New Methods in the Classical Economics of Uncertainty: Characterizing Utility Functions [1]

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Abstract

In this paper, we provide an instrument to solve a large set of problems related to the effects of uncertainty on preferences and risk-taking. This instrument – called the Diffidence Theorem – is particularly well fitted to analyze the interaction between independent risks. Beside providing a simple tool to explain well-known results, it allows to solve many problems that have been considered as untractable up to now. For example, we characterize the set of utility functions for which opening up a new asset market raises current consumption, or if it increases the aversion towards other independent risky prospects.

1 Introduction

Since the important papers by Kilhstrom, Romer and Williams [1981] and Pratt and Zeckhauser [1987], our understanding of the behaviour of agents facing more than one source of risk has made much progress. A common feature of analyses of the interaction of independent risks is the existence of paradoxes that can be solved by putting more restrictions to the utility function. For example, Eeckhoudt and Kimball [1992] and Kimball [1993] showed that adding a nonmarketable zero-mean risk to wealth can induce a risk-averse agent to purchase more of another independent risky asset. Another common feature in this literature is the complexity of the proof of most results.

Still, many problems that are considered in this literature share the same technical structure. In many instances, they can be written under the following form: which are the conditions on functions f_1 and f_2 that guarantee that

$$\forall \tilde{x} : \quad Ef_1(\tilde{x}) \leq Ef_1(\tilde{\omega}) \Longrightarrow Ef_2(\tilde{x}) \leq Ef_2(\tilde{\omega}), \tag{1}$$

for any $\tilde{\omega}$ belonging to a specific set of random variables. In our companion paper (Gollier and Kimball [1994]), we examined dual problems of the same form in which risks – rather than preferences – are compared. As in this companion paper, we develop a standard technique that is based on a fundamental theorem which is called here the "Diffidence Theorem". This theorem does not only allow to systematize the way by which existing concepts of comparing risk attitudes are characterized, but it also allows for characterizing new concepts that are defined in this paper. We claim that this theorem provides a very simple instrument to solve relatively sophisticated problems. This paper owes much to Pratt and Zeckhauser [1987] who provided the basic instrument to prove the Diffidence Theorem. This instrument was also used by Kimball [1993] and by Gollier and Pratt [1993].

Let us consider a very simple illustration of problem (1). There are several ways to define the concept of "more risk aversion". For example, an agent with utility function u_2 is more risk-averse than another with utility function u_1 if and only if any risk that is rejected by the latter is also rejected by the former, independent of the sure common wealth level of the agents. In mathematical terms, the following condition can serve as a definition for u_2

being more risk-averse than u_1:

$$\forall \tilde{x} : \quad Eu_1(w + \tilde{x}) \leq u_1(w) \Longrightarrow Eu_2(w + \tilde{x}) \leq u_2(w), \qquad (2)$$

for any reference wealth level w. To mention just a few other existing concepts that can be solved using the Diffidence Theorem, we have the concept of risk aversion, prudence, decreasing absolute risk aversion, decreasing absolute prudence, proper risk aversion (see Pratt and Zeckhauser [1987]), weak proper risk aversion (Gollier and Pratt [1993]), and standard risk aversion (see Kimball [1993]). Some of these concepts require the use of two random variables for which a Bivariate Diffidence Theorem is provided.

In this paper, we also consider new concepts having the same structure as in (1). Under which condition does an increase in nonmarketable background risk raise the equilibrium risk free rate in the economy? Under which condition does opening up a new asset market raise current consumption? Does it reduce the demand for another independent risky asset? All these questions can be solved by using the Diffidence Theorem.

In section 2, we prove the Diffidence Theorem and its Corollary. Most results that are presented in this section are dual to those presented in our companion paper. We illustrate the use of the Diffidence Theorem by presenting many applications in section 3. Section 4 is devoted to the Bivariate Diffidence Theorem and its applications. Other extensions are considered in section 5.

2 The Diffidence Theorem

In this section, we characterize the set of real-valued functions f_2 that satisfy condition (1) for a given real-valued function f_1 and a given reference random variable $\tilde{\omega}$. The proof of the Diffidence Theorem relies on the following Lemma.[1]

Lemma 1 *Condition (1) is satisfied for any distribution of $\tilde{x}$ if and only if it is satisfied (A) for all one-point distributions, and (B) for all two-point distributions that satisfy condition $Ef_1(\tilde{x}) \leq Ef_1(\tilde{\omega})$ as an equality.*

[1]The proof of Theorem 3 in Pratt and Zeckhauser [1987] is based on the same basic idea.

Proof: Necessity is obvious. To see sufficiency, note that condition (1) is equivalent to

$$\max_{\tilde{x}} \quad Ef_2(\tilde{x}) - Ef_2(\tilde{\omega}) \quad \leq 0 \tag{3}$$
$$\text{subject to } Ef_1(\tilde{x}) - Ef_1(\tilde{\omega}) \leq 0.$$

Both the objective function and the constraint of this problem are linear in probabilities. Thus it is a linear programming problem on the unit simplex. If the region in which the constraint is not violated is nonempty, the maximum value of the objective must be achieved on one of the vertices of the multidimensional polyhedron formed by slicing off the portion of the unit simplex that violates the constraint. These vertices represent two types of random variables: (A) degenerate distributions with all of the mass on one value of x, and (B) two-point distributions that satisfy the constraint with equality. ■

This result should be related to the well-established fact that condition $Ef_2(\tilde{x}) \leq Ef_2(\tilde{\omega})$ holds for any $\tilde{x}$ if and only if it holds for any degenerate (one-point) random variable. The fact that a condition on $\tilde{x}$ is introduced, i.e. $\tilde{x}$ must satisfy condition $Ef_1(\tilde{x}) \leq Ef_1(\tilde{\omega})$, forces us to consider not only degenerate random variables, but also all two-point random variables.

The Diffidence Theorem is a direct consequence of the Lemma.

