduces to the first-order case considered in the paper. More details about this approach are provided in Appendix A.

2. THE INPUT OBSERVER

In this section, we describe the design of an input observer for system (1), (2) in the case when θ is accurately known. Specifically, we consider an input observer in the form proposed in Reference [4],

$$\hat{u}(t) = \varepsilon(t) + \alpha_0 y(t) + \theta^{\mathrm{T}}\phi(t)$$

$$\dot{\phi} = -\alpha_0\phi - \alpha_0 z$$

$$\dot{\varepsilon} = -\alpha_0\varepsilon - \alpha_0^2 y \qquad (6)$$

Here $\alpha_0 > 0$ is the observer gain and the states of the observer are ϕ and ε. Similar filters have been previously employed in the composite adaptation literature [13].

We next study the observer properties when it is initialized according to the following procedure:

$$\phi(0) = z(0)$$

$$\varepsilon(0) = \tfrac{1}{2}(\underline{u}(0) + \bar{u}(0)) - \alpha_0 y(0) - \hat{\theta}^{\mathrm{T}} z(0) \qquad (7)$$

where $\hat{\theta} \in \Theta^0$. Note that with $\hat{\theta} = \theta$, it follows that $\hat{u}(0) = \tfrac{1}{2}(\underline{u}(0) + \bar{u}(0))$, which is the best estimate of $u(0)$. The subsequent developments require, however, that we also characterize the worst-case error when the observer is initialized with $\hat{\theta} \neq \theta$. The observer error bounds are obtained in the following.

Proposition 1
If (1)–(6) hold, then for any $\phi(0)$, $z(0)$ the error $\hat{u}(t) - u(t)$ is ultimately bounded (as $t \to \infty$) in an interval $[-2\alpha_0\mu - (h/\alpha_0), 2\alpha_0\mu + (h/\alpha_0)]$. If, in addition, (7) holds then the error between $\hat{u}(t)$ and $u(t)$ satisfies the following bound:

$$|\hat{u}(t) - u(t)| \leqslant R_{\alpha_0\mu}(t)$$

$$R_{\alpha_0\mu}(t) = \alpha_0\mu + \left(e^{-\alpha_0 t} \left(\frac{3}{4} (\bar{u}(0) - \underline{u}(0))^2 + 3\alpha_0^2\mu^2 + 3 \max_{\theta,\hat{\theta}\in\Theta^0} ((\theta - \hat{\theta})^{\mathrm{T}} z(0))^2 \right) + \left(\alpha_0\mu + \frac{h}{\alpha_0} \right)^2 \right)^{1/2}$$

$$(8)$$

Proof
Let

$$\tilde{\hat{u}} = \varepsilon(t) + \alpha_0 x(t) + \theta^{\mathrm{T}}\phi(t)$$

$$= \hat{u}(t) - \alpha_0\zeta(t)$$

and

$$V = \tfrac{1}{2}(\tilde{\hat{u}} - u)^2$$

In the proof, we make repeated use of the identities $(a + b + c)^2 \leqslant 3a^2 + 3b^2 + 3c^2, 2|ab| \leqslant (a^2/d) + b^2 d$ that hold for any a, b, c and $d > 0$. By differentiating V with respect to time, we obtain

$$\dot{V} = (-\alpha_0\varepsilon - \alpha_0^2 x - \alpha_0^2\zeta + \alpha_0\theta^{\mathrm{T}}z + \alpha_0 u + \theta^{\mathrm{T}}(-\alpha_0\phi) + \theta^{\mathrm{T}}(-\alpha_0 z) - \dot{u})(\tilde{\hat{u}} - u)$$

$$= (-\alpha_0(\varepsilon + \alpha_0 x + \theta^{\mathrm{T}}\phi - u) - \alpha_0^2\zeta - \dot{u})(\tilde{\hat{u}} - u)$$

$$= -\alpha_0(\tilde{\hat{u}} - u)^2 + (-\alpha_0^2\zeta - \dot{u})(\tilde{\hat{u}} - u)$$

$$\leqslant -\frac{\alpha_0}{2}(\tilde{\hat{u}} - u)^2 + \frac{1}{2\alpha_0}\left(\alpha_0^2\mu + \sup_t |\dot{u}(t)| \right)^2$$

$$= -\alpha_0 V + \frac{1}{2\alpha_0}\left(\alpha_0^2\mu + \sup_t |\dot{u}(t)| \right)^2$$

Recall that if for some constant r,

$$\dot{V} \leqslant -\alpha_0 V + r$$

then

$$V(t) \leqslant e^{-\alpha_0 t} V(0) + (1 - e^{-\alpha_0 t})\frac{r}{\alpha_0} \leqslant e^{-\alpha_0 t} V(0) + \frac{r}{\alpha_0}$$

Using this and (4), we obtain

$$V(t) \leqslant e^{-\alpha_0 t} V(0) + \frac{(\alpha_0^2\mu + \sup_t |\dot{u}(t)|)^2}{2\alpha_0^2}$$

$$|\tilde{\hat{u}}(t) - u(t)| \leqslant \sqrt{e^{-\alpha_0 t}(\tilde{\hat{u}}(0) - u(0))^2 + \frac{(\alpha_0^2\mu + h)^2}{\alpha_0^2}}$$

This inequality, (2) and $\hat{u}(t) = \tilde{u} + \alpha_0\zeta(t)$ show that the error $\hat{u}(t) - u(t)$ is ultimately bounded (as $t \to \infty$) in an interval $[-2\alpha_0\mu - (h/\alpha_0), 2\alpha_0\mu + (h/\alpha_0)]$. To demonstrate (8), we need to upper bound $(\tilde{u}(0) - u(0))^2$ under the assumption that (7) holds. From (7), we have

$$\tilde{u}(0) = \tfrac{1}{2}(\underline{u}(0) + \bar{u}(0)) - \alpha_0\zeta(0) + (\theta - \hat{\theta})^{\mathrm{T}}z(0)$$

Using (3), we conclude that

$$(\tilde{u}(0) - u(0))^2 \leqslant \left(\left(\tfrac{1}{2}(\underline{u}(0) + \bar{u}(0)) - u(0) \right) - \alpha_0\zeta(0) + (\theta - \hat{\theta})^{\mathrm{T}}z(0) \right)^2$$

$$\leqslant \frac{3}{4}(\bar{u}(0) - \underline{u}(0))^2 + 3\alpha_0^2\mu^2 + 3 \max_{\theta,\hat{\theta}\in\Theta^0}((\theta - \hat{\theta})^{\mathrm{T}}z(0))^2$$

and (8) follows. The proof is complete. $\qquad\square$

Remark 1
According to Proposition 1, the error $\hat{u}(t) - u(t)$ is guaranteed to be ultimately bounded (as $t \to \infty$) in an interval $[-2\alpha_0\mu - (h/\alpha_0), 2\alpha_0\mu + (h/\alpha_0)]$. Consequently, increasing the observer gain, in general, improves the accuracy of estimating the input but it may make the observer more sensitive to measurement noise. The interval ultimately bounding the error and $R_{\alpha_0\mu}(t)$ for large t are minimized if $\alpha_0 = \sqrt{h/(2\mu)}$.

The asymptotic properties of the error bound are summarized in the following.

Proposition 2
Suppose the assumptions of Proposition 1 (including (7)) hold and $\alpha_0 \to \infty$, $\mu \to 0$, $\alpha_0\mu \to 0$. Then, $R_{\alpha_0\mu}(t) \to 0$, and $\hat{u}(t) \to u(t)$ $\forall t > 0$.

Proposition 2 follows immediately from the error bound (8). It implies that a large observer gain (relative to the rate of change of the input) and negligible noise guarantee accurate input estimation by the observer. Note, however, that in practical applications the sampling rate would limit the observer gain even if there is no noise.

A disadvantage of the error bound (8) is that it scales proportionally to the measurement noise. The following proposition shows that the estimation error can be small even if the measurement noise is large if this noise is slowly varying (i.e. low frequency). In fact, in this case the noise magnitude, μ, is irrelevant for the error bound.

