MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02

Review B: Coordinate Systems

Coordinate Systems

B.1 Cartesian Coordinates

A coordinate system consists of four basic elements:

(1) Choice of origin

(2) Choice of axes

(3) Choice of positive direction for each axis

(4) Choice of unit vectors for each axis

We illustrate these elements below using Cartesian coordinates.

(1) Choice of Origin

Choose an origin O. If you are given an object, then your choice of origin may coincide with a special point in the body. For example, you may choose the mid-point of a straight piece of wire.

(2) Choice of Axis

Now we shall choose a set of axes. The simplest set of axes are known as the Cartesian axes, x-axis, y-axis, and the z-axis. Once again, we adapt our choices to the physical object. For example, we select the x-axis so that the wire lies on the x-axis, as shown in Figure B.1.1:

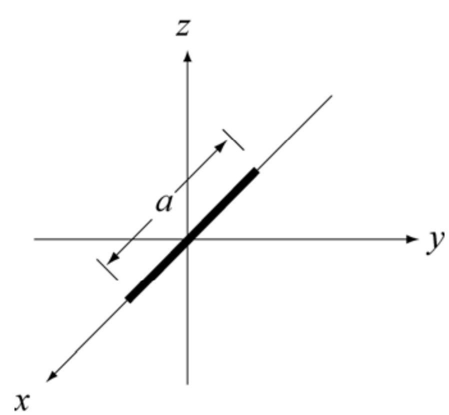

Figure B.1.1 A wire lying along the *x*-axis of Cartesian coordinates.

Then each point P in space our S can be assigned a triplet of values (x_P, y_P, z_P), the Cartesian coordinates of the point P. The ranges of these values are: $-\infty < x_P < +\infty$, $-\infty < y_P < +\infty$, $-\infty < z_P < +\infty$.

The collection of points that have the same the coordinate y_P is called a level surface. Suppose we ask what collection of points in our space S have the same value of $y = y_P$. This is the set of points $S_{y_P} = \{(x, y, z) \in S \text{ such that } y = y_P\}$. This set S_{y_P} is a plane, the x-z plane (Figure B.1.2), called a level set for constant y_P. Thus, the y-coordinate of any point actually describes a plane of points perpendicular to the y-axis.

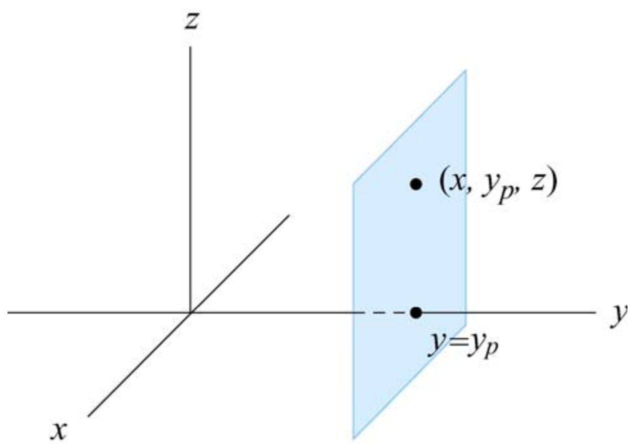

Figure B.1.2 Level surface set for constant value y_P.

(3) Choice of Positive Direction

Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol + along the positive axis. Conventionally, Cartesian coordinates are drawn with the x-y plane corresponding to the plane of the paper. The horizontal direction from left to right is taken as the positive x-axis, and the vertical direction from bottom to top is taken as the positive y-axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes. The endpoints of the wire now have the coordinates $(a/2, 0, 0)$ and $(-a/2, 0, 0)$.

(4) Choice of Unit Vectors

We now associate to each point P in space, a set of three unit directions vectors $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_P| = 1$, $|\hat{\mathbf{j}}_P| = 1$, and $|\hat{\mathbf{k}}_P| = 1$. We assign the direction of $\hat{\mathbf{i}}_P$ to point in the direction of the increasing x-coordinate at the point P. We define the directions for $\hat{\mathbf{j}}_P$ and $\hat{\mathbf{k}}_P$ in the direction of the increasing y-coordinate and z-coordinate respectively. (Figure B.1.3).

