Chapter 14

α Decay

Note to students and other readers: This Chapter is intended to supplement Chapter 8 of Krane’s excellent book, “Introductory Nuclear Physics”. Kindly read the relevant sections in Krane’s book first. This reading is supplementary to that, and the subsection ordering will mirror that of Krane’s, at least until further notice.

14.1 Why α Decay Occurs

See Krane’s 8.1.

14.2 Basic α Decay Processes

An α decay is a nuclear transformation in which a nucleus reduces its energy by emitting an α-particle.

\[ {}_Z^A X_N \rightarrow {}_{Z-2}^{A-4} X'_{N-2} + {}_2^4 \text{He}_2 , \]

or, more compactly :

\[ {}_Z^A X \rightarrow X' + \alpha . \]

The resultant nucleus, \( X' \) is usually left in an excited state, followed, possibly, by another α decay, or by any other form of radiation, eventually returning the system to the ground state.
The energetics of $\alpha$ decay

The $\alpha$-decay process is “fueled” by the rest mass energy difference of the initial state and final state. That is, using a relativistic formalism:

\[
\begin{align*}
E_i &= E_f \\
E_{\alpha} c^2 &= m_{X'} c^2 + T_{X'} + m_{\alpha} c^2 + T_{\alpha} \\
Q &= T_{X'} + T_{\alpha} \quad \text{where} \\
Q &= m_{X} c^2 - m_{X'} c^2 - m_{\alpha} c^2 \\
Q/c^2 &\approx m(A_X) - m(A-4X') - m(^4\text{He}), \quad (14.1)
\end{align*}
\]

where $T_{X'}$ and $T_{\alpha}$ are the kinetic energies of the nucleus and the $\alpha$-particle following the decay. Note that atomic masses may be substituted for the nuclear masses, as shown in the last line above. The electron masses balance in the equations, and there is negligible error in ignoring the small differences in electron binding energies.

The line of flight of the decay products are in equal and opposite directions, assuming that $X$ was at rest. Conservation of energy and momentum apply. Thus we may solve for $T_{\alpha}$ in terms of $Q$ (usually known). $X'$ is usually not observed directly. Solving for $T_{\alpha}$ by eliminating $X'$:

\[
\begin{align*}
Q &= T_{\alpha} + T_{X'} \\
|\vec{p}_{\alpha}| &= |\vec{p}_{X'}| \\
p_{\alpha}^2 &= p_{X'}^2 \\
2m_{\alpha} T_{\alpha} &= 2m_{X'} T_{X'} \\
(m_{\alpha}/m_{X'}) T_{\alpha} &= T_{X'} . \quad (14.3)
\end{align*}
\]

Using (14.3) and (14.2) to eliminate $T_{X'}$ results in:

\[
\begin{align*}
Q &= T_{\alpha}(1 + m_{\alpha}/m_{X'}) \quad \text{or} \quad (14.4) \\
T_{\alpha} &= \frac{Q}{(1 + m_{\alpha}/m_{X'})} \quad (14.5) \\
T_{\alpha} &\approx \frac{Q}{(1 + 4/A')} \quad \text{or} \quad (14.6) \\
T_{\alpha} &\approx \frac{Q}{(1 + 4/A)} \quad \text{or} \quad (14.7) \\
T_{\alpha} &\approx Q(1 - 4/A) \quad (14.8)
\end{align*}
\]
Equations (14.4) and (14.5) are exact within a non-relativistic formalism. Equation (14.6) is an approximation, but a good one that is suitable for all α-decay’s, including \(^8\text{Be} \rightarrow 2\alpha\). Equation (14.7) is suitable for all the other α-emitters, while (14.8) is only suitable for the heavy emitters, since it assumes \(A \gg 4\).

We see from (14.8) that the typical recoil energy, for heavy emitters is:

\[
T_{X'} = Q - T_\alpha \approx (4/A)Q .
\]  

For a typical α-emitter, this recoil energy \((Q = 5 \text{ MeV}, A = 200)\) is 100 keV. This is not insignificant. α-emitters are usually found in crystalline form, and that recoil energy is more than sufficient to break atomic bonds, and cause a microfracture along the track of the recoil nucleus.