Theorem 1 *(Diffidence Theorem) To escape triviality, assume that there exists at least one $x \in R$ such that $f_1(x) \leq Ef_1(\tilde{\omega})$. Condition (1) for any random variable $\tilde{x}$ with support in $[a, b]$ is equivalent to the condition that there exists an $m \in R_+$ such that*

$$\forall x \in [a, b]: \quad f_2(x) - Ef_2(\tilde{\omega}) \leq m[f_1(x) - Ef_1(\tilde{\omega})]. \tag{4}$$

Proof: Notice first that the sufficiency of (4) is obvious. Let ϕ_1 and ϕ_2 denote $Ef_1(\tilde{\omega})$ and $Ef_2(\tilde{\omega})$, respectively. By assumption, a solution exists to program (3). So, by the Lemma, we know that the condition (1) holds if and only if
$\forall x \in [a, b]$:

$$f_1(x) \leq \phi_1 \quad \Longrightarrow \quad f_2(x) \leq \phi_2,$$

and (B) $\forall x_1, x_2 \in [a, b], \forall p \in [0, 1]$:

$$pf_1(x_1) + (1-p)f_1(x_2) = \phi_1 \quad \Longrightarrow \quad pf_2(x_1) + (1-p)f_2(x_2) \leq \phi_2. \quad (6)$$

The remaining task is to show that the pair of conditions (A) and (B) implies condition (4). Since condition (A) already guarantees condition (B) when $f_1(x_1) = f_1(x_2) = \phi_1$, in considering condition (B) one can assume without loss of generality that $f_1(x_1) < \phi_1$ and $f_1(x_2) > \phi_1$. The first condition in (6) is equivalent to

$$p = \frac{\phi_1 - f_1(x_2)}{f_1(x_1) - f_1(x_2)}.$$

Substituting p in the second condition of (6) makes (6) equivalent to

$$(\phi_1 - f_1(x_2))(f_2(x_1) - \phi_2) \geq (\phi_1 - f_1(x_1))(f_2(x_2) - \phi_2),$$

for any x_1 and x_2 in $[a, b]$ such that $f_1(x_1) < \phi_1$ and $f_1(x_2) > \phi_1$. After some manipulations, the above inequality is itself equivalent to

$$\frac{f_2(x_2) - \phi_2}{f_1(x_2) - \phi_1} \leq \frac{f_2(x_1) - \phi_2}{f_1(x_1) - \phi_1}, \quad (7)$$

for any x_1 and x_2 in $[a, b]$ such that $f_1(x_1) < \phi_1$ and $f_1(x_2) > \phi_1$. This is equivalent to the fact that there exists a real number m for which

$$\frac{f_2(x_2) - \phi_2}{f_1(x_2) - \phi_1} \leq m \leq \frac{f_2(x_1) - \phi_2}{f_1(x_1) - \phi_1}, \quad (8)$$

for any x_1 and x_2 in $[a, b]$ such that $f_1(x_1) < \phi_1$ and $f_1(x_2) > \phi_1$. This is in turn equivalent to condition (4). So, condition (4) means that condition (B) is satisfied. In conclusion, by the Lemma, condition (1) is equivalent to condition (4) plus condition (A). This pair of conditions is obviously equivalent to condition (4) with m being nonnegative. ■

It is noteworthy that condition (4), with m positive or negative, forces condition (1) to hold for all two-point random variables $\tilde{x}$ that satisfy condition $Ef_1(\tilde{x}) \leq Ef_1(\tilde{\omega})$ as an equality. The other condition – m nonnegative – forces condition (1) for all degenerate (one-point) random variables. It is thus easy to verify that if the first inequality in (1) is replaced by an equality, the conclusion is the same except that m can be any real number.[2]

The benefit of this result with respect to the Lemma is that one have to verify a unidimensional condition. But the cost is the necessity to look for an m that satisfies it. An important additional simplification can be obtained if there exists a scalar x_0 such that $f_1(x_0) = Ef_1(\tilde{\omega})$ *and* $f_2(x_0) = Ef_2(\tilde{\omega})$. This is the case for example if $\tilde{\omega}$ is degenerated (at x_0). In this case, there is a unique candidate for m that is stated in the following Corollary.

Corollary 1 *Assume that there exists a scalar x_0 such that $f_1(x_0) = Ef_1(\tilde{\omega})$ and $f_2(x_0) = Ef_2(\tilde{\omega})$. Then, if $f_1'(x_0) \neq 0$ and $f_2'(x_0)$ exists, the unique candidate for m in (4) is $\frac{f_2'(x_0)}{f_1'(x_0)}$. Also, if f_1 and f_2 are twice differentiable at x_0, a necessary condition for (1) is*

$$f_2''(x_0) \leq \frac{f_2'(x_0)}{f_1'(x_0)} f_1''(x_0). \tag{9}$$

Proof: Condition (4) means that function $\xi(x)$ must be nonpositive, with

$$\xi(x) = f_2(x) - f_2(x_0) - m[f_1(x) - f_1(x_0)].$$

But $\xi(x_0) = 0$, so the global maximum of ξ must be at x_0. The necessary condition for a maximum must hold: $f_2'(x_0) - mf_1'(x_0) = 0$, yielding the first result. Condition (9) is a direct consequence of the second-order condition $\xi''(x_0) \leq 0$. This concludes the proof. ■

In the remaining of the paper, we will assume that the conditions presented in the above Corollary holds, so that our basic problem can be rewritten as

$$\forall \tilde{x}: \quad Ef_1(\tilde{x}) \leq f_1(x_0) \Longrightarrow Ef_2(\tilde{x}) \leq f_2(x_0), \tag{10}$$

with the necessary and sufficient condition

$$\forall x \in [a,b]: \quad f_2(x) - f_2(x_0) \leq \frac{f_2'(x_0)}{f_1'(x_0)}[f_1(x) - f_1(x_0)], \tag{11}$$

[2]Also, the change of the direction of the first and/or the second inequality in (1) can easily be considered by redefining f_i by $-f_i$, for $i = 1$ and/or 2.

together with the necessary condition $\frac{f_2'(x_0)}{f_1'(x_0)} \geq 0$. Notice that this condition implies that $f_2(x) \leq f_2(x_0)$ whenever $f_1(x) \leq f_1(x_0)$, for any value x close enough to x_0. This is related to the fact that condition $m \geq 0$ takes care of one-point distributions.