Proposition 3
Suppose (1)–(6) hold and, in addition, $\dot{\zeta}$ is a Lebesgue-measurable function such that $\sup_t |\dot{\zeta}(t)| \leqslant \eta$. Then for any $\phi(0), z(0)$ the error $\hat{u}(t) - u(t)$ is ultimately bounded (as $t \to \infty$) in an interval $[-\eta - (h/\alpha_0), \eta + (h/\alpha_0)]$. If, in addition, (7) holds then the error between $\hat{u}(t)$ and $u(t)$ satisfies the following bound:

$$|\hat{u}(t) - u(t)| \leqslant \bar{R}_{\alpha_0\eta}(t)$$

$$\bar{R}_{\alpha_0\eta}(t) = \left(\mathrm{e}^{-\alpha_0 t} \left(\frac{1}{2}(\bar{u}(0) - \underline{u}(0))^2 + 2 \max_{\theta,\hat{\theta}\in\Theta^0}((\theta - \hat{\theta})^{\mathrm{T}}z(0))^2 \right) + \left(\eta + \frac{h}{\alpha_0} \right)^2 \right)^{1/2} \qquad (9)$$

Proof
The proof follows similarly to Proposition 1 with

$$V = \tfrac{1}{2}(\hat{u} - u)^2$$

Then,

$$\dot{V} = (\hat{u} - u)(\alpha_0(u - \hat{u}) + \alpha_0\dot{\zeta} - \dot{u})$$

$$= -\alpha_0(u - \hat{u})^2 + (\hat{u} - u)(\alpha_0\dot{\zeta} - \dot{u})$$

$$\leqslant -\alpha_0 V + \frac{(\alpha_0\eta + h)^2}{2\alpha_0}$$

Hence,

$$V(t) \leqslant \mathrm{e}^{-\alpha_0 t} V(0) + \frac{(\eta + (h/\alpha_0))^2}{2}$$

and using

$$\hat{u}(0) = \varepsilon(0) + \alpha_0 y(0) + \theta^{\mathrm{T}}\phi(0) = \tfrac{1}{2}(\underline{u}(0) + \bar{u}(0)) + (\theta - \hat{\theta})^{\mathrm{T}}\phi(0)$$

Equation (9) follows similar to the proof of Proposition 2. The proof is complete.   □

Bound (9) shows, in particular, that arbitrary constant measurement offsets due to sensor drifts, i.e. $y = x + \zeta$, where $\dot{\zeta}(t) \equiv 0$ can be handled by the observer irrespective of the offset magnitude.

Remark 2
According to Proposition 3, the error $\hat{u}(t) - u(t)$ is guaranteed to be ultimately bounded (as $t \to \infty$) in an interval $[-\eta - (h/\alpha_0), \eta + (h/\alpha_0)]$. This interval and $\bar{R}_{\alpha_0\eta}(t)$ for large t decrease as α_0 increases.

The asymptotic properties of the alternative error bound can be summarized in the following.

Proposition 4
Suppose the assumptions of Proposition 3 (including (7)) hold and $\alpha_0 \to \infty$. Then, $\bar{R}_{\alpha_0\eta}(t) \to \eta$ for any $t > 0$. Furthermore, if $\alpha_0 \to \infty$, $\eta \to 0$ then $\bar{R}_{\alpha_0\eta}(t) \to 0$ for $t > 0$.

An additional insight into the results in Propositions 1 and 3 can be obtained if we pre-multiply both sides of (1) and (2) by the transfer function $\alpha_0 s/(s + \alpha_0)$. Then,

$$\frac{\alpha_0 s}{s + \alpha_0} y - \frac{\alpha_0 s}{s + \alpha_0}\zeta = -\theta^{\mathrm{T}}\phi + \frac{\alpha_0}{s + \alpha_0}u$$

After algebraic manipulations, this implies

$$\alpha_0 y + \varepsilon + \theta^{\mathrm{T}}\phi = \frac{\alpha_0}{s + \alpha_0}u + \frac{\alpha_0 s}{s + \alpha_0}\zeta$$

Hence,

$$u - \hat{u} = \frac{s}{s + \alpha_0}u - \frac{\alpha_0 s}{s + \alpha_0}\zeta$$

This easily derived expression confirms and additionally explains the nature of results in Propositions 1 and 3.

Remark 3

In the derivations of the error bounds and throughout the paper, we assumed that the noise only affects the measurements of x and that the signal z is noise-free. If the noise affects z, similar bounds can be derived. Suppose, for example, $\xi(t) \in \Sigma$ is the noise additive to $z(t)$, where $\Sigma \subset \mathbf{R}^m$ is a compact set. Suppose also that the measurement noise $\zeta(t)$ in (2) can be decomposed into the sum of two terms, $\zeta(t) = \zeta^1(t) + \zeta^2(t)$, where $\sup_{t \geqslant 0} |\zeta^1(t)| \leqslant \mu$ and $\sup_{t \geqslant 0} |\dot{\zeta}^2(t)| \leqslant \eta$. This assumption on the measurement noise generalizes the two cases which we previously treated separately, and it may lead to less conservative bounds. Define

$$\kappa = \max_{\theta \in \Theta^0, \xi \in \Sigma} |\theta^{\mathrm{T}} \xi|$$

Then (7) implies the following error bound:

$$|\hat{u}(t) - u(t)| \leqslant \tilde{R}_{\alpha_0 \mu \eta \kappa}(t)$$

where

$$\tilde{R}_{\alpha_0 \mu \eta \kappa}(t) = \alpha_0 \mu + \left(\mathrm{e}^{-\alpha_0 t} \left[(\bar{u}(0) - \underline{u}(0))^2 + 4\alpha_0^2 \mu^2 + 4 \max_{\theta, \hat{\theta} \in \Theta^0, \xi(0) \in \Sigma} (((\theta - \hat{\theta})^{\mathrm{T}} z(0))^2 + ((\theta - \hat{\theta})^{\mathrm{T}} \xi(0))^2) \right] \right.$$

$$\left. + \left[\alpha_0 \mu + \kappa + \eta + \frac{h}{\alpha_0} \right]^2 \right)^{1/2}$$

This more general bound, $\tilde{R}_{\alpha_0 \mu \eta \kappa}(t)$, can be used in the subsequent developments in exactly the same manner as either $R_{\alpha_0 \mu}(t)$ or $\bar{R}_{\alpha_0 \eta}(t)$.

Remark 4

Our formulation of the input observer (6) and the derivations of the above error bounds have been performed in continuous-time. In the actual implementation of (6), inadequate (i.e. slow) sampling and approximation of (6) by a discrete-time system, such as

$$\hat{u}(t) = \varepsilon_d(t) + \alpha_0 y(t) + \theta^{\mathrm{T}} \phi_d(t)$$

$$\phi_d(t + \Delta t) = \mathrm{e}^{-\alpha_0 \Delta t} \phi_d(t) - (1 - \mathrm{e}^{-\alpha_0 \Delta t}) z(t)$$

$$\varepsilon_d(t + \Delta t) = \mathrm{e}^{-\alpha_0 \Delta t} \varepsilon_d(t) - (1 - \mathrm{e}^{-\alpha_0 \Delta t}) \alpha_0 y(t)$$

may result in additional and nonnegligible errors. These errors due to sampling (easily shown to be in the order of the sampling period, Δt, under appropriate assumptions) can be estimated and included in the input observer error bounds. Their treatment and the issues of optimally selecting the sampling rate, to balance good input observer accuracy with the containable computing effort, will be addressed in the future work.

We next consider the case when both u and θ are unknown.

3. ON-LINE PARAMETER ESTIMATION

If the parameter θ is not known, then in the input observer (6) we replace θ by its estimate $\hat{\theta}$ so that

$$\hat{u}(t) = \varepsilon(t) + \alpha_0 y(t) + \hat{\theta}^{\mathrm{T}} \phi(t)$$

$$\dot{\phi} = -\alpha_0 \phi - \alpha_0 z$$

$$\dot{\varepsilon} = -\alpha_0 \varepsilon - \alpha_0^2 y \tag{10}$$

Since $\hat{\theta}$ affects only the observer output, $\hat{u}$, and not the observer dynamics (i.e. the equations for $\dot{\phi}$ and $\dot{\varepsilon}$), we may view (10) as an infinite family of *concurrent* observers for $\hat{u}$, each corresponding to a particular value of $\hat{\theta}$.