**Relativistic effects?**

One may do a fully relativistic calculation, from which it is found that:

\[
T_\alpha = \frac{Q \left(1 + \frac{1}{2} \frac{Q}{m_{X'} c^2}\right)}{\left(1 + \frac{m_\alpha}{m_{X'}} + \frac{Q}{m_{X'} c^2}\right)}, \tag{14.10}
\]

\[
T_{X'} = \frac{\left(\frac{m_\alpha}{m_{X'}}\right)Q \left(1 + \frac{1}{2} \frac{Q}{m_\alpha c^2}\right)}{\left(1 + \frac{m_\alpha}{m_{X'}} + \frac{Q}{m_{X'} c^2}\right)} . \tag{14.11}
\]

Even in the worst-case scenario (low-\(A\)) this relativistic correction is about \(2.5 \times 10^{-4}\). Thus the non-relativistic approximation is adequate for determining \(T_\alpha\) or \(T_{X'}\).

### 14.3 α Decay Systematics

**As \(Q\) increases, \(t_{1/2}\) decreases**

This is, more or less, self-evident. More “fuel” implies faster decay.

The “smoothest” example of this “law” is seen in the α decay of the even-even nuclei. Shell model variation is minimized in this case, since no pair bonds are being broken. See Figure 14.1, where \(\log_{10}(t_{1/2})\) is plotted \(vs.\) \(Q\). Geiger and Nuttal proposed the following phenomenological fit for \(\log_{10}(t_{1/2}(Q))\):
\[ \log_{10} \lambda = C - DQ^{-1/2} \quad \text{or} \quad (14.12) \]
\[ \log_{10} t_{1/2} = -C' + DQ^{-1/2} , \quad (14.13) \]

where \( C \) and \( D \) are fitting constants, and \( C' = C - \log_{10}(\ln 2) \). Odd-odd, even-odd and odd-even nuclei follow the same general systematic trend, but the data are much more scattered, and their half-lives are 2–1000 times that of their even-even counterparts.

**Prediction of \( Q \) from the semiempirical mass formula**

The semiempirical mass formula can be employed to estimate \( Q_\alpha \), as a function of \( Z \) and \( A \).

\[
Q = B(Z - 2, A - 4) + B(^{4}\text{He}) - B(Z, A) \\
\approx 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 4a_c Z A^{-1/3}(1 - Z/3A) - 4a_{sym}(1 - 2Z/A)^2 + 3a_p A^{-7/4} .
\]

A plot of \( Q(Z, A) \) using (14.14) is given in Figure 14.2.

These trends, from experimental data, are seen in Figure 14.3. However, there is also evidence of the impact of the shell closing at \( N = 126 \), that is not seen in Figure 14.3.

### 14.4 Theory of \( \alpha \) Emission

Figure 14.4 shows 3 potentials that are used in the estimation of barrier penetration probabilities, for determining the halflife of an \( \alpha \)-emitter. The simplest potential, the rectilinear box potential, shown by the dashed line, although crude, maybe used to explain the phenomenon of \( \alpha \) decay.

**The simplest theory of \( \alpha \) emission**

In this section we solve for the decay probability using the simplest rectilinear box potential. This potential is characterized by:
Figure 14.2: $Q(Z, A)$ using (14.14). The dashed line is for Pb ($Z = 82$), while the lower dotted is for Os ($Z = 76$). The upper dotted line is for Lr ($Z = 103$). Each separate $Z$ has its own line with higher $Z$'s oriented to the right.

$$V_1(r < R_N) = -V_0$$
$$V_2(R_N < r < b) = V_C$$
$$V_3(r > b) = 0,$$

where $V_0$ and $V_C$ are constants.

The 3-D radial wavefunctions (assuming that the $\alpha$ is in an $s$-state), are of the form, $R_i(r) = u_i(r)/r$ take the form:

Figure 14.3: Paste in Krane's Figure 8.2.
where

\begin{align*}
  k_1 &= \sqrt{\frac{2m(V_0 + Q)}{\hbar}} \\
  k_2 &= \sqrt{\frac{2m(V_C - Q)}{\hbar}} \\
  k_3 &= \sqrt{\frac{2mQ}{\hbar}} .
\end{align*}

Turning the mathematical crank, we arrive at the transmission coefficient:
14.4. THEORY OF $\alpha$ EMISSION

\[ T = \left[ \frac{1}{4} \left( 2 + \frac{k_1}{k_3} + \frac{k_3}{k_1} \right) + \frac{\sinh^2[2k_2(b - R_N)]}{4} \left( \frac{k_1 k_3}{k_2^2} + \frac{k_2}{k_1 k_3} + \frac{k_1}{k_3} + \frac{k_3}{k_1} \right) \right]^{-1} \]  

(14.18)