Notice also that another necessary condition is stated in the Corollary. This condition is hereafter called the "local" diffidence condition. Indeed, it is the necessary and sufficient for (10) to hold for any *small* risk around x_0, using the property that

$$Ef_i(\tilde{x}) - f_i(x_0) \cong f_i'(x_0)E(\tilde{x} - x_0) + 0.5f_i''(x_0)E(\tilde{x} - x_0)^2.$$

The local diffidence condition (9) is in general strictly weaker than the global necessary condition (11), i.e. something that is true for any small risks is not necessarily true for any (large) risks.

In general, the problem faced by researchers is a bit more complicated than (1). More specifically, functions f_i may depend upon an arbitrary parameter w, leading to problem

$$\forall w \forall \tilde{x}: \quad Ef_1(w, \tilde{x}) \leq f_1(w, x_0) \Longrightarrow Ef_2(w, \tilde{x}) \leq f_2(w, x_0). \tag{12}$$

In most cases, w designates an intial wealth level that is arbitrary. A direct extension of the Diffidence Theorem is that condition (12) has the following *bivariate* necessary and sufficient condition (NSC)

$$\forall x \in [a, b], \forall w: \quad f_2(w, x) - f_2(w, x_0) \leq \frac{\frac{\partial f_2}{\partial x}(w, x_0)}{\frac{\partial f_1}{\partial x}(w, x_0)}[f_1(w, x) - f_1(w, x_0)], \tag{13}$$

with necessary conditions

$$\text{NC1: } \forall w: \ \frac{\frac{\partial f_2}{\partial x}(w, x_0)}{\frac{\partial f_1}{\partial x}(w, x_0)} \geq 0 \tag{14}$$

and

$$\text{NC2: } \forall w: \ \frac{\partial^2 f_2}{\partial x^2}(w, x_0) \leq \frac{\frac{\partial f_2}{\partial x}(w, x_0)}{\frac{\partial f_1}{\partial x}(w, x_0)} \frac{\partial^2 f_1}{\partial x^2}(w, x_0). \tag{15}$$

3 Applications of the Diffidence Theorem

There exist examples of problems of type (12) for which the combination of the necessary conditions (14) and (15) is sufficient. This leads to a characterization of the solution that is simpler than the bivariate characterization (13). These examples are gathered in the first part of this section.

3.1 Applications with a univariate necessary and sufficient condition

3.1.1 Risk aversion

Probably the simple application of the Diffidence Theorem is the characterization of risk aversion. Agent with increasing utility u_1 is risk averse if and only if:

$$E\tilde{x} \leq 0 \quad \Longrightarrow \quad Eu_1(w+\tilde{x}) \leq u_1(w), \tag{16}$$

for all $\tilde{x}$ and all w. Applying our technology yields the following conditions:

$$\text{NSC: } \forall x, w: \; u_1(w+x) - u_1(w) \leq u_1'(w)x \tag{17}$$

$$\text{NC1: } \forall w: \; u_1'(w) \geq 0 \tag{18}$$

$$\text{NC2: } \forall w: \; u_1''(w) \leq 0 \tag{19}$$

where NSC, NC1 and NC2 designate respectively the necessary and sufficient condition (13), necessary condition (14) and necessary condition (15) applied to the corresponding problem. Notice that $m(w) = u'(w)$ is naturally positive, so it makes no difference to replace the first inequality in (16) by an equality. Also, the necessary condition (19) is obviously sufficient for the necessary and sufficient condition (17). As Pratt [1964] observed, the local risk aversion condition is sufficient for global risk aversion: risk aversion can be defined by the rejection of any *small* zero-mean background risk.[3]

[3] One can also define risk aversion by requiring that the solution of the first-order condition for the portfolio problem $\max_\alpha Eu(w+\alpha\tilde{x})$ raises expected utility. Mathematically, $E\tilde{x}u_1'(w+\tilde{x}) = 0 \Rightarrow Eu_1(w+\tilde{x}) \geq u_1(w)$. The necessary and sufficient condition is $u_1(w+x) - u_1(x) \geq xu_1'(w+x)$ for all w and x. It is easy to verify that this is equivalent to $u_1'' < 0$.

One can weaken the concept of risk aversion by requiring that condition (16) holds only for a given w. The underlying question is to find the properties of the utility function of an agent from whose we observe that he rejects any (un)fair lottery *at his/her current wealth* w. It yields the necessary and sufficient condition (17)for any x, but for that given w. It means that the utility curve u lies entirely below the line tangent to u_1 at w. We call this condition "central risk aversion" around w.

3.1.2 Prudence

Kimball [1990] introduced the concept of prudence. A prudent agent reduces his/her current consumption when a zero-mean risk is added to future incomes. Under time-separable preferences, a prudent agent with future utility of consumption u_1 is thus characterized by the following condition:

$$E\tilde{x} = 0 \quad \Longrightarrow \quad Eu_1'(w + \tilde{x}) \geq u_1'(w), \tag{20}$$

for all $\tilde{x}$ and all w. By symmetry with the previous application (replace u_1 by $-u_1'$), we have the following conditions:

$$\text{NSC: } \forall x, w: \ u_1'(w + x) - u_1'(w) \geq u_1''(w)x \tag{21}$$

$$\text{NC1: irrelevant (first condition is an equality)} \tag{22}$$

$$\text{NC2: } \forall w: \ u_1'''(w) \geq 0 \tag{23}$$

All the remarks made for the previous application also apply here. In particular, the local diffidence condition is also global. We can define the concept of central prudence around w. It is defined by the fact that, for a given future consumption w, adding a zero-mean risk to it incites the centrally prudent agent to reduce his/her current consumption. Its characterization is given by (21) for the specified future consumption w.