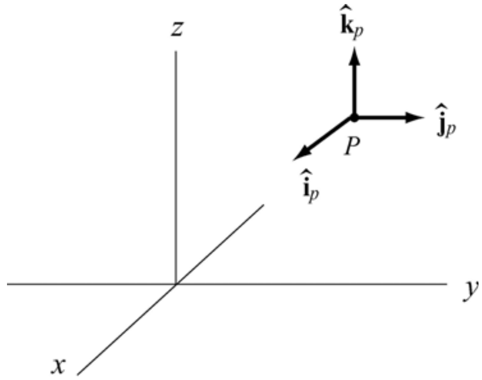

Figure B.1.3 Choice of unit vectors.

B.1.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\vec{\mathbf{s}}$ between two points P_1 and P_2 (Figure B.1.4). In Cartesian coordinates this vector can be decomposed into

$$d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{B.1.1}$$

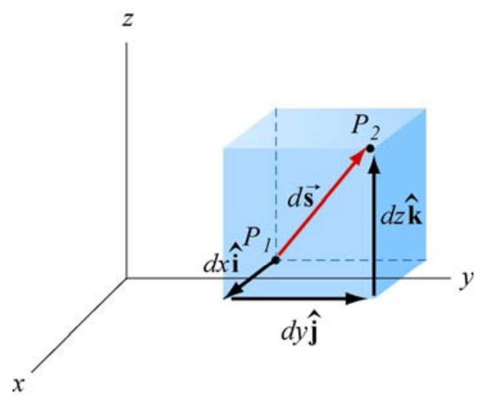

Figure B.1.4 Displacement between two points

B.1.2 Infinitesimal Area Element

An infinitesimal area element of the surface of a small cube (Figure B.1.5) is given by

$$dA = (dx)(dy) \tag{B.1.2}$$

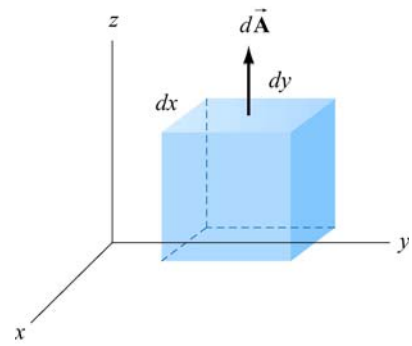

Figure B.1.5 Area element for one face of a small cube

Area elements are actually vectors where the direction of the vector $d\vec{\mathbf{A}}$ is perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface, defined by the right-hand rule. So for the above, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = dx\,dy\,\hat{\mathbf{k}} \tag{B.1.3}$$

B.1.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.1.6) in Cartesian coordinates is given by

$$dV = dx\,dy\,dz \tag{B.1.4}$$

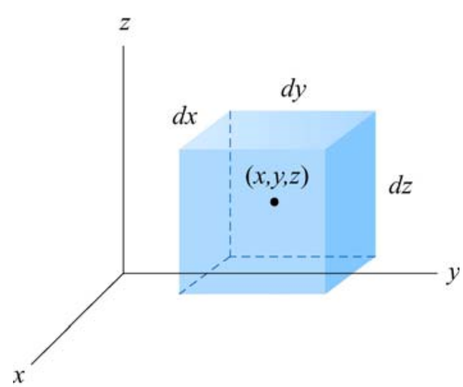

Figure B.1.6 Volume element in Cartesian coordinates.

B.2 Cylindrical Coordinates

We first choose an origin and an axis we call the z-axis with unit vector $\hat{\mathbf{z}}$ pointing in the increasing z-direction. The level surface of points such that $z = z_P$ define a plane. We shall choose coordinates for a point P in the plane $z = z_P$ as follows.

The coordinate ρ measures the distance from the z-axis to the point P. Its value ranges from $0 \le \rho < \infty$. In Figure B.2.1 we draw a few contours that have constant values of ρ. These "level contours" are circles. On the other hand, if z were not restricted to $z = z_P$, as in Figure B.2.1, the level surfaces for constant values of ρ would be cylinders coaxial with the z-axis.