Note: We solved a similar problem in NERS 311, but with $V_0 = 0$, that is, $k_1 = k_3$. The factor $k_2(b - R_N) \approx 35$ for typical $\alpha$ emitters. Thus, we can simplify to:

\[ T = \frac{16e^{-2k_2(b - R_N)}}{\left( \frac{k_1 k_3}{k_2^2} + \frac{k_2}{k_1 k_3} + \frac{k_1}{k_3} + \frac{k_3}{k_1} \right)} \]  

(14.19)

Recall that the transmission coefficient is the probability of escape by a single $\alpha$-particle. To calculate the transmission rate, we estimate a “frequency factor”, $f$, that counts the number of instances, per unit time, that a $\alpha$, with velocity $v_\alpha$ presents itself at the barrier as an escape candidate. There are several estimates for $f$:

<table>
<thead>
<tr>
<th>$f$</th>
<th>Source</th>
<th>Estimate (s$^{-1}$)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_\alpha/R_N$</td>
<td>Krane</td>
<td>$\approx 10^{21}$</td>
<td>Too low, by $10^1$–$10^2$</td>
</tr>
<tr>
<td>$v_\alpha/(2R_N)$</td>
<td>Others</td>
<td>$\approx 5 \times 10^{20}$</td>
<td>Too low, by $10^1$–$10^2$</td>
</tr>
<tr>
<td>Fermi’s Golden Rule # 2</td>
<td></td>
<td>$\approx 10^{24}$</td>
<td>Too high, by $10^1$–$10^2$</td>
</tr>
</tbody>
</table>

The correct answer, determined by experiment, lies in between these two extremes. However approximate our result it, it does show an extreme sensitivity to the shape of the Coulomb barrier, through the exponential factor in (14.19).

**Gamow’s theory of $\alpha$ decay**

Gamow’s theory of $\alpha$ decay is based on an approximate solution$^1$ to the Schrödinger equation. Gamow’s theory gives:

\[ T = \exp \left[ -2 \left( \frac{2m}{\hbar^2} \right)^{1/2} \int_{R_N}^{b} dr \sqrt{V(r) - Q} \right], \]  

(14.20)

where $b$ is that value of that defines the $r$ where $V(r) = Q$, on the far side of the barrier.

If we apply Gamow’s theory to the potential of the previous section, we obtain:

$^1$The approximation Gamow used, is a semi-classical approximation to the Schrödinger equation, called the WKB (Wentzel-Kramers-Brillouin) method. The WKB method works best when the potential changes slowly with position, and hence the frequency of the wavefunction, $k(x)$, also changes slowly. This is not the case for the nucleus, due to its sharp nuclear edge. Consequently, it it thought that Gamow’s solution can only get to within a factor or 2 or 3 of the truth. In nuclear physics, a factor of 2 or 3 is often thought of as “good agreement”!
\[
T_{\text{exact}} = \frac{16}{\left(\frac{k_1 k_3}{k_2^2} + \frac{k_2^2}{k_1 k_3} + \frac{k_3}{k_1} + \frac{k_1}{k_3}\right)} T_{\text{Gamow}}. \tag{14.21}
\]

That factor in front is about 2–3 for most \(\alpha\) emitters. This discrepancy is usually ignored, considering the large uncertainly in the \(f\) factor.

**Krane’s treatment of \(\alpha\)-decay**

Krane starts out with (14.20), namely:

\[
T = \exp \left[ -2 \left( \frac{2m}{\hbar^2} \right)^{1/2} \int_a^b dr \sqrt{V(r) - Q} \right],
\]

where

\[
V(x) = \frac{2(Z - 2)e^2}{4\pi \epsilon_0 x}, \quad V(a) \equiv B = \frac{2(Z - 2)e^2}{4\pi \epsilon_0 a}, \quad a = R_0(A - 4)^{1/3},
\]

\[
V(b) \equiv Q = \frac{2(Z - 2)e^2}{4\pi \epsilon_0 b}. \tag{14.22}
\]

That is, the \(\alpha\) moves in the potential of the daughter nucleus, \(B\) is the height of the potential at the radius of the daughter nucleus, and \(b\) is the radius where that potential is equal to \(Q\). (See Krane’s Figure 8.3 on page 251.)

We note that \(a/b = Q/B\).