3.1.3 More diffidence

We say that individual u_2 is more diffident than individual u_1 if the former rejects a larger set of lotteries than the latter. Mathematically, it is written as:

$$\forall w, \tilde{x}: \quad Eu_1(w + \tilde{x}) \leq u_1(w) \quad \Longrightarrow \quad Eu_2(w + \tilde{x}) \leq u_2(w). \tag{24}$$

Applying the Diffidence Theorem yields the following conditions:

$$\text{NSC: } \forall x, w : \frac{u_2(w+x) - u_2(w)}{u_2'(w)} \leq \frac{u_1(w+x) - u_1(w)}{u_1'(w)} \tag{25}$$

$$\text{NC1: } \forall w : \frac{u_2'(w)}{u_1'(w)} \geq 0 \tag{26}$$

$$\text{NC2: } \forall w : \frac{u_2''(w)}{u_2'(w)} \leq \frac{u_1''(w)}{u_1'(w)} \tag{27}$$

The necessary condition (26) is always satisfied. So, it makes no difference to consider problem (24) or the same one in which the first inequality is replaced by an equality. Notice also that the second necessary condition is equivalent to u_2'/u_1' being decreasing. This in turn is equivalent to

$$\frac{u_2'(w+\xi)}{u_1'(w+\xi)}\xi \leq \frac{u_2'(w)}{u_1'(w)}\xi,$$

or

$$\frac{u_2'(w+\xi)}{u_2'(w)}\xi \leq \frac{u_1'(w+\xi)}{u_1'(w)}\xi,$$

for all w and ξ. If ξ is positive, simplify the above condition by ξ and integrate ξ between 0 and $x > 0$. It yields sufficient condition (25). The same property holds for $\xi < 0$. In conclusion, the necessary condition (27) implies the necessary and sufficient condition (25): u_2 is more diffident than u_1 iff it is more risk-averse, i.e. $A_2(w) \geq A_1(w)$ for all w, with $A_i(w) = -u_i''(w)/u_i'(w)$. This is an example in which the local diffidence condition (15) is also global.

Associated to the concept of "more diffidence" is the concept of "central diffidence". Individual u_2 is centrally more diffident around w than individual u_1 iff u_2 rejects more lotteries than u_1, both individuals having the same initial wealth w. It is characterized by condition (25) for this specific w. It implies that u_2 is *locally* more risk-averse than u_1, but not necessarily globally. If we normalize u_1 and u_2 such that $u_1(w) = u_2(w)$ and $u_1'(w) = u_2'(w)$, central more diffidence means that the u_2-curve is below the u_1-curve. If u_1 is linear, we get back the notion of central risk aversion. In fact, the concept of "central diffidence" comparative version of the notion of "central risk aversion".

Notice that we can consider the following similar problem:

$$\forall w, \tilde{x}: \quad Eu_1'(w+\tilde{x}) \geq u_1'(w) \quad \Longrightarrow \quad Eu_2'(w+\tilde{x}) \leq u_2'(w). \tag{28}$$

Under which condition on preferences does any risk that increases precautionary savings of agent u_1 also increase precautionary savings of agent u_2? If w is arbitrary, an argument symmetric to the one presented above implies that this is possible if and only if $-u_2'$ is more risk averse than $-u_1'$, i.e. u_2 is more prudent than u_1 in the sense of Kimball [1990], or $P_2(w) \geq P_1(w)$ for all w, with $P_i(w) = -u_i'''(w)/u_i''(w)$. If we restrict the analysis to a given w, the necessary and sufficient condition for such a property to hold is written as:

$$\text{NSC: } \forall x, w: \ \frac{u_2'(w+x) - u_2'(w)}{u_2''(w)} \leq \frac{u_1'(w+x) - u_1'(w)}{u_1''(w)}, \tag{29}$$

together with risk aversion for both agents at w.

3.1.4 Decreasing absolute risk aversion

There are several ways to define the concept of decreasing absolute risk aversion. One of them is given by the condition that prudence is stronger than risk aversion, i.e. any undesirable risk on future incomes reduces current consumption:

$$\forall w, \tilde{x}: \quad Eu_1(w+\tilde{x}) \leq u_1(w) \quad \Longrightarrow \quad Eu_1'(w+\tilde{x}) \geq u_1'(w). \tag{30}$$

This problem is equivalent to problem (24) by defining $u_2 = -u_1'$. The same conclusion applies, with a necessary and sufficient condition that the coefficient of absolute risk aversion of $-u_1'$ be larger than the one of u_1, i.e. $P_1(w) \geq A_1(w)$ for all w. It is easily seen that this is equivalent to decreasing absolute risk aversion of u_1 since $A_1'(w) = A_1(w)[A_1(w) - P_1(w)] \leq 0$. A local concept can be easily associated to decreasing absolute risk aversion by following the same line than in the previous subsections.[4]

[4]A more natural way to define decreasing absolute risk aversion of u_1 is the condition that $u_1(w_2 + .)$ being more diffident than $u_1(w_1 + .)$, for any $w_2 < w_1$: a reduction in wealth enlarges the set of undesirable lotteries. Our results on more diffidence directly yields necessary and sufficient condition $A_1(w_2) \geq A_1(w_1)$ for all $w_2 < w_1$, or equivalently, A_1 decreasing.

3.1.5 Decreasing absolute prudence

Kimball [1993] defined decreasing absolute prudence of u_1 by P_1 being uniformly negative. This is equivalent to decreasing absolute risk aversion of $-u_1'$, or, as seen above, to

$$\forall w, \tilde{x}: \quad Eu_1'(w+\tilde{x}) \geq u_1'(w) \quad \Longrightarrow \quad Eu_1''(w+\tilde{x}) \leq u_1''(w). \tag{31}$$

Gollier [1994] provides a simple interpretation of this condition that is based on how a marketable risk should be shared when an agent in the pool bears another nonmarketable risk. From our discussion on the concept of more diffidence, condition (31) – which means that u_1'' is more diffident than $-u_1'$ – holds iff u_1'' be more risk-averse than $-u_1'$. It is easy to verify that this condition is equivalent to P_1 being nonincreasing.

