Brief Introduction and Discussion of Error Propagation

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1 Definitions

$$< x >\equiv \frac{1}{N}\sum_i^N x_i \tag{1}$$

$$\sigma_x \equiv \sqrt{\frac{1}{N}\sum_i^N (x_i - < x >)^2} \tag{2}$$

By squaring the term $(x_i - < x >)$ we get

$$\sigma_x = \sqrt{\frac{1}{N}\sum_i^N ({x_i}^2 - 2x_i < x > + < x >^2)} = \sqrt{\frac{1}{N}\sum_i^N ({x_i}^2) - \frac{1}{N}\sum_i^N (2x_i < x >) + \frac{1}{N}\sum_i^N (< x >^2)}.$$

We now note that $< x >$ and $< x^2 >$ are constants, which enables us to write

$$\Longrightarrow \sigma_x = \sqrt{< x^2 > -2 < x > \frac{1}{N}\sum_i^N (x_i) + < x >^2 \frac{1}{N}\sum_i^N (1)}.$$

$$\Longrightarrow \sigma_x = \sqrt{< x^2 > -2 < x >< x > + < x >^2} = \sqrt{< x^2 > -2 < x >^2 + < x >^2}.$$

And finally, we arrive at an alternate expression for σ_x:

$$\sigma_x \equiv \sqrt{< x^2 > - < x >^2}. \tag{3}$$

2 Theory

Let's begin by defining a function $f = F(x, y, z, \ldots)$ where f is a value for the Function F with specific values of x,y,z, $\ldots$ The end goal of this discussion is to come up with an expression σ_f, which would tell us the uncertainty in f.

First off, let's say that the value of f is found from various measurements. If we had enough real measurements, finding σ_f would be straight forward by either using equation 2 or equation 3. The problem is that you may not have a number of measurements of f to do statistics on. Also, you may want to predict the error that you have in your experiment before you even start. The rest of this discussion should help you to estimate this error, given that you know errors ($\sigma_x, \sigma_y, \sigma_z, \ldots$)in the **independent**[1] parameters (x,y,z, $\ldots$) of the function F.

In equation 2 (with respect to your function F, we refered specifically to the deviation of f as the deviation from the mean, ie. $(f_i - < f >)$. We will assume that the deviations are small about the mean. In other words, we can approximate the deviation of f $(f_i - < f >)$ as the first order terms in the Taylor expansion:

$$(f_i - < f >) = (x_i - < x >)\frac{\partial F}{\partial x} + (y_i - < y >)\frac{\partial F}{\partial y} + (z_i - < z >)\frac{\partial F}{\partial z} + \ldots. \tag{4}$$

By subsituting this expression for the deviation of f due to deviations in x,y,z, $\ldots$ into equation 2, we get:

$$\sigma_f = \sqrt{\sum_i^N \Big((x_i - < x >)\frac{\partial F}{\partial x} + (y_i - < y >)\frac{\partial F}{\partial y} + (z_i - < z >)\frac{\partial F}{\partial z} + \ldots\Big)^2}.$$

After squaring and gathering terms we obtain

$$\begin{aligned}\sigma_f = \sqrt{\vphantom{\sum}}\Big(&\sum_i^N (x_i - < x >)^2(\frac{\partial F}{\partial x})^2 + (y_i - < y >)^2(\frac{\partial F}{\partial y})^2 + (z_i - < z >)^2(\frac{\partial F}{\partial z})^2 + \ldots \\ &+ 2(x_i - < x >)(y_i - < y >)(\frac{\partial F}{\partial x})\frac{\partial F}{\partial y}) \\ &+ 2x_i - < x >)(z_i - < z >)(\frac{\partial F}{\partial x})\frac{\partial F}{\partial z}) \\ &+ 2y_i - < y >)(z_i - < z >)(\frac{\partial F}{\partial y})\frac{\partial F}{\partial z}) + \ldots\Big).\end{aligned}$$

[1] Note that independent is bolded here. This analysis depends explicitly on the parameters of F being seperate and independent of each other

Or more generally,

$$\sigma_f = \sqrt{\sum_{x_\xi}^{M}(\frac{\partial F}{\partial x_\xi})^2 {\sigma_{x_\xi}}^2 + 2\sum_{x_\xi}^{M}\sum_{x_\eta}^{M}{}_{\xi\neq\eta}(\frac{\partial F}{\partial x_\xi}\frac{\partial F}{\partial x_\eta}){\sigma_{x_\xi x_\eta}}^2}, \tag{5}$$

where ${\sigma_{x_\xi x_\eta}}^2$, called the correlation error between variables x_ξ and x_η, is given by

$$ {\sigma_{x_\xi x_\eta}}^2 \equiv \sum_{i}^{N}(x_{\xi i} - < x_\xi >)(x_{\eta i} - < x_\eta >)$$

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Because we assume deviation is random (not systematic), if x_ξ and x_η are **independent** of each other, then $(x_{\xi i} - < x_\xi >)(x_{\eta i} - < x_\eta >)$ will be , on average, much smaller than σ_ξ^2 or σ_η^2. Therefore, we can neglect the sum of cross terms. In other words, as long as the deviation $(x_{\xi i} - < x_\xi >)$ is n̲ot dependent on any other parameter x_η and since we assume random, first order deviation, ${\sigma_{x_\xi x_\eta}}^2 \ll \sigma_\xi$ and ${\sigma_{x_\xi x_\eta}}^2 \ll \sigma_\eta$. After neglecting the correlation error terms, we arrive at what is known as Gauss' law of propagation of errors (which is not really a law):

$$\sigma_f = \sqrt{\sum_{x_\xi}^{M}(\frac{\partial F}{\partial x_\xi})^2 {\sigma_{x_\xi}}^2}. \tag{6}$$

3 Conclusions

All of the rules and formulæ which introduced in the lab manual of come from equation 6. Because they are all derived from this "law", the different variables used **must** be independent of each other.

References

[1] http://www.physics.valpo.edu/courses/p310/errors_ch3.2/sld001.htm.