Substituting the potential in (14.22) into (14.20) results in:

\[
T = \exp \left\{ -2 \left( \frac{2m'_\alpha e'^2}{Q(\hbar c)^2} \right)^{1/2} \frac{zZ'e^2}{4\pi \epsilon_0} \left[ \arccos(\sqrt{x}) - \sqrt{x(1-x)} \right] \right\}, \tag{14.23}
\]

where \(x \equiv a/b = Q/B\). Note that the reduced mass has been used:

\[
m'_\alpha = \frac{m_\alpha m_{X'}}{m_\alpha + m_{X'}} \approx m_\alpha \left(1 - 4/A\right). \tag{14.24}
\]
This “small” difference can result in a change in $T$ by a factor of 2–3, even for heavy nuclei!

Krane also discusses the approximation to (14.23) that results in his equation (8.18). This comes from the Taylor expansion:

$$\arccos(\sqrt{x}) - \sqrt{x(1-x)} \longrightarrow \frac{\pi}{2} - 2\sqrt{x} + O(x^{3/2}).$$

This is only valid for small $x$. Typically $x \approx 0.3$, and use of Krane’s (8.18) involves too much error. So, stick with the equation given below.

Factoring in the frequency factor, one can show that:

$$t_{1/2} = \ln(2) \frac{a}{c} \sqrt[3]{\frac{m_\alpha c^2}{2(V_0 + Q)}} \times \exp \left\{ 2 \left( \frac{2m'_\alpha c^2}{Q(\hbar c)^2} \right)^{1/2} \frac{zZ'e^2}{4\pi\varepsilon_0} \left[ \arccos(\sqrt{x}) - \sqrt{x(1-x)} \right] \right\}. \quad (14.25)$$

### 14.4.1 Comparison with Measurements

In this section we employ the simplest form of $f$ and compute the half-life for $\alpha$-decay as follows:

$$t_{1/2} = \ln(2) \frac{a}{c} \sqrt[3]{\frac{m_\alpha c^2}{2(V_0 + Q)}} \exp \left\{ 2 \left( \frac{2m'_\alpha c^2}{Q(\hbar c)^2} \right)^{1/2} \frac{zZ'e^2}{4\pi\varepsilon_0} \left[ \arccos(\sqrt{x}) - \sqrt{x(1-x)} \right] \right\}. \quad (14.26)$$

The data are shown in the following table, where the half-lives of the even-even isotopes of Th ($Z = 90$) are shown. The calculations were performed using a nuclear radius of $a = 1.25A^{1/3}$ (fm), and $V_0 = 35$ (MeV). The absolute comparison exhibits the same trends for both experiment and calculations, with the calculations being overestimated by 2–3 orders of magnitude. This is most likely due to a gross underestimate of $f$. The relative comparisons are in much better shape, showing discrepancies of about a factor of 2–3, quite a success for such a crude theory. We note that small changes in $Q$ result in enormous differences in the results. In this table $Q$ changes by about a factor of 2, while the half-lives span about 23 orders of magnitude. The probability of escape is greatly influenced by the height and width of the Coulomb barrier. Besides this dependence, the only other variation in the comparison relates to the nuclear radius. This also affects the barrier since the nuclear radius is proportional to $A^{1/3}$. This hints that the remaining discrepancy, at least for the relative comparison, is related to the fine details of the shape of the barrier, perhaps mostly in the vicinity of the inner turning point. A more refined shape of the Coulomb barrier would likely
yield better results, as well would a higher-order WKB analysis that would account more precisely, for that shape variation. Additionally, the $\alpha$-particle was treated as if it were a point charge in this analysis. A refined calculation should certainly take this effect into account.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$Q$ (MeV)</th>
<th>$t_{1/2}$ (s) abs. meas.</th>
<th>$t_{1/2}$ (s) abs. calc.</th>
<th>$t_{1/2}$ (s) rel. meas.</th>
<th>$t_{1/2}$ (s) rel. calc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>220</td>
<td>8.95</td>
<td>$10^{-5}$</td>
<td>$10^{-3}$</td>
<td>$5.4 \times 10^{-9}$</td>
<td>$3.6 \times 10^{-9}$</td>
</tr>
<tr>
<td>222</td>
<td>8.13</td>
<td>$2.8 \times 10^{-3}$</td>
<td>$2.1 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-6}$</td>
<td>$7.3 \times 10^{-7}$</td>
</tr>
<tr>
<td>224</td>
<td>7.31</td>
<td>$1.04$</td>
<td>$1.1 \times 10^2$</td>
<td>$5.6 \times 10^{-4}$</td>
<td>$3.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>226</td>
<td>6.45</td>
<td>$1854$</td>
<td>$2.9 \times 10^5$</td>
<td>$\equiv 1$</td>
<td>$\equiv 1$</td>
</tr>
<tr>
<td>228</td>
<td>5.52</td>
<td>$6.0 \times 10^7$</td>
<td>$1.2 \times 10^{10}$</td>
<td>$3.2 \times 10^4$</td>
<td>$4.2 \times 10^4$</td>
</tr>
<tr>
<td>230</td>
<td>4.77</td>
<td>$2.5 \times 10^{12}$</td>
<td>$6.0 \times 10^{14}$</td>
<td>$1.3 \times 10^9$</td>
<td>$2.1 \times 10^9$</td>
</tr>
<tr>
<td>232</td>
<td>4.08</td>
<td>$4.4 \times 10^{17}$</td>
<td>$1.8 \times 10^{20}$</td>
<td>$2.4 \times 10^{14}$</td>
<td>$6.2 \times 10^{14}$</td>
</tr>
</tbody>
</table>