3.1.6 More risk aversion

Alternative to the definition in subsection 3.1.3, one can say that u_2 is more risk averse than u_1 iff u_2 purchases less of the risky asset than u_1, independent of the common wealth level w (see Pratt [1964]). As is well-known (see for example Dionne, Eeckhoudt and Gollier [1993]), if we assume that u_2 is risk-averse, then this problem is fully characterized by the following condition:

$$\forall w, \tilde{x}: \quad E\tilde{x}u_1'(w+\tilde{x}) = 0 \quad \Longrightarrow \quad E\tilde{x}u_2'(w+\tilde{x}) \leq 0. \tag{32}$$

Using the Diffidence Theorem and its Corollary, it yields the following conditions:

$$\text{NSC: } \forall x, w: \ \frac{xu_2'(w+x)}{u_2'(w)} \leq \frac{xu_1'(w+x)}{u_1'(w)} \tag{33}$$

$$\text{NC1: } \forall w: \ \frac{u_2'(w)}{u_1'(w)} \geq 0 \tag{34}$$

$$\text{NC2: } \forall w: \ \frac{u_2''(w)}{u_2'(w)} \leq \frac{u_1''(w)}{u_1'(w)}. \tag{35}$$

Again, the first necessary condition (34) is always satisfied, so it is indifferent to consider problem (32) with the first condition being an equality or a "less or equal" condition. Also, as said above, the first necessary condition (35) means that u_2'/u_1' is decreasing, i.e. u_2' decreases at a larger rate than u_1' when wealth is increased. This is equivalent to the sufficient condition (33). This is again an example in which the unidimensional necessary condition (15) is also sufficient.[5]

Associated to this problem is the concept of "centrally more risk-averse", which is condition (32) for a specific w. In other words, u_2 is centrally more risk-averse around w than u_1 if agent u_2 purchases less units of the risky asset than agent u_1, independent of their common initial wealth w. The necessary and sufficient condition is condition (33) for this specific w.

3.1.7 Effect of opening up an asset market on consumption

How would current consumption be affected by the opening up of an asset market? To answer this question consider a simple model with two periods and time-additive preferences characterized by discounted utility functions v and u_1. The agent is endowed with discounted wealth W at date 1. Initially, there is only one asset whose price is normalized to unity and that pays a sure $\$(1+r)$ at date 2. In such a situation, the agent purchases an amount $s = s_1$ of the risk free asset that maximizes his discounted utility $v(W-s)+u_1((1+r)s)$. This yields first-order condition $-v'(W-s_1)+(1+r)u_1'((1+r)s_1) = 0$. Now, assume that a market for a risky asset is opened. The excess return of the risky asset over the risk free rate at date 2 is $\tilde{x}$. The problem of the investor is now to select at date 1 the demand α for the risky asset and the demand $s-\alpha$ for the risk free asset that solves the following problem:

$$\max_{(s,\alpha)} \; v(W-s) + Eu_1((1+r)s + \alpha\tilde{x}).$$

For $s = s_1$ fixed, the first-order condition on α is $E\tilde{x}u_1'((1+r)s_1 + \alpha^*\tilde{x})$. Assuming risk aversion, the optimal total saving at date 1 if the new market is opened will be less than s_1 iff $-v'(W-s_1) + (1+r)Eu_1'((1+r)s_1 + \alpha^*\tilde{x})$ is negative. Using the first-order condition on s_1 makes this condition

[5] A third characterization of decreasing absolute risk aversion is that a reduction in wealth reduces the demand of the risky asset. Using standard comparative static techniques, this means that $E\tilde{x}u_1''(w+\tilde{x}) \geq 0$ whenever $E\tilde{x}u_1'(w+\tilde{x}) = 0$. Using the result above with $u_2 = -u_1'$ yields the necessary and sufficient condition that $-u_1'$ be more risk-averse than u_1, a condition that we know to be equivalent to decreasing absolute risk aversion.

equivalent to $Eu_1'((1+r)s_1+\alpha^*\tilde{x}) \leq u_1'((1+r)s_1)$. Assuming without loss of generality that $\alpha^* = 1$, this means that opening up a new market increases consumption if and only

$$\forall w, \tilde{x}: \quad E\tilde{x}u_1'(w+\tilde{x}) = 0 \quad \Longrightarrow \quad Eu_1'(w+\tilde{x}) \leq u_1'(w), \tag{36}$$

with w representing $(1+r)s_1$ that is arbitrary if r or v is arbitrary. Applying the Diffidence Theorem yields the following conditions:

$$\text{NSC: } \forall x, w: \ u_1'(w+x) - u_1'(w) \leq \frac{u_1''(w)}{u_1'(w)} x u_1'(w+x) \tag{37}$$

NC1: irrelevant (first condition is an equality)

$$\text{NC2: } \forall w: \ u_1'''(w) \leq 2\frac{[u_1''(w)]^2}{u_1'(w)}. \tag{38}$$

Necessary condition (38) is equivalent to

$$-\frac{u_1'''(w)}{u_1''(w)} \leq -2\frac{u_1''(w)}{u_1'(w)}, \tag{39}$$

or, to $P_1 \leq 2A_1$.

These conditions allow us to prove the following Proposition. Its proof relies on showing that conditions (39) and (37) are equivalent.

Proposition 1 *Opening up a new asset market raises current consumption if and only if absolute prudence is smaller than twice absolute risk aversion.*

Proof: Integrating condition (39) indicates that this necessary condition is equivalent to $-u_1''/u_1'^2$ being increasing. Integrating again yields the necessary condition that $g(w) = 1/u_1'(w)$ be convex. It implies that $g(w+x) \geq g(w) + xg'(w)$ for all w and x. This is equivalent to the sufficient condition (37). ■

By symmetry, opening up a new asset market reduces current consumption if absolute prudence is larger than twice absolute risk aversion. This raises the question of whether P_1 be less or larger than $2A_1$ in the real world. The intuitively appealing condition of decreasing absolute risk aversion is not helpful since it simply means that P_1 is larger than A_1. Let us consider the case of power utility functions, i.e. functions with constant relative risk aversion γ. This family contains both functions with $P_1 \geq 2A_1$ and functions with $P_1 \leq 2A_1$, with the limit case represented by the logarithmic function whose γ is unity. Indeed, $A_1(w) = \frac{\gamma}{w}$ and $P_1(w) = \frac{\gamma+1}{w}$, yielding $P_1 \leq 2A_1$ if and only if $\gamma \geq 1$. An investor would increase his current consumption after a new market is opened if he is more risk-averse than the logarithm investor. The logarithm agent would not change is current consumption. This is another illustration of the myopia of the logarithmic agents (see for example Mossin [1968]).