The available information, besides (3) and (5), is either $|\zeta(t)| \leqslant \mu$ or $|\dot{\zeta}(t)| \leqslant \eta$. Suppose, for example, that $|\zeta(t)| \leqslant \mu$ and the conditions of Proposition 1 hold (the case when $|\dot{\zeta}(t)| \leqslant \eta$ and the conditions of Proposition 3 hold can be treated in the same way). Then, if in (10) we had $\hat{\theta} = \theta$ (i.e. we guessed the parameter right), we would have

$$u(t) - R_{\alpha_0 \mu}(t) \leqslant \hat{u}(t) \leqslant u(t) + R_{\alpha_0 \mu}(t)$$

Using the *a priori* bound (3), we obtain

$$\underline{u}(t) - R_{\alpha_0 \mu}(t) \leqslant \hat{u}(t) \leqslant \bar{u}(t) + R_{\alpha_0 \mu}(t) \tag{11}$$

Due to (10), inequality (11) is equivalent to

$$\underline{u}(t) - R_{\alpha_0 \mu}(t) - \varepsilon(t) - \alpha_0 y(t) \leqslant \hat{\theta}^{\mathrm{T}} \phi(t) \leqslant \bar{u}(t) + R_{\alpha_0 \mu}(t) - \varepsilon(t) - \alpha_0 y(t) \tag{12}$$

Definition
A set $\Theta(t)$ of plausible parameter estimates at time $t > 0$ is the set of all $\hat{\theta} \in \mathbf{R}^m$ yielding (12).

Note that $\Theta(t)$ is nonempty for each $t > 0$ because $\theta \in \Theta(t)$. Setting $\Theta(0) = \Theta^0$, we denote

$$F(t) \stackrel{\triangle}{=} \bigcap_{0 \leqslant \tau \leqslant t} \Theta(\tau) \tag{13}$$

Since $\theta \in \Theta(t)$ for $t \geqslant 0$, $\theta \in F(t)$.

Remark 5
Suppose the time-rate of change of the noise is bounded by η but the upper bound on the noise magnitude is unknown, as in Proposition 3. Then, (12), (13) still hold but in (12), $R_{\alpha_0 \mu}(t)$ should be replaced by $\bar{R}_{\alpha_0 \eta}(t)$, defined in (9).

Note that, in general, $\hat{\theta} \in F(t)$ does not uniquely identify $\hat{\theta}$. The selection of a specific $\hat{\theta} \in F(t)$ at time t can be made in a number of different ways. One choice, motivated by the set-membership estimation literature (see e.g. Reference [14]), is to use the *Chebyshev centre* of $F(t)$. This is the point for which the maximum distance to any other point within $F(t)$ is minimized.

Since $F(t_1) \subset F(t_2)$ for $t_1 \geqslant t_2$, $F(t)$ provides tighter and tighter overbounds of the unknown parameter $\{\theta\}$ as t increases. We next seek conditions guaranteeing that $F(t)$ for

$t > 0$ provides as tight overbound of θ as possible and converges to $\{\theta\}$ as the observer gain, α_0, increases.

4. ASYMPTOTIC PROPERTIES

Consider the asymptotic properties of the set $F(t)$, defined by (13), as $\alpha_0 \to \infty, \mu \to 0, \mu\alpha_0 \to 0$. Subtracting $\theta^{\mathrm{T}}\phi(t)$ from each part of inequality (12), we obtain an equivalent inequality,

$$(\underline{u}(t) - u(t)) + (u(t) - \varepsilon(t) - \alpha_0 y(t) - \theta^{\mathrm{T}}\phi(t)) - R_{\alpha_0\mu}(t)$$

$$\leqslant (\hat{\theta} - \theta)^{\mathrm{T}}\phi(t) \leqslant (\bar{u}(t) - u(t)) + (u(t) - \varepsilon(t) - \alpha_0 y(t) - \theta^{\mathrm{T}}\phi(t)) + R_{\alpha_0\mu}(t)$$

Applying Proposition 1, we obtain the estimate

$$(\underline{u}(t) - u(t)) - 2R_{\alpha_0\mu}(t) \leqslant (\hat{\theta} - \theta)^{\mathrm{T}}\phi(t) \leqslant (\bar{u}(t) - u(t)) + 2R_{\alpha_0\mu}(t) \tag{14}$$

Inequality (14) leads to the tightest constraints on $(\hat{\theta} - \theta)$ at the time instants when the bounds $\underline{u}(t)$ and $\bar{u}(t)$ are tight. Specifically, suppose there exist time instants $t_1^1, \ldots, t_1^{n_1}$ and $t_2^1, \ldots, t_2^{n_2}$ such that $u(t_1^i) = \underline{u}(t_1^i)$, $i = 1, \ldots, n_1$, $u(t_2^i) = \bar{u}(t_2^i)$, $i = 1, \ldots, n_2$. At these time instants, (14) implies

$$\phi^{\mathrm{T}}(t_1^i)(\hat{\theta} - \theta) \geqslant -2R_{\alpha_0\mu}(t_1^i), \quad i = 1, \ldots, n_1$$

$$\phi^{\mathrm{T}}(t_2^j)(\hat{\theta} - \theta) \leqslant 2R_{\alpha_0\mu}(t_2^j), \quad j = 1, \ldots, n_2 \tag{15}$$

Theorem 1
Suppose $F(t)$ is defined by (13) and there exist time instants $t_1^1, \ldots, t_1^{n_1} > 0$ and $t_2^1, \ldots, t_2^{n_2} > 0$ such that

$$u(t_1^i) = \underline{u}(t_1^i), \quad i = 1, \ldots, n_1$$

$$u(t_2^j) = \bar{u}(t_2^j), \quad j = 1, \ldots, n_2 \tag{16}$$

Define a matrix

$$S = \begin{bmatrix} z^{\mathrm{T}}(t_1^1) \\ z^{\mathrm{T}}(t_1^2) \\ \vdots \\ z^{\mathrm{T}}(t_1^{n_1}) \\ -z^{\mathrm{T}}(t_2^1) \\ \vdots \\ -z^{\mathrm{T}}(t_2^{n_2}) \end{bmatrix} \in \mathbf{R}^{(n_1+n_2)\times m} \tag{17}$$

Suppose the only solution to the system of linear inequalities

$$S\tilde{\theta} \leqslant 0, \quad \tilde{\theta} \in \mathbf{R}^m \tag{18}$$

is $\tilde{\theta} = 0$. Suppose further that z in (1) is differentiable with $\dot{z}$ being bounded. Then for $t > \max\{t_1^1, \ldots, t_1^{n_1}, t_2^1, \ldots, t_2^{n_2}\}$ it follows that $F(t) \rightarrow \{\theta\}$ and $\hat{u}(t) \rightarrow u(t)$ as $\alpha_0 \rightarrow \infty$, $\mu \rightarrow 0$, and $\alpha_0 \mu \rightarrow 0$.

Proof
From (15), the following set inclusion holds for $t > \max\{t_1^1, \ldots, t_1^{n_1}, t_2^1, \ldots, t_2^{n_2}\}$:

$$F(t) \subset \{\theta\} + Q^1 \bigcup Q^2(\alpha_0, \mu) \tag{19}$$

where

$$Q^1 = \{\tilde{\theta} \in \mathbf{R}^m : \exists \hat{\theta} \in \Theta^0, \; \theta \in \Theta^0 \text{ such that } \tilde{\theta} = \hat{\theta} - \theta$$

$$z^{\mathrm{T}}(t_1^i)\tilde{\theta} \leqslant 0, \; -z(t_2^j)^{\mathrm{T}}\tilde{\theta} \leqslant 0, \; i = 1, \ldots, n_1, \; j = 1, \ldots, n_2\}$$

$$Q^2(\alpha_0, \mu) = \{\tilde{\theta} \in \mathbf{R}^m : \exists \hat{\theta} \in \Theta^0, \; \theta \in \Theta^0 \text{ such that}$$

$$\tilde{\theta} = \hat{\theta} - \theta, \; 0 < z^{\mathrm{T}}(t_1^i)\tilde{\theta} \leqslant 2R_{\alpha_0\mu}(t_1^i) - (\phi(t_1^i) + z(t_1^i))^{\mathrm{T}}\tilde{\theta}$$

$$0 < -z^{\mathrm{T}}(t_2^j)\tilde{\theta} \leqslant 2R_{\alpha_0\mu}(t_2^j) - (\phi(t_2^j) + z(t_2^j))^{\mathrm{T}}\tilde{\theta}$$

$$i = 1, \ldots, n_1, \; j = 1, \ldots, n_2.\}$$

Since $S\tilde{\theta} \leqslant 0$ implies $\tilde{\theta} = 0$, it follows that $Q^1 = \{0\}$ and S has full column rank. The latter and the compactness of Θ^0 imply that $Q^2(\alpha_0, \mu)$ is bounded. As $\alpha_0 \rightarrow \infty$, it follows, with the help of integration by parts, that