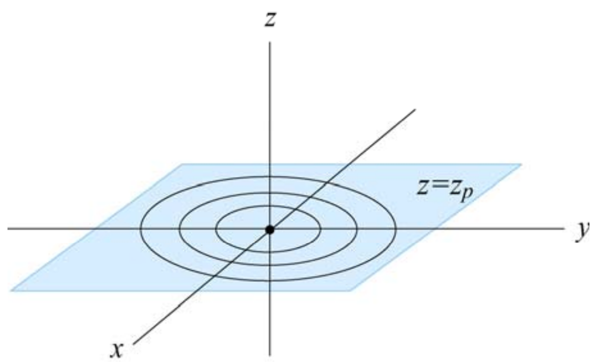

Figure B.2.1 Level surfaces for the coordinate ρ.

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angular coordinate. We choose a "reference ray," a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our reference ray is the positive x-direction). We define the angle coordinate for the point P as follows. We draw a ray from the origin to P. We define ϕ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point P, (see Figure B.2.2):

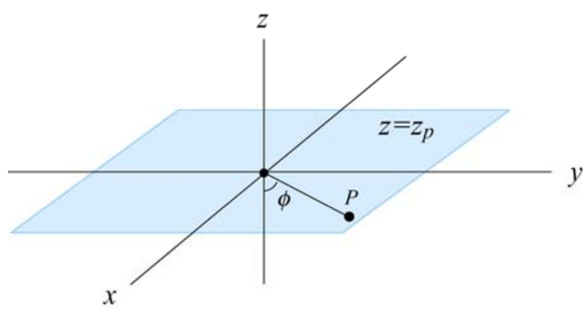

Figure B.2.2 The angular coordinate

All the other points that lie on a ray from the origin to infinity passing through P have the same value of ϕ. For any arbitrary point, ϕ can take on values from $0 \le \phi < 2\pi$. In Figure B.2.3 we depict other "level surfaces" for the angular coordinate.

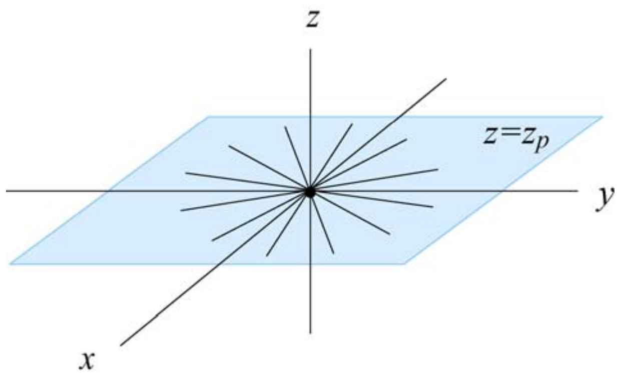

Figure B.2.3 Level surfaces for the angle coordinate.

The coordinates (ρ,ϕ) in the plane $z = z_P$ are called *plane polar coordinates*. We choose two unit vectors in the plane at the point P as follows. We choose $\hat{\boldsymbol{\rho}}$ to point in the direction of increasing ρ, radially away from the *z*-axis. We choose $\hat{\boldsymbol{\varphi}}$ to point in the direction of increasing ϕ. This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure B.2.4. This coordinate system is called a "cylindrical coordinate system." Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane.

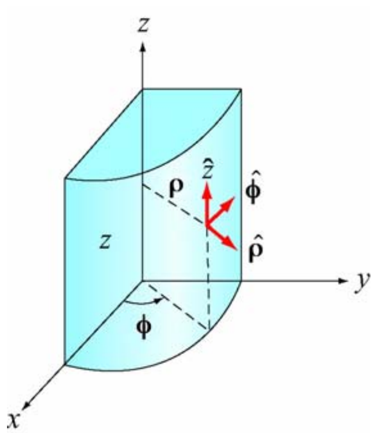

Figure B.2.4 Cylindrical coordinates

When referring to any arbitrary point in the plane, we write the unit vectors as $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$, keeping in mind that they may change in direction as we move around the plane, keeping $\hat{\mathbf{z}}$ unchanged. If we need to make a reference to this time changing property, we will write the unit vectors as explicit functions of time, $\hat{\boldsymbol{\rho}}(t)$ and $\hat{\boldsymbol{\varphi}}(t)$.