3.1.8 Effect of opening up an asset market on the attitude toward another small risk

Would opening up a new asset market induce risk-averse investors to reject a lottery that they would have accepted otherwise? In this section, we solve this problem for the case of small lotteries. In other terms, we determine whether opening up a new market does increase local risk aversion. The problem is written as follows:

$$\forall w, \tilde{x}: \quad E\tilde{x}u_1'(w+\tilde{x}) = 0 \quad \Longrightarrow \quad -\frac{Eu_1''(w+\tilde{x})}{Eu_1'(w+\tilde{x})} \geq -\frac{u_1''(w)}{u_1'(w)}, \tag{40}$$

Using the Diffidence Theorem yields the following conditions:

$$\text{NSC: } \forall x, w: \ A_1(w+x) \geq A_1(w) + xA_1'(w) \tag{41}$$

$$\text{NC1: irrelevant (first condition is an equality)}$$

$$\text{NC2: } \forall w: \ A_1''(w) \geq 0. \tag{42}$$

Obviously, conditions (41) and (42) coincide. This proves the following Proposition.

Proposition 2 *Opening up a new market raises local risk aversion if and only if absolute risk aversion is convex.*

The index of absolute risk aversion of all familiar utility functions is convex. In fact, all familiar utility functions have an harmonic risk aversion. Notice that this condition is not sufficient to guarantee that opening a new asset market increases risk aversion globally. So for example, it does not guarantee that opening a market reduces the demand for other independent assets.

3.2 Applications with no univariate necessary and sufficient condition

3.2.1 Effect of a fair background risk on the equilibrium risk free rate

Consider again the two-period model examined above, but in a general equilibrium framework. All agents in the economy share the same time-separable utility functions v and u_1, and they are all endowed with income w_1 and w_2 at dates 1 and 2 respectively. Assuming the existence of a market for a risk free asset, the equilibrium rate in the economy obtains:

$$1 + r^* = \frac{v'(w_1)}{u_1'(w_2)}.$$

Following Weil [1992], assume alternatively that agents all bear an unfair idiosyncratic risk on their period 2 incomes. Namely, the period 2 income is $w_2 + \tilde{x}$, with $E\tilde{x} \leq 0$. If there is no market to trade this risk in the economy, the equilibrium risk free rate is

$$1 + \hat{r} = \frac{v'(w_1)}{Eu_1'(w_2 + \tilde{x})}.$$

Under risk aversion and prudence, any idiosyncratic unfair background risk reduces the equilibrium risk free rate in the economy. The relative discrepancy in the equilibrium risk free rate of the two models is measured by ratio

$$\frac{1 + r^*}{1 + \hat{r}} = \frac{Eu_1'(w_2 + \tilde{x})}{u_1'(w_2)}. \tag{43}$$

If $E\tilde{x} = 0$, this ratio also measures the relative error of a calibrator who would estimate the risk free rate by considering a (wrong) representative agent model (see Weil [1989] for more details).

In this subsection, we are interested in characterizing the set of utility functions u_2 such that the effect of any idiosyncratic zero-mean background risk $\tilde{x}$ has a larger effect on the equilibrium risk free rate in the economy with u_2 than in the economy with u_1. Mathematically, this problem is written as

$$\forall w, \tilde{x}: \quad E\tilde{x} \leq 0 \quad \Longrightarrow \quad \frac{Eu_2'(w + \tilde{x})}{u_2'(w)} \geq \frac{Eu_1'(w + \tilde{x})}{u_1'(w)}, \tag{44}$$

Thus, our problem simplifies to determining when does an unfair risk raise expected marginal utility by a bigger percentage for one utility than another. Applying the Diffidence Theorem yields the following conditions:

$$\text{NSC: } \forall x, w: \ \frac{u_1'(w+x)}{u_1'(w)} - \frac{u_2'(w+x)}{u_2'(w)} \leq x \left[\frac{u_1''(w)}{u_1'(w)} - \frac{u_2''(w)}{u_2'(w)} \right] \tag{45}$$

$$\text{NC1: } \forall w: \ \frac{u_1''(w)}{u_1'(w)} - \frac{u_2''(w)}{u_2'(w)} \geq 0 \tag{46}$$

$$\text{NC2: } \forall w: \ \frac{u_1'''(w)}{u_1'(w)} - \frac{u_2'''(w)}{u_2'(w)} \leq 0. \tag{47}$$

This application of the Diffidence Theorem appears to be the first example for which the local diffidence condition (15) is not global. Simple sufficient conditions are provided in the next Proposition.

Proposition 3 *Consider problem (44) that guarantees that introducing an unfair idiosyncratic background risk has a larger effect on the risk free rate in the economy with utility function* u_2 *than in the economy with utility function* u_1. *Two necessary conditions are respectively* $A_2 \geq A_1$ *and* $A_2P_2 \geq A_1P_1$. *Each of the two following univariate conditions are sufficient when combined with condition* $A_2 \geq A_1$:

(i) $u_2'' - \frac{u_2'}{u_1'}u_1''$ *is increasing;*

(ii) $A_2(P_2 - 0.5A_2)$ *is uniformly larger than* $A_1(P_1 - 0.5A_1)$.

Proof: See the Appendix

Sufficient condition (i) is weaker than sufficient condition (ii), but it has the disadvantage of not being separable.