$$\phi(t_1^i) + z(t_1^i) = \mathrm{e}^{-\alpha_0 t_1^i} z(0) - \int_0^{t_1^i} \mathrm{e}^{-\alpha_0(t_1^i - \tau)}\alpha_0 z(\tau)\,\mathrm{d}\tau + z(t_1^i)$$

$$= \mathrm{e}^{-\alpha_0 t_1^i} z(0) - z(t_1^i) + \int_0^{t_1^i} \mathrm{e}^{-\alpha_0(t_1^i - \tau)}\dot{z}(\tau)\,\mathrm{d}\tau + z(t_1^i)$$

$$\rightarrow 0$$

for $i = 1, \ldots, n_1$. Similarly, as $\alpha_0 \rightarrow \infty$, it follows that $\phi(t_2^j) + z(t_2^j) \rightarrow 0, j = 1, \ldots, n_2$. Further, as $\alpha_0 \rightarrow \infty$, $\mu \rightarrow 0$, $\alpha_0 \mu \rightarrow 0$ it follows that $R_{\alpha_0\mu}(t_1^i) \rightarrow 0$ and $R_{\alpha_0\mu}(t_2^j) \rightarrow 0$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$. Thus $Q^2(\alpha_0, \mu) \rightarrow \{0\}$. This completes the proof. $\qquad\square$

Remark 6
Suppose that the time-rate of change of the noise is bounded by η, as in Proposition 3. Then, with $F(t)$ defined by (12), (13), where $R_{\alpha_0\mu}$ is replaced by $\bar{R}_{\alpha_0\eta}$ in (12), and under the remaining assumptions of Theorem 1, the result of Theorem 1 holds as $\alpha_0 \rightarrow \infty, \eta \rightarrow 0$.

Remark 7
Theorem 1 indicates that accurate input and parameter estimation becomes possible when the unknown input *touches* its bounds a sufficient number of times. Note that the knowledge of what these time instants are is not required.

Figure 1 illustrates graphically conditions (17)–(18) in the case $m = 2$, $n_1 = 2$, $n_2 = 1$. As Figure 1(a) shows, it may be possible to accurately estimate (in the sense of Theorem 1)
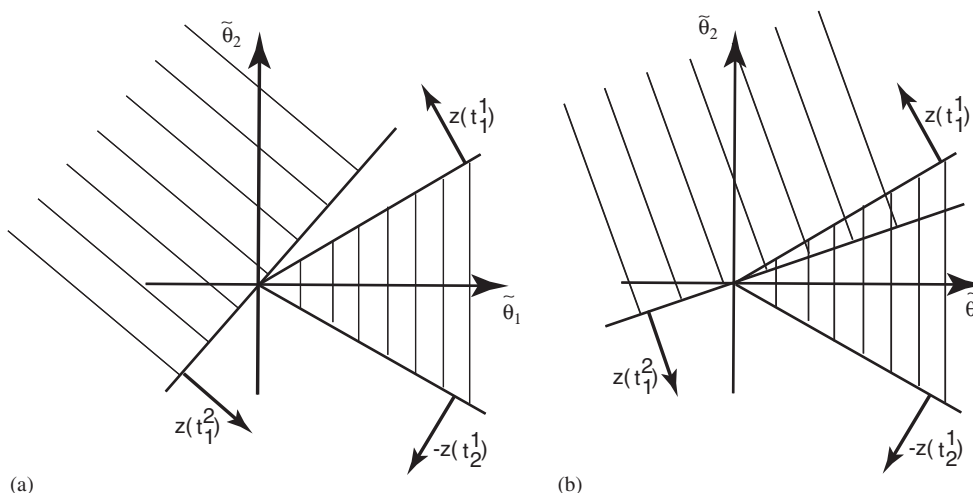
Figure 1. Graphical illustrations of conditions (17)–(18) for $m = 2$, $n_1 = 2$ and $n_2 = 1$: (a) conditions hold; and (b) conditions do not hold.

two unknown parameters if the unknown input touches its bounds just three times. Figure 1(b) indicates, however, that three touches may not always be enough.

In the general case, the *inequality consequence*, $S\tilde{\theta} \leqslant 0$ implies $\tilde{\theta} = 0$, can be checked using a linear programming solver (this is a feasibility problem) or, more elegantly, via the Farkas lemma. Specifically, the necessary and sufficient condition for it to hold is the existence of a matrix $X \in \mathbf{R}^{(n_1+n_2) \times 2m}$ with all *nonnegative* elements such that

$$S^{\mathrm{T}} X = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \tag{20}$$

The MATLAB$^{\parallel}$ function *lsqnonneg* can be used for a straightforward computational check if X satisfying (20) with all nonnegative elements exist.

We note that a condition from the identification literature, that the matrix $J = \sum_{i=1}^{n_1} z(t_1^i) z^{\mathrm{T}}(t_1^i) + \sum_{j=1}^{n_2} z(t_2^j) z^{\mathrm{T}}(t_2^j)$ is positive-definite, is only necessary but, as simple examples in the case $m = 1$ show, not sufficient for $S\tilde{\theta} \leqslant 0$ to imply $\tilde{\theta} = 0$. However, if $F(t)$ is defined by (13), J is positive-definite and $n_1 = n_2$, $t_1^i = t_2^i$, $i = 1, \ldots, n_1$ (i.e. $\underline{u}(t_1^i) = u(t_1^i) = \bar{u}(t_1^i)$), then $F(t) \to \{\theta\}$ and $\hat{u}(t) \to u(t)$ for $t \geqslant \max\{t_1^1, \ldots, t_1^{n_1}\}$ as $\alpha_0 \to \infty$, $\mu \to 0$, $\alpha_0 \mu \to 0$. The same result holds if $F(t)$ is defined as in Remark 7 and $\alpha_0 \to \infty$, $\eta \to 0$.

Suppose now that for any $T > 0$ there exist time instants $t_1^i > T$, $i = 1, \ldots, n_1$, and $t_2^j > T$, $j = 1, \ldots, n_2$, which satisfy the conditions of Theorem 1 (i.e. conditions of Theorem 1 hold

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$^{\parallel}$MATLAB is a registered trademark of the MathWorks Inc. of Natick, MA.

persistently). Then the asymptotic bounds on $F(t)$ as $t \to \infty$ can be characterized on the basis of (19). In particular, for any $\delta > 0$, there exist $T > 0$ such that if the matrix Φ is defined by (17) with z replaced by ϕ and is of full column rank, and if $t > \max\{t_1^i, t_2^j, i = 1, \ldots, n_1, j = 1, \ldots, n_2\}$, then

$$F(t) \subset \{\theta\} + (\delta + 4\alpha_0\mu + (2h/\alpha_0))\|(\Phi^{\mathrm{T}}\Phi)^{-1}\Phi^{\mathrm{T}}\|\mathscr{B}_\infty$$

where $\|\cdot\|$ denotes 1-matrix norm and B_∞ is the ∞-norm unit ball in $\mathbf{R}^m$.

5. COMPUTATIONAL DETAILS

While the on-line implementation of the input observer (10) is straightforward, the on-line implementation of (13) warrants further discussion.

Firstly, the continuous-time version of (13) can be replaced by a sampled-data version, $F(t) \approx F_d(k)$:

$$F_d(k) = \bigcap_{t=0,\Delta,\ldots,k\cdot\Delta} \Theta(t), \quad k \cdot \Delta \leqslant t < (k+1) \cdot \Delta$$

$$\Theta(0) = \Theta^0 \tag{21}$$

where Δ is the time period between two updates of F_d. In a recursive form,

$$F_d(k+1) = F_d(k) \bigcap \Theta((k+1) \cdot \Delta), \quad F_d(0) = \Theta^0 \tag{22}$$

Note that $F(t) \in F_d(k)$ for $k \cdot \Delta \leqslant t < (k+1) \cdot \Delta$. Furthermore, if the assumptions of Theorem 1 (or Remark 6) hold for $t_1^i = k_1^i\Delta$, $t_2^j = k_2^j\Delta$, $k_1^i, k_2^j \in \mathbf{Z}^+$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$ then the conclusion of Theorem 1 (or Remark 6, respectively) holds for $F_d(k)$, $k \geqslant \max\{k_1^1, \ldots, k_1^{n_1}, k_2^1, \ldots, k_2^{n_2}\}$.