If you are given polar coordinates (ρ,ϕ) of a point in the plane, the Cartesian coordinates (x, y) can be determined from the coordinate transformations:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi \tag{B.2.1}$$

Conversely, if you are given the Cartesian coordinates (x, y), the polar coordinates (ρ, ϕ) may be represented as

$$\rho = +(x^2 + y^2)^{1/2}, \qquad \phi = \tan^{-1}(y/x) \tag{B.2.2}$$

Note that $\rho \geq 0$ so you always need to take the positive square root. Note also that $\tan\phi = \tan(\phi + \pi)$. Suppose that $0 \leq \phi \leq \pi/2$, then $x \geq 0$ and $y \geq 0$. Then the point $(-x, -y)$ will correspond to the angle $\phi + \pi$.

The unit vectors also are related by the coordinate transformations

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}, \qquad \hat{\boldsymbol{\varphi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \tag{B.2.3}$$

Similarly,

$$\hat{\mathbf{i}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\varphi}}, \qquad \hat{\mathbf{j}} = \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\varphi}} \tag{B.2.4}$$

The crucial difference between cylindrical coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point P_1 in the plane. The unit vectors in Cartesian coordinates $(\hat{\mathbf{i}}_1, \hat{\mathbf{j}}_1)$ at the point P_1 have the same magnitude and point in the same direction as the unit vectors $(\hat{\mathbf{i}}_2, \hat{\mathbf{j}}_2)$ at P_2. Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

$$\hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2, \quad \hat{\mathbf{j}}_1 = \hat{\mathbf{j}}_2 \tag{B.2.5}$$

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure B.2.5.

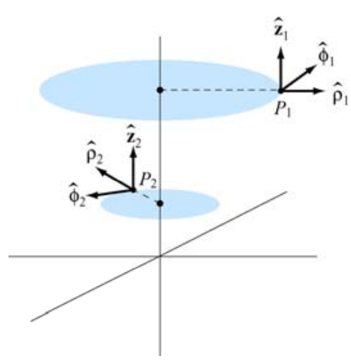

Figure B.2.5 Unit vectors at two different points in polar coordinates.

B.2.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\vec{\mathbf{s}}$ between two points P_1 and P_2 (Figure B.2.6). This vector can be decomposed into

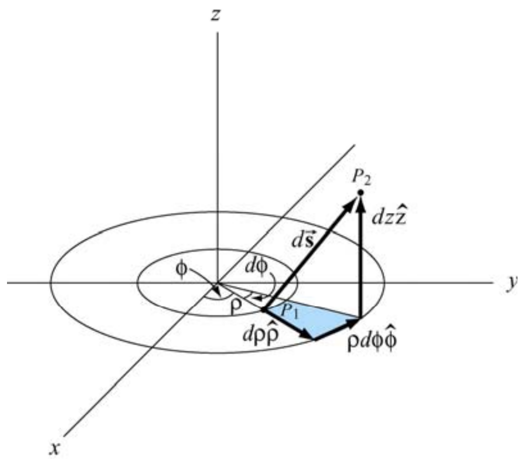

Figure B.2.6 displacement vector $d\vec{\mathbf{s}}$ between two points

$$d\vec{\mathbf{s}} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\phi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{k}} \tag{B.2.6}$$

B.2.2 Infinitesimal Area Element

Consider an infinitesimal area element on the surface of a cylinder of radius ρ (Figure B.2.7).

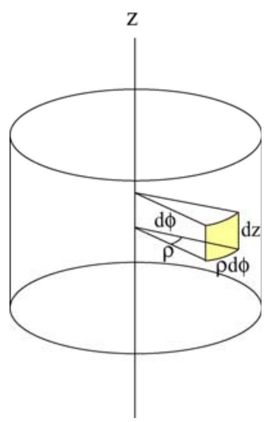

Figure B.2.7 Area element for a cylinder

The area of this element has magnitude

$$dA = \rho\,d\phi\,dz \tag{B.2.7}$$

Area elements are actually vectors where the direction of the vector $d\vec{\mathbf{A}}$ points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = \rho\, d\phi\, dz\, \hat{\boldsymbol{\rho}} \qquad \text{(B.2.8)}$$

Consider an infinitesimal area element on the surface of a disc (Figure B.2.8) in the x-y plane.