3.2.2 Acceptance of a portfolio

Consider the one-period model with two assets, one risk free and one risky. An investor with utility u_1 selects the portfolio that maximizes his expected utility, yielding $E\tilde{x}u_1'(w + \alpha^*\tilde{x}) = 0$. Under which condition is this portfolio desirable for an individual u_2 with the same wealth w? This problem can be encountered by a fund manager or a life insurer who would select a portfolio that maximizes the welfare of the representative customer, assuming that customers have no possibility to rebalance their risk on the market. Normalizing α^* to unity, the condition is:

$$\forall w, \tilde{x}: \quad E\tilde{x}u_1'(w+\tilde{x}) = 0 \quad \Longrightarrow \quad Eu_2(w+\tilde{x}) \geq u_2(w), \tag{48}$$

Applying the Diffidence Theorem yields the following conditions:

$$\text{NSC: } \forall x, w: \; u_2(w+x) - u_2(w) \geq \frac{u_2'(w)}{u_1'(w)} x u_1'(w+x) \tag{49}$$

NC1: irrelevant (first condition is an equality)

$$\text{NC2: } \forall w: \; \frac{u_2''(w)}{u_2'(w)} \geq 2\frac{u_1''(w)}{u_1'(w)}. \tag{50}$$

We obtain the following Proposition.[6]

Proposition 4 *A necessary condition for u_2 to like the optimal portfolio of u_1 is that the absolute risk aversion of u_2 be less than twice the absolute risk aversion of u_1, i.e. $A_2 \leq 2A_1$. A sufficient condition is that the absolute prudence of u_1 is larger than the absolute risk aversion of u_2, but less than twice the absolute risk aversion of u_1, i.e. $A_2 \leq P_1 \leq 2A_1$.*

Proof: Necessity of $A_2 \leq 2A_1$ is a rewriting for (50). Sufficiency of $A_2 \leq P_1 \leq 2A_1$ comes from the fact that the second inequality implies that $[E\tilde{x}u_1'(w+\tilde{x}) = 0 \Longrightarrow Eu_1'(w+\tilde{x}) \leq u_1'(w)]$ whereas the first inequality implies that $[Eu_1'(w+\tilde{x}) \leq u_1'(w) \Longrightarrow Eu_2(w+\tilde{x}) \geq u_2(w)]$. ■

Despite this problem corresponds to problem (37) with $u_2 \equiv -u_1'$ – a problem for which the necessary condition (15) is also sufficient – it is not true here that the necessary condition $A_2 \leq 2A_1$ is sufficient. To get sufficiency, we do not only have to verify that $A_2 \leq 2A_1$ but also that the absolute prudence of agent u_1 is in $[A_2, 2A_1]$.

[6]If we consider the problem of agent u_2 to always dislike the optimal portfolio of agent u_1, we get by symmetry the necessary condition $A_2 \geq 2A_1$, and the sufficient condition $A_2 \geq P_1 \geq 2A_1$.

3.2.3 Intensifiability

Gollier and Pratt [1993] defined the concept of "intensifiability" as follows: u_1 has an intensifiable risk aversion if adding an unfair background risk to wealth raises the aversion to any other independent risk, namely:

$$\forall w, \tilde{x} : \quad E\tilde{x} \leq 0 \quad \Longrightarrow \quad -\frac{Eu_1''(w+\tilde{x})}{Eu_1'(w+\tilde{x})} \geq -\frac{u_1''(w)}{u_1'(w)}, \tag{51}$$

or equivalently, assuming risk aversion,

$$\forall w, \tilde{x} : \quad E\tilde{x} \leq 0 \quad \Longrightarrow \quad \frac{Eu_1''(w+\tilde{x})}{u_1''(w)} \geq \frac{Eu_1'(w+\tilde{x})}{u_1'(w)}, \tag{52}$$

Intensifiability is an application of problem (44) by defining $u_2 \equiv -u_1'$. Therefore the necessary and sufficient condition

$$u_1'(w+x)\left[A_1(w+x) - A_1(w)\right] \geq x u_1'(w) A_1'(w), \tag{53}$$

the necessary condition $P_1 \geq A_1$ and the sufficient condition $A_1'' \geq A_1' A_1$ that are obtained by Gollier and Pratt [1993] are directly obtained from respectively conditions (45), (46) and $[u_2'' - \frac{u_2'}{u_1'} u_1'']' \geq 0$.

3.2.4 Semi properness

Pratt and Zeckhauser [1987] introduced the notion of properness that guarantees that any undesirable risk $\tilde{x}$ can never be made desirable by adding another independent undesirable risk to wealth. This notion is reconsidered in the next section. Properness means that indirect utility function $v(w) = Eu_1(w+\tilde{x})$ must be more diffident than utility function u_1. In this subsection, we introduce the notion of semi properness. Utility function u_1 is semi proper if adding any undesirable background risk to wealth makes the agent more risk-averse to any other independent risk:

$$\forall w, \tilde{x} : \quad Eu_1(w+\tilde{x}) \leq u(w) \quad \Longrightarrow \quad -\frac{Eu_1''(w+\tilde{x})}{Eu_1'(w+\tilde{x})} \geq -\frac{u_1''(w)}{u_1'(w)}, \tag{54}$$

It means that adding a risk to wealth increases risk aversion at any level of wealth where this risk is undesirable. As it is apparent, this notion is a weaker restriction than properness, since indirect utility function v is required to be locally more risk-averse than u_1 at w, a condition we know to be necessary, but not sufficient, for "more diffidence". To the contrary, it is a stronger restriction than intensifiability. This comes from the fact that the same condition – $-\frac{Eu_1''(w+\tilde{x})}{Eu_1'(w+\tilde{x})} \geq -\frac{u_1''(w)}{u_1'(w)}$ – is required to hold for a *larger* set of random variables under semi properness than under intensifiability.