Secondly, for real-time implementation it is highly desirable that the representation of $F_d(k)$, update (22) and the computation of the Chebyshev centre of $F_d(k)$ (since $\hat{\theta}(t)$ for $k \cdot \Delta \leqslant t < (k+1)\Delta$ is set equal to the Chebyshev centre of $F_d(k)$) are as simple as possible.

Standard algorithms in the set-membership literature can be used for on-line construction of easily computable overbounds of $F_d(k)$. Well-developed procedures exist based on bounding methods for an intersection of an ellipsoid and a strip [15] or a parallelotope and a strip [16]. See also Reference [1].

In our work, motivated by specific automotive applications, we followed a similar approach where we even further sacrificed the tightness of the overbound for the simplicity of implementation and computations, which are important considerations for the automotive applications where computational resources (chronometrics and memory) are very limited.

Specifically, in our approach we replace the updates of $F_d(k)$ with the updates of an m-dimensional rectangle, $P(k)$, which overbounds $F_d(k)$, i.e. $F_d(k) \subset P(k)$. In the case $m = 1$ a simple algorithm of [2] is sufficient. These computational procedures are reviewed in more detail in Appendix B.

6. POWERTRAIN CONTROL APPLICATIONS

The first example is based on an internal combustion engine intake manifold filling and emptying dynamics,

$$\dot{p} = \frac{R \cdot T_m}{V_m}(W_{\text{th}} + W_{\text{egr}} - W_{\text{cyl}}) \tag{23}$$

where p is the measured intake manifold pressure, W_{th} is the flow through the engine throttle measured by a mass flow sensor, W_{egr} is the flow through the exhaust gas recirculation (EGR) valve and W_{cyl} is the mean-value of the flow into the engine cylinders. The flows W_{egr} and W_{cyl} are not measured. The R, V_m and T_m are, respectively, the gas constant, the intake manifold volume, and the gas temperature in the intake manifold. See Figure 2.

Both W_{cyl} and W_{egr} need to be accurately estimated for good air–fuel ratio and spark control, and this estimation is therefore essential for vehicle drivability, fuel economy and emissions reduction. Furthermore, an EGR flow estimate can be used for a feedforward compensation in partial air pressure control system [17]. The cylinder flow, W_{cyl}, can be estimated according to the following equation [18]:

$$W_{\text{cyl}} = W_{\text{cyl},0} \triangleq \frac{N}{RT_m}(k_{\text{cyl},0} + k_{\text{cyl},1} \cdot p) \tag{24}$$

where N is the engine speed and $k_{\text{cyl},0}$, $k_{\text{cyl},1}$ are parameters. With aging, soot deposits may form in the intake runners (this is especially a problem in direct injection spark ignition engines, see Reference [18]). These deposits can reduce the flow area in ways dependent on unique driving and environmental factors for each vehicle, and difficult to predict in advance. The part-to-part manufacturing variabilities can also render estimate (24) inaccurate. Thus there is a benefit to using on-line adaptation to improve the cylinder flow estimate over time. For example, an adaptive model may have the form,

$$W_{\text{cyl}} = \theta_1 \cdot W_{\text{cyl},0} \tag{25}$$

where θ_1 is a constant parameter to be estimated on-line. The model for the EGR flow, W_{egr}, can be developed on the basis of the compressible flow 'orifice' equation,

$$W_{\text{egr}} = f_{\text{egr}}\left(\frac{p}{p_{\text{ex}}}\right) \cdot A_{\text{egr}} \cdot \frac{p_{\text{ex}}}{\sqrt{T_{\text{ex}}}} \tag{26}$$
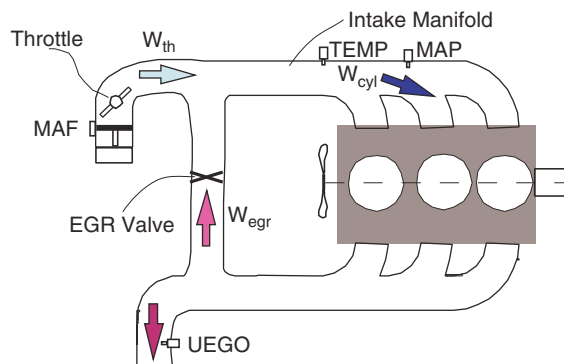

Figure 2. Internal combustion engine with exhaust gas recirculation (EGR).

where p_{ex} is the exhaust pressure, T_{ex} is the exhaust temperature, A_{egr} is the effective flow area of the EGR valve which is a function of measured EGR valve position, and f_{egr} is a known function of the pressure ratio across the EGR valve:

$$f_{egr}(x) = \begin{cases} \gamma^{1/2}\left(\dfrac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)} & \text{if } x \leqslant 0.5 \\[2mm] x^{1/\gamma}\left\{\dfrac{2\gamma}{\gamma-1}[1 - x^{(\gamma-1)/\gamma}]\right\}^{1/2} & \text{if } x > 0.5 \end{cases}$$

Here γ, the ratio of specific heats, is considered to be constant ($\gamma = 1.4$). In reality, W_{egr} is difficult to estimate based on (26) because p_{ex} and T_{ex} are not measured, and A_{egr} may change over time due to soot deposits [18]. Additionally, (26) only applies for small openings of the EGR valve, and for large openings A_{egr} exhibits engine speed dependence. Thus a natural approach is to treat W_{egr} as an unknown input and estimate it on-line simultaneously with θ_1 in (25). An adaptive multiplier, θ_2, can also be introduced to compensate mass air flow sensor drifts so that $W_{th} = \theta_2 \cdot W_{th,0}$. Specifically, if we let

$$x = p, \quad z_1 = -\frac{RT_m}{V_m} W_{cyl,0}, \quad z_2 = \frac{RT_m}{V_m} W_{th,0}$$

$$u = \frac{RT_m}{V_m} W_{egr}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{27}$$

then the system takes the standard form (1).

We assume that in (5),

$$\Theta^0 = [0.7, 1.9] \times [0.9, 1.7]$$

so that the nominal parameter estimates are $\hat{\theta}^1 = \hat{\theta}^2 = 1.3$. Note that this problem is challenging since all three flows W_{cyl}, W_{th} and W_{egr} are uncertain and the measurements of pressure, p, are noisy. Our estimation algorithm will be applied to trajectories of the pressure and of the three flows resulting from changing throttle and the EGR valve positions during a 12 s time period. These trajectories are shown in Figure 3.

The *a priori* knowledge of bounds on the unknown input is not an unreasonable assumption. For example, in the EGR flow estimation problem, while (26) cannot be used to accurately estimate W_{egr} it still can be used to accurately estimate the upper and lower bounds on W_{egr}.

The bounds on the EGR flow, $\underline{W}_{egr}$ and $\bar{W}_{egr}$, are also shown in Figure 3. These bounds are estimated on-line from Equation (26) using measured intake manifold pressure, p, and known ranges of exhaust pressure, p_{ex}, temperature, T_{ex}, and EGR valve effective flow area, A_{egr}. In (3), $\underline{u} = (RT_m/V_m)\underline{W}_{egr}$, $\bar{u} = (RT_m/V_m)\bar{W}_{egr}$. The nominal cylinder flow estimate, $\hat{W}_{cyl} = \hat{\theta}_1 W_{cyl,0}$ with $\hat{\theta}_1 = 1.3$ is significantly higher than the actual cylinder flow, W_{cyl}, because the latter corresponds to $W_{cyl} = \theta_1 W_{cyl,0}$ with $\theta_1 = 0.8$. Similarly, the nominal throttle flow estimate, $\hat{W}_{th} = \hat{\theta}_2 W_{th,0}$ with $\hat{\theta}_2 = 1.3$ is significantly higher than the actual throttle flow, $W_{th} = \theta_2 W_{th,0}$ with $\theta_2 = 1.0$.