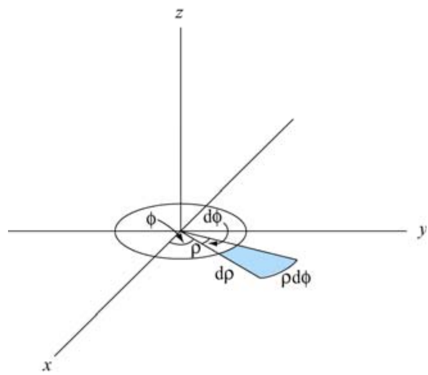

Figure B.2.8 Area element for a disc.

This area element is given by the vector

$$d\vec{\mathbf{A}} = \rho d\phi\, d\rho\, \hat{\mathbf{k}} \qquad \text{(B.2.9)}$$

B.2.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.2.9) is given by

$$dV = \rho d\phi\, d\rho\, dz \qquad \text{(B.2.10)}$$

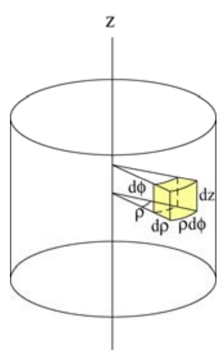

Figure B.2.9 Volume element for a cylinder.

B.3 Spherical Coordinates

We first choose an origin. Then we choose a coordinate, r, that measures the radial distance from the origin to the point P. The coordinate r ranges in value from $0 \le r < \infty$. The set of points that have constant value for r are spheres ("level surfaces").

Any point on the sphere can be defined by two angles (θ, ϕ) and r. We will define these angles with respect to a choice of Cartesian coordinates (x, y, z). The angle θ is defined to be the angle between the positive z-axis and the ray from the origin to the point P. Note that the values of θ only range from $0 \le \theta \le \pi$. The angle ϕ is defined (in a similar fashion to polar coordinates) as the angle in the between the positive x-axis and the projection in the x-y plane of the ray from the origin to the point P. The coordinate angle ϕ can take on values from $0 \le \phi < 2\pi$.

The spherical coordinates (r, θ, ϕ) for the point P are shown in Figure B.3.1. We choose the unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ at the point P as follows. Let $\hat{\mathbf{r}}$ point radially away from the origin, and $\hat{\boldsymbol{\theta}}$ point tangent to a circle in the positive θ direction in the plane formed by the z-axis and the ray from the origin to the point P. Note that $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ. We choose $\hat{\boldsymbol{\varphi}}$ to point in the direction of increasing ϕ. This unit vector points tangent to a circle in the $xy-$plane centered on the z-axis. These unit vectors are also shown in Figure B.3.1.

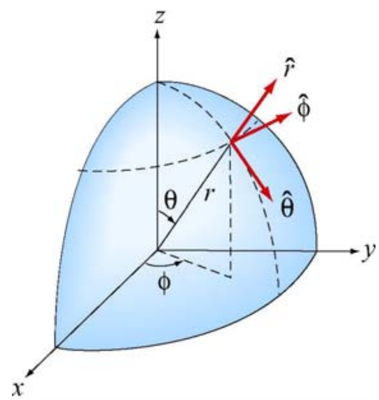

Figure B.3.1 Spherical coordinates

If you are given spherical coordinates (r, θ, ϕ) of a point in the plane, the Cartesian coordinates (x, y, z) can be determined from the coordinate transformations

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \qquad \text{(B.3.1)}$$

Conversely, if you are given the Cartesian coordinates (x, y, z), the spherical coordinates (r,θ,ϕ) can be determined from the coordinate transformations

$$
\begin{aligned}
r &= +(x^2 + y^2 + z^2)^{1/2} \\
\theta &= \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right) \\
\phi &= \tan^{-1}(y/x)
\end{aligned}
\tag{B.3.2}
$$

The unit vectors also are related by the coordinate transformations

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\varphi}} &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}
\end{aligned}
\tag{B.3.3}
$$

These results can be understood by considering the projection of $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}})$ into the unit vectors $(\hat{\boldsymbol{\rho}},\hat{\mathbf{k}})$, where $\hat{\boldsymbol{\rho}}$ is the unit vector from cylindrical coordinates (Figure B.3.2),