Table 14.1: Half-lives of Th isotopes, absolute and relative comparisons of measurement and theory.

Cluster decay probabilities

If $\alpha$ decay can occur, surely $^8$Be and $^{12}$C decay can occur as well. It is just a matter of relative probability. For these decays, the escape probabilities are given approximately by:

$$T_{^8\text{Be}} = T_{\alpha}^2$$
$$T_{^{12}\text{C}} = T_{\alpha}^3$$
$$T_{^z\text{x}} = T_{\alpha}^{z/2}.$$ (14.27)

The last estimate is for a $^z\text{x}$ cluster, with $z$ protons and an atomic mass of $a$.

14.5 Angular momentum and parity in $\alpha$ decay

Angular momentum

If the $\alpha$-particle carries off angular momentum, we must add the repulsive potential associated with the centrifugal barrier to the Coulomb potential, $V_C(r)$:

$$V(r) = V_C(r) + \frac{l(l+1)\hbar^2}{2m' \alpha r^2},$$ (14.28)
14.5. ANGULAR MOMENTUM AND PARITY IN $\alpha$ DECAY

represented by the second term on the right-hand side of (14.28).

The effect on $^{90}$Th, with $Q = 4.5$ MeV is:

<table>
<thead>
<tr>
<th>$l$</th>
<th>$T_i/T_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.84</td>
</tr>
<tr>
<td>2</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.36</td>
</tr>
<tr>
<td>4</td>
<td>0.18</td>
</tr>
<tr>
<td>5</td>
<td>0.078</td>
</tr>
<tr>
<td>6</td>
<td>0.028</td>
</tr>
</tbody>
</table>

So, as $l \uparrow$, $T \downarrow$.

Conservation of angular momentum and parity

$\alpha$ decay’s must satisfy the constraints given by the conservation of total angular momentum:

$$
\vec{\bar{I}}_i = \vec{\bar{I}}_f + \vec{l}_\alpha \\
\Pi_i = \Pi_f \times \Pi_\alpha,
$$

where $i$ represents the parent nucleus, and $f$ represents the daughter nucleus. Since the $\alpha$-particle is a $0^+$ nucleus, (14.29) simplifies to:

$$
\vec{\bar{I}}_i = \vec{\bar{I}}_f + \vec{l}_\alpha \\
\Pi_i = \Pi_f \times (-1)^{l_\alpha},
$$

where $l_\alpha$ is the orbital angular momentum carried off by the $\alpha$-particle. If $I_i$ is non-zero, the $\alpha$ decay is able to populate any excited state of the daughter, or go directly to the ground state.

If the initial state has total spin 0, with few exceptions it is a $0^+$. In this case, (14.30) becomes.

$$
\vec{0} = \vec{\bar{I}}_f + \vec{l}_\alpha \\
1 = \Pi_f \times (-1)^{l_\alpha},
$$

or.

$$
\vec{\bar{I}}_f = \vec{l}_\alpha \\
\Pi_f^+ = (-1)^{l_\alpha},
$$
Thus the only allowed daughter configurations are: $0^+, 1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-, 8^+, 9^- \cdots$

All other combinations are absolutely disallowed.

The $\alpha$ decay can show these allowed transitions quite nicely. A particularly nice example is the case where the transition is $0^+ \rightarrow 0^+$, where the low-lying rotational band, and higher energy phonon structure are explicitly revealed through $\alpha$ decay. (See Figure 8.7 in Krane.)

**Angular intensity of $\alpha$ decays for elliptic nuclei**

This is well described in Krane, pages 260–261.

### 14.6 $\alpha$-decay spectroscopy

Not covered in NERS312.