The upper bound on the measurement noise in (2) was assumed to be $\mu = 1$ kPa, and $h = 101.87$ in (4) was estimated by model simulation. The input observer gain α_0 was set based on Remark 1 as $\alpha_0 = (h/(2\mu))^{0.5} = 7.14$. The time-dependent input observer error bound,
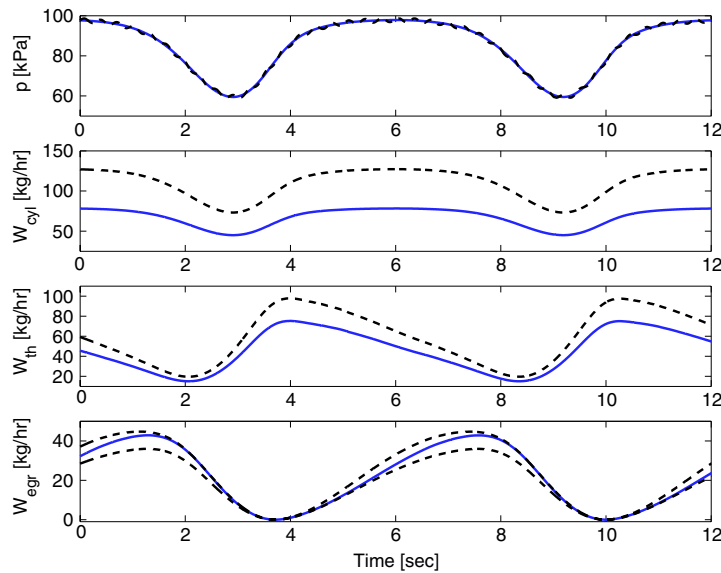
Figure 3. Time histories (from (a) at the top to (d) at the bottom) of: (a) $x = p$ (solid) and noisy measurement y (dashed); (b) W_{cyl} (solid) and $\hat{W}_{\mathrm{cyl}} = \hat{\theta}_1 W_{\mathrm{cyl},0}$ with $\hat{\theta}_1 = 1.3$ (dashed); (c) W_{th} (solid) and $\hat{W}_{\mathrm{th}} = \hat{\theta}_2 W_{\mathrm{th},0}$ with $\hat{\theta}_1 = 1.3$ (dashed); and (d) W_{egr} (solid), $\underline{W}_{\mathrm{egr}}$ (dashed) and $\bar{W}_{\mathrm{egr}}$ (dashed).

$R_{\alpha_0\mu}(t)$, was calculated according to (8). Figure 4 shows the time-dependent input observer error bound, $R_{\alpha_0\mu}(t)$, calculated according to (8).

Figure 5 shows that the simultaneous input and parameter estimation can be accomplished quite well by our algorithm. The estimated EGR flow, $\hat{W}_{\mathrm{egr}}$, converges to the actual EGR flow, W_{egr}, very fast and, in fact, as Figure 6 shows, the area of the rectangle $P(k)$, which bounds plausible parameter estimates at $t = k\Delta$, is reduced by a factor of 13.5 within just 12 s. The parameter estimates, $\hat{\theta}_1, \hat{\theta}_2$ converge to a small neighbourhood of θ_1, θ_2 within the same time period.

Other powertrain control applications include the speed dynamics of an internal combustion engine which are described by the following equation:

$$J\dot{\omega} = \tau_{\mathrm{eng}} - \tau_{\mathrm{load}}$$

where J is a known inertia of the crankshaft and τ_{load} is an unknown load torque on the crankshaft. The engine torque, τ_{eng}, may be nominally characterized by a model, $\tau_{\mathrm{eng},0}$, but aging and piece-to-piece variability can render this nominal model inaccurate. Assuming that the uncertainty can be adequately captured by an offset and a multiplier, so that $\tau_{\mathrm{eng}} = \theta_1 \cdot \tau_{\mathrm{eng},0} + \theta_2$, and by defining

$$x = \omega, \quad u = -\frac{1}{J}\tau_{\mathrm{load}}, \quad z_1 = \frac{1}{J}\tau_{\mathrm{eng},0}, \quad z_2 = \frac{1}{J} \cdot 1$$

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
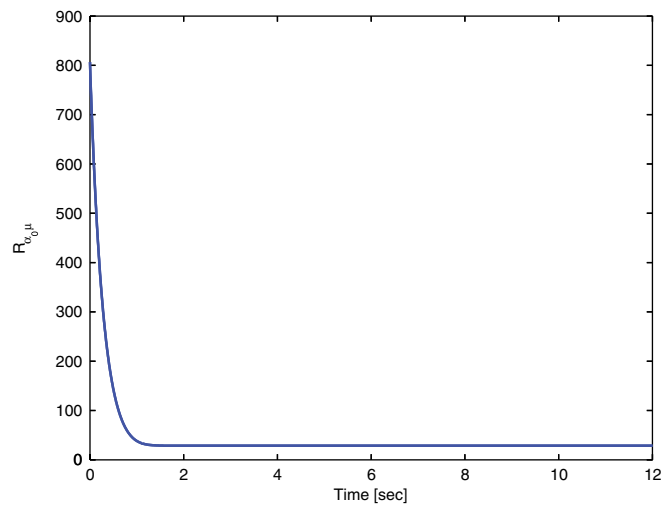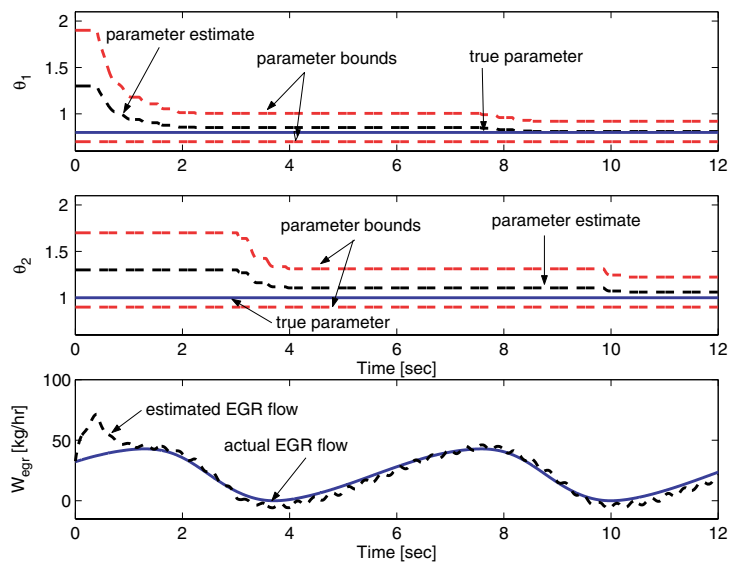
Figure 4. Time history of $R_{\alpha_0\mu}(t)$.

Figure 5. Time histories (from (a) at the top to (c) at the bottom) of: (a) true parameter θ_1 (solid), parameter estimate $\hat{\theta}_1 = c_1$ (dashed), parameter bounds $c_1 + l_1$ and $c_1 - l_1$ (dashed); (b) true parameter θ_2 (solid), parameter estimate $\hat{\theta}_2 = c_2$ (dashed), parameter bounds $c_2 + l_2$ and $c_2 - l_2$ (dashed); and (c) actual EGR flow W_{egr} (solid) and estimated EGR flow, $\hat{W}_{egr}$ (dashed).

the system is transformed into the standard form (1). The accurate knowledge of τ_{eng} is required for engine and powertrain control, in particular, for drivability and engine/transmission coordination during gear shifts. Additionally, the estimate of τ_{load} can be included in the feedforward compensation to improve idle speed or cruise control. Experiments are currently being pursued for this application that will be reported elsewhere.
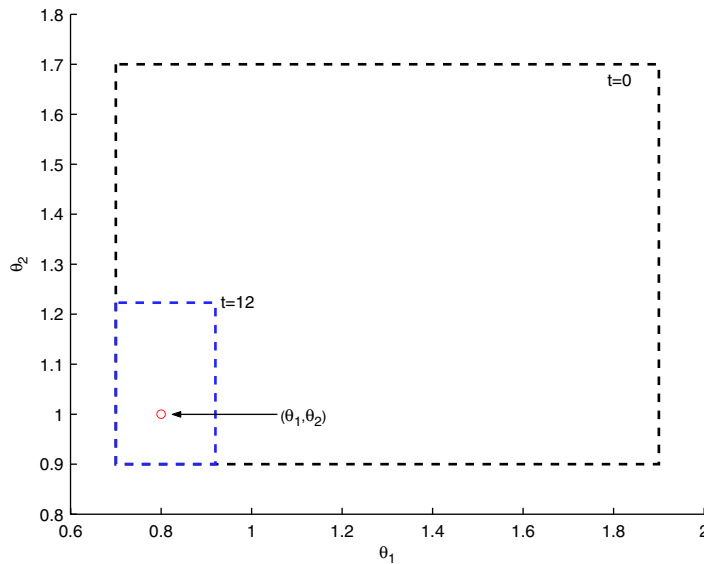
Figure 6. The rectangles $P(0)$ and $P(k)$ at $t = k\Delta = 12$ s.