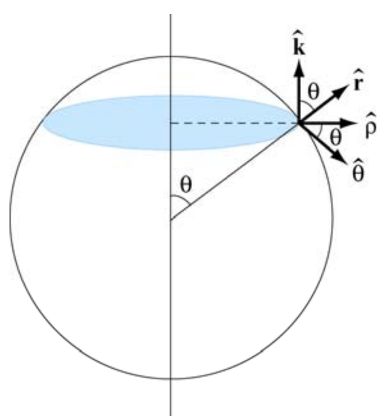

Figure B.3.2 Cylindrical and spherical unit vectors

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{k}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\mathbf{k}}
\end{aligned}
\tag{B.3.4}
$$

We can use the vector decomposition of $\hat{\boldsymbol{\rho}}$ into the Cartesian unit vectors $(\hat{\mathbf{i}},\hat{\mathbf{j}})$:

$$
\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \tag{B.3.5}
$$

To find the inverse transformations we can use the fact that

$$\hat{\boldsymbol{\rho}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} \tag{B.3.6}$$

to express

$$\begin{aligned}\hat{\mathbf{i}} &= \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\varphi}}\\ \hat{\mathbf{j}} &= \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\varphi}}\end{aligned} \tag{B.3.7}$$

as

$$\begin{aligned}\hat{\mathbf{i}} &= \cos\phi\sin\theta\,\hat{\mathbf{r}} + \cos\phi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}\\ \hat{\mathbf{j}} &= \sin\phi\sin\theta\,\hat{\mathbf{r}} + \sin\phi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}\end{aligned} \tag{B.3.8}$$

The unit vector $\hat{\mathbf{k}}$ can be decomposed directly into $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}})$ with the result that

$$\hat{\mathbf{k}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \tag{B.3.9}$$

B.3.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\,\vec{\mathbf{s}}$ between two points (Figure B.3.3). This vector can be decomposed into

$$d\,\vec{\mathbf{s}} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\varphi}} \tag{B.3.10}$$

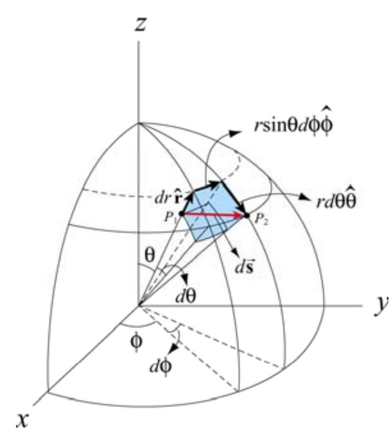

Figure B.3.3 Infinitesimal displacement vector $d\,\vec{\mathbf{s}}$.

B.3.2 Infinitesimal Area Element

Consider an infinitesimal area element on the surface of a sphere of radius r (Figure B.3.4).

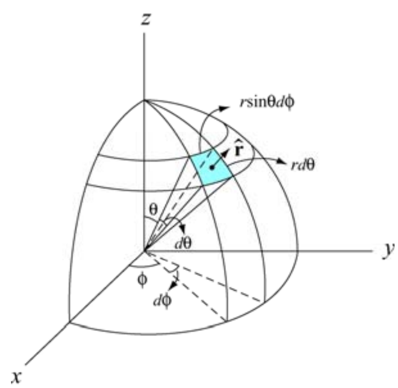

Figure B.3.4 Area element for a sphere.

The area of this element has magnitude

$$dA = (rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta d\theta d\phi \qquad \text{(B.3.11)}$$

points in the radially direction (outward from the surface of the sphere). So for the surface of the sphere, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}} \qquad \text{(B.3.12)}$$

B.3.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.3.5) is given by

$$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr \qquad \text{(B.3.13)}$$

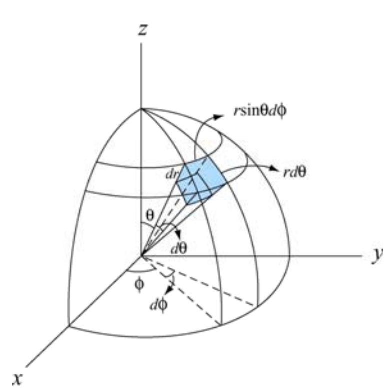

Figure B.3.5 Infinitesimal volume element.