7. CONCLUDING REMARKS

In this paper, we addressed a problem of simultaneously estimating an input and an unknown parameter in a dynamic system. This problem is not solvable without special assumptions. We showed that this problem can be treated by combining a conventional input observer with a set-membership algorithm for parameter estimation. The set-membership algorithm provides a set of plausible parameter estimates consistent with the *a priori* available information including bounds on the unknown input and expected input observer errors for the case with no parametric uncertainty. The potential applications to engine control have been discussed and illustrated with a simulation example.

APPENDIX A: GENERALIZATION TO HIGHER ORDER SYSTEMS

The theoretical developments in this paper mainly concern a first-order system which was relevant to a particular class of powertrain control applications. We now discuss how certain classes of higher order systems can be treated.

Suppose a higher order system can be transformed into the form,

$$\dot{x} = Z^{\mathrm{T}}(t)\theta + u(t)$$
$$y(t) = x(t) + \zeta(t) \tag{A1}$$

where x is an n-vector state, $Z(t)$ is an $m \times n$ matrix, θ is an m-vector of constant unknown parameters, u is an n-vector unknown input, y is an n-vector measured output, and ζ is an

n-vector noise. In this case, μ, $\underline{u}(t)$, $\bar{u}(t)$ and h are n-vectors, and (2), (3), (4) are understood to apply to each vector component.

System (A1) can be equivalently viewed as a system of n differential equations of the form,

$$\dot{x}_1 = Z_{11}(t)\theta_1 + Z_{21}(t)\theta_2 + \cdots + Z_{m1}(t)\theta_m + u_1(t)$$
$$\dot{x}_2 = Z_{12}(t)\theta_1 + Z_{22}(t)\theta_2 + \cdots + Z_{m2}(t)\theta_m + u_2(t)$$
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A2)$$
$$\dot{x}_n = Z_{1n}(t)\theta_1 + Z_{2n}(t)\theta_2 + \cdots + Z_{mn}(t)\theta_m + u_n(t)$$

where Z_{ij} is the (i,j)th entry of the matrix Z. Each of these differential equations is in the standard form (1) and can be used as a basis for generating a set-membership estimate (13), i.e. $\hat{\theta} \in F_i(t)$, at time t. Combining the information for $i = 1, \ldots, n$, we obtain $\hat{\theta} \in \bigcap_{i=1,\ldots,n} F_i(t)$.

As a specific example of the transformation into form (A1), consider a general nonlinear system,

$$\dot{\chi} = f(\chi) + g(\chi)v + p(\chi)\theta$$

where $\chi(t) \in \mathbf{R}^q$ is the state, $v(t) \in \mathbf{R}^n$ is the unknown input, and $\theta \in \Theta^0 \subset \mathbf{R}^m$ is the vector of unknown parameters. The objective is to simultaneously estimate $v(t)$ and θ. Assuming that the state, χ, can be measured, let $x(t) \in \mathbf{R}^n$ be some output of the system so that,

$$x = h(\chi)$$

Note that the dimensionality of x is the same as that of v. Then,

$$\dot{x} = \frac{\partial h}{\partial \chi}(f(\chi) + p(\chi)\theta + g(\chi)v) \qquad\qquad (A3)$$

There are two approaches for converting (A3) into form (A1). In the first approach, we define

$$Z = \left(\frac{\partial h}{\partial \chi}p(\chi)\right)^{\mathrm{T}}, \quad u = \frac{\partial h}{\partial \chi}(f(\chi) + g(\chi)v)$$

and obtain a system of form (A1). If the $n \times n$ matrix $((\partial h/\partial \chi)g(\chi(t)))$ is invertible then the estimate of $v(t)$, $\hat{v}(t)$, can be uniquely computed from the estimate of $u(t)$, $\hat{u}(t)$, as $\hat{v}(t) = ((\partial h/\partial \chi)g(\chi(t)))^{-1}(\hat{u}(t) - (\partial h/\partial \chi)f(\chi(t)))$. In the second approach to estimating $v(t)$ and θ, we view the term, $(\partial h/\partial \chi)f(\chi)$ in (A3), as $(\partial h/\partial \chi)f(\chi) \cdot \bar{\theta}$, where $\bar{\theta} = 1$ is an additional constant parameter to be estimated along with θ. In that case, to convert (A3) to form (A1) we can define $u = ((\partial h/\partial \chi)g(\chi))v$,

$$Z = \begin{bmatrix} \left(\dfrac{\partial h}{\partial \chi}p(\chi)\right)^{\mathrm{T}} \\[2mm] \left(\dfrac{\partial h}{\partial \chi}f(\chi)\right)^{\mathrm{T}} \end{bmatrix}, \quad \tilde{\theta} = \begin{bmatrix} \theta \\ \bar{\theta} \end{bmatrix}$$

and impose (5) in the form $\tilde{\theta} \in \tilde{\Theta}^0 = \Theta^0 \times \{1\}$. If the $n \times n$ matrix $((\partial h/\partial \chi)g(\chi(t)))$ is invertible, then the estimate of $v(t)$, $\hat{v}(t)$, can be uniquely computed from the estimate of $u(t)$, $\hat{u}(t)$, as $\hat{v}(t) = ((\partial h/\partial \chi)g(\chi(t)))^{-1}\hat{u}(t)$.

To provide a brief illustration of a practical problem which can be treated with these ideas, suppose in the flow estimation problem treated in Section 6, the air flow sensor is *slow* and its

dynamics are

$$\dot{W}_{\text{th},m} + aW_{\text{th},m} = aW_{\text{th}} \tag{A4}$$

where $W_{\text{th},m}$ is the air flow sensor measurement, W_{th} is the actual flow of the air through the throttle and $1/a$ is the air flow meter time constant. In this case, one may not use $W_{\text{th},m}$ in place of W_{th} in (23) when estimating W_{egr} and unknown parameters. Suppose, however, that a nominal model for the throttle flow, $W_{\text{th},0}$, is available. For instance, such a nominal model can be based on an orifice equation (see Reference [19]), and it determines the throttle flow as a function of pressure drop across the throttle and throttle position. To compensate for aging and piece-to-piece variability, an adaptive model for the throttle flow, W_{th}, can be introduced as follows:

$$W_{\text{th}} = \theta_2 W_{\text{th},0} + \theta_3$$

where θ_2 and θ_3 are the offset and the multiplier modifiers, respectively. The parameterization of the cylinder flow is as in Section 6, i.e. $W_{\text{cyl}} = \theta_1 W_{\text{cyl},0}$. Then the problem of simultaneously estimating θ_1, θ_2, θ_3 and W_{egr} on the basis of (23) and (A4) can be solved using the higher order case approach described in this appendix.

APPENDIX B: A SET-MEMBERSHIP COMPUTATIONAL ALGORITHM

In our approach, we replace the updates of $F_d(k)$ with the updates of an m-dimensional rectangle, $P(k)$, which overbounds $F_d(k)$, i.e. $F_d(k) \subset P(k)$. The centre of $P(k)$ is denoted by $c(k) \in \mathbf{R}^m$ and its half-sides are $l_1(k), \ldots, l_m(k)$, i.e.

$$P(k) = \{\theta = [\theta_1, \ldots, \theta_m]^{\text{T}} \in \mathbf{R}^m : c_i(k) - l_i(k) \leqslant \theta_i \leqslant c_i(k) + l_i(k), \quad i = 1, \ldots, m\} \tag{B1}$$

Note that the Chebyshev centre of $P(k)$ is $c(k)$ and thus $\hat{\theta}(t) = c(k)$ for $k \cdot \Delta \leqslant t < (k+1) \cdot \Delta$.

A new rectangle, $P(k)$, is formed recursively so that $P(k-1) \bigcap \Theta(k\Delta) \subset P(k)$. The computationally straightforward procedure to update $c(k)$ and $l(k)$ amounts to tightening each side of the rectangle $P(k-1)$ to overbound the set $P(k-1) \bigcap \Theta(k\Delta)$.

Referring to (12), note first that $\Theta(k\Delta) = \{\theta \in \mathbf{R}^m : s(k)\theta + r_1(k) \leqslant 0, -s(k)\theta + r_2(k) \leqslant 0\}$, where

$$s(k) = \phi^{\text{T}}(k\Delta)$$

$$r_1(k) = -\bar{u}(k\Delta) - R_{\alpha_0\mu}(k\Delta) + \varepsilon(k\Delta) + \alpha_0 y(k\Delta)$$

$$r_2(k) = \underline{u}(k\Delta) - R_{\alpha_0\mu}(k\Delta) - \varepsilon(k\Delta) - \alpha_0 y(k\Delta)$$

Therefore, $\Theta(k\Delta)$ is the set 'in-between' two parallel hyper-planes $H_1(k) = \{\theta \in \mathbf{R}^m : s(k)\theta + r_1(k) = 0\}$ and $H_2(k) = \{\theta \in \mathbf{R}^m : s(k)\theta - r_2(k) = 0\}$. To determine whether an $(m-1)$-dimensional face of the rectangle can be tightened, we determine all intersections of H_1 and H_2 with the extended edges of the rectangle perpendicular to that face. The extended edges are *straight lines* that pass through the edges. A face can be tightened if these intersection points, which are explicitly and easily computable, and the opposite face lie in the same half-space. Figure A1 illustrates the procedure in the case $m = 2$. Note that the result is independent of the order in which the sides are tightened.
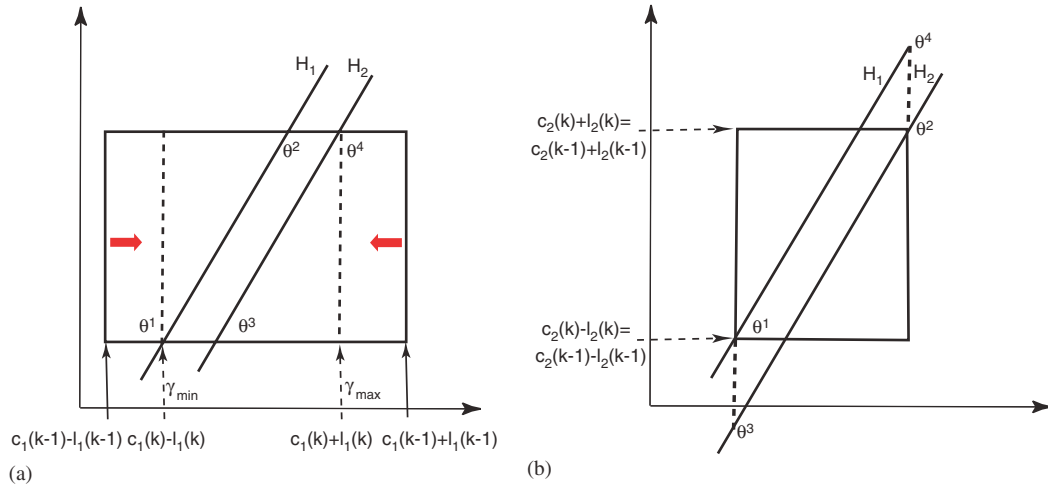
Figure A1. Algorithm for updating $P(k-1)$ to $P(k)$ for $m = 2$: (a) update of $c_1(k-1), l_1(k-1)$ to $c_1(k), l_1(k)$, both vertical faces are tightened; and (b) update of $c_2(k-1), l_2(k-1)$ to $c_2(k), l_2(k)$, the horizontal faces cannot be tightened.

This algorithm for updating $c(k-1)$ and $l(k-1)$ to, respectively, $c(k)$ and $l(k)$ is now explicitly specified.

Algorithm for updating $P(k-1)$ to $P(k)$, $k > 0$

Step 1: If $s(k) \neq 0$ proceed to Step 2. If $s(k) = 0, r_1(k) \leqslant 0, r_2(k) \leqslant 0$ assign $c(k) := c(k-1)$, $l(k) := l(k-1)$, Exit. If $s(k) = 0$ and either $r_1(k) > 0$ or $r_2(k) > 0$, then $P(k) = \emptyset$, Exit and follow the Reset procedure.

Step 2: Set $i = 0$, $c(k) := c(k-1)$, $l(k) := l(k-1)$.

Step 3: Set $i = i + 1$.

Step 4: If $s_i(k) = 0$ then Goto Step 3. If $s_i(k) \neq 0$, compute all intersection points, θ_i^q, $q = 1, \ldots, 2^m$, of H_1 and H_2 with the extended edges parallel to the ith ort in $\mathbf{R}^m$:

$$\theta_i^q = \frac{1}{s_i(k)}(-s_1(k)(c_1(k) \pm l_1(k)) - \cdots - s_{i-1}(k)(c_{i-1}(k) \pm l_{i-1}(k))$$

$$- s_{i+1}(k)(c_{i+1}(k-1) \pm l_{i+1}(k-1)) - \cdots$$

$$- s_m(k)(c_m(k-1) \pm l_m(k-1)) - r_1(k))$$

$$\theta_i^q = \frac{1}{s_i(k)}(-s_1(k)(c_1(k) \pm l_1(k)) - \cdots - s_{i-1}(k)(c_{i-1}(k) \pm l_{i-1}(k))$$

$$- s_{i+1}(k)(c_{i+1}(k-1) \pm l_{i+1}(k-1)) - \cdots - s_m(k)(c_m(k-1) \pm l_m(k-1)) + r_2(k))$$

Step 5: Set

$$\gamma_{\max} = \min\{\max\{\theta_i^q, \ q = 1, \ldots, 2^m\}, c_i(k-1) + l_i(k-1)\}$$

$$\gamma_{\min} = \max\{\min\{\theta_i^q, \ q = 1, \ldots, 2^m\}, c_i(k-1) - l_i(k-1)\}$$

and

$$c_i(k) = \tfrac{1}{2}(\gamma_{\max} + \gamma_{\min})$$

$$l_i(k) = \tfrac{1}{2}(\gamma_{\max} - \gamma_{\min})$$

Step 6: If $l_i(k) < 0$, then $P(k) = \emptyset$, Exit and follow the Reset procedure. Otherwise, Goto Step 7.

Step 7: While $i < m$, Goto Step 3. Otherwise, Exit.

The algorithm can be initialized with $c(0)$ and $l(0)$ for which the corresponding $P(0)$ in (B1) overbounds Θ^0, i.e. $\Theta^0 \subset P(0)$. For $m = 1$, a different, even simpler to implement algorithm for updating $\hat{\theta}(t)$ has been reported in Reference [2]. That algorithm, however, tends to keep $\hat{\theta}(t)$ at or near the boundary of $F_d(k)$ and not close to its Chebyshev centre.

The Reset procedure is used if $P(k) = \emptyset$ for some k. This cannot happen under the ideal assumptions, but it may happen in the real application if the assumptions are violated, for example, as a result of an abrupt change in the unknown parameter θ or due to the bounds not being satisfied. To account for possible abrupt parameter changes, the Reset procedure assigns $c(k) = c(k-1)$, $l(k) = l(0)$ thereby reconstituting the rectangle $P(k)$ to its original size and centring it around the latest valid parameter estimate, $c(k-1)$. Strategies can also be devised to revise the bounds μ, h, $\underline{u}$ and $\bar{u}$ on-line if $P(k) = \emptyset$ tends to occur frequently.

For the case $m = 1$, an alternative and even easier to implement parameter estimation procedure has been proposed in Reference [2]. In Reference [2], if $s(k) \neq 0$ then the update law for the parameter estimate, $\hat{\theta}(k\Delta)$, takes the following form:

$$\hat{\theta}(k\Delta) = \begin{cases} \dfrac{r_2(k)}{s(k)} & \text{if } s(k)\hat{\theta}((k-1)\Delta) < r_2(k) \\[2mm] \dfrac{-r_1(k)}{s(k)} & \text{if } s(k)\hat{\theta}((k-1)\Delta) > -r_1(k) \\[2mm] \hat{\theta}((k-1)\Delta) & \text{if } r_2(k) \leqslant s(k)\hat{\theta}((k-1)\Delta) \leqslant -r_1(k) \end{cases}$$

If $s(k) = 0$ then $\hat{\theta}(k\Delta) = \hat{\theta}((k-1)\Delta)$, since, under our assumptions, $-r_1(k) \geqslant r_2(k)$. This procedure guarantees that $\hat{\theta}(k\Delta) \in F_d(k\Delta)$, although, in general, $\hat{\theta}(k\Delta)$ is not the Chebyshev centre of $F_d(k\Delta)$.

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