Let us characterize semi properness:

$$\text{NSC: } \forall x, w: \ \frac{A_1(w+x) - A_1(w)}{A_1'(w)} \leq \frac{u_1(w+x) - u_1(w)}{u_1'(w+x)} \tag{55}$$

$$\text{NC1: } \forall w: \ A_1'(w) \leq 0 \tag{56}$$

$$\text{NC2: } \forall w: \ \frac{A_1''}{A_1'} \leq A_1(w). \tag{57}$$

Proposition 5 *Adding an undesirable risk to wealth raises risk aversion to any other independent risk is possible only if absolute risk aversion is decreasing and* $A_1'' \geq A_1'A_1$. *A sufficient condition is that absolute risk aversion be decreasing, and be more concave than the utility function in the sense of Arrow-Pratt, i.e.* $-A_1''/A_1' \geq A_1$, *or* $A_1'' \geq -A_1'A_1$.

Proof: The sufficient condition is proved by first observing that if A_1 is more risk-averse than u_1, it is also more diffident than it. It implies that

$$\frac{A_1(w+x) - A_1(w)}{A_1'(w)} \leq \frac{u_1(w+x) - u_1(w)}{u_1'(w)},$$

for all w and x. Second, by risk aversion, it is easy to verify that

$$\frac{u_1(w+x) - u_1(w)}{u_1'(w)} \leq \frac{u_1(w+x) - u_1(w)}{u_1'(w+x)}$$

for all w and x. Combining these two conditions yields the sufficient condition (55). ■

4 The Bivariate Diffidence Theorem

Other problems involve more than one random variables. For example, Pratt and Zeckhauser [1987] defined proper risk aversion by requiring that any undesirable risk can never be made desirable by adding another independent undesirable risk to wealth. Mathematically, (fixed-wealth) properness is defined by the following property: $\forall w, \tilde{x}, \tilde{y}$:

$$Eu_1(w+\tilde{x}) \leq u_1(w) \text{ and } Eu_1(w+\tilde{y}) \leq u_1(w) \Longrightarrow Eu_1(w+\tilde{y}+\tilde{x}) \leq Eu_1(w+\tilde{x}), \tag{58}$$

where $\tilde{x}$ and $\tilde{y}$ are independent random variables. There is a simple way to solve this problem by using the Diffident Theorem twice. Indeed, by definition, u_1 is proper if and only if $v(w) = Eu_1(w+\tilde{x})$ is more diffident than u_1 for any undesirable risk $\tilde{x}$ at w, and for any w. Thus, an equivalent definition of properness is: $\forall w, y, \tilde{x} : Eu_1(w+\tilde{x}) \leq u_1(w)$ implies

$$\frac{Eu_1(w+y+\tilde{x}) - Eu_1(w+\tilde{x})}{Eu_1'(w+\tilde{x})} \leq \frac{u_1(w+y) - u_1(w)}{u_1'(w)}. \tag{59}$$

Applying again the Diffidence Theorem on this condition for every y yields the necessary and sufficient condition obtained by Pratt and Zeckhauser [1987, Theorem 4].

The general structure of the problem that can be solved by the next Theorem is as follows: [7]

$$\forall \tilde{x}, \tilde{y} : \quad Ef_1(\tilde{x}) = f_1(0) \text{ and } Ef_2(\tilde{y}) = f_2(0) \quad \Longrightarrow \quad Eh(\tilde{x}, \tilde{y}) \leq 0. \tag{60}$$

In the following Bivariate Diffidence Theorem, we systematize the technique that has been presented to characterize properness.

Theorem 2 *Suppose that function $h(x,y)$ satisfies $h(x,0) \equiv 0$ and $h(0,y) \equiv 0$. Then, as long as $f_1'(0) \neq 0$ and $f_2'(0) \neq 0$, condition (60) is equivalent to the condition that*

[7] To be concise, we limit the analysis to a degenerate reference point $\tilde{\omega}$, and equality conditions. Other formulations can easily be characterized by following the same rules as above.

$$h(x,y) - \delta_1(x)\frac{\partial h}{\partial x}(0,y) + \delta_2(y)\frac{\partial h}{\partial y}(x,0) + \delta_1(x)\delta_2(y)\frac{\partial^2 h}{\partial x \partial y}(0,0) \leq 0, \quad (61)$$

for all x and y, with $\delta_i(t) = \frac{f_i(t)-f_i(0)}{f_i'(0)}$.

Proof: Given the assumptions, for any $\tilde{y}$ satisfying the second condition in (60), any $\tilde{x}$ satisfying the first condition in (60) implies $Eh(\tilde{x},\tilde{y}) \leq 0$. Therefore, for any such $\tilde{y}$, the univariate Diffidence Theorem (the version with equality) can be applied and

$$E_y\left[h(x,\tilde{y}) - \delta_1(x)\frac{\partial h}{\partial x}(0,\tilde{y})\right] \leq 0 \quad (62)$$

for all x. But then the second condition in (60) implies (62), which allows us to apply the univariate Diffidence Theorem again to obtain (61). ■

Note: $h(x,y)$ satisfying $h(x,0) \equiv 0$ and $h(0,y) \equiv 0$ is equivalent to:

$$h(x,y) = j(x,y) - j(x,0) - j(0,y) + j(0,0)$$

for some function $j(x,y)$. It is clear that for any j this gives an h with the stipulated property. On the other hand, if one has h in hand, it is easy to find a j – just take h itself as j.

One could consider a lot of applications of the Bivariate Diffidence Theorem. For example, if the first condition in (58) is replaced by $Eu_1'(w+\tilde{x}) \geq u_1'(w)$, we get the notion of standard risk aversion introduced by Kimball [1993]. If it is replaced by condition $E\tilde{x} \leq 0$, we get intensifiability. It would also be easy to apply the Theorem to the following problem: $\forall w, \tilde{x}, \tilde{y}$:

$$E\tilde{x}u_1'(w+\tilde{x}) = 0 \text{ and } Eu_1(w+\tilde{y}) = u_1(w) \Rightarrow Eu_1(w+\tilde{y}+\tilde{x}) \leq Eu_1(w+\tilde{x}). \quad (63)$$

This is the problem of guaranteeing that opening up a new asset market makes undesirable any risk for which one was indifferent prior to the opening. Obviously, this condition is stronger than (40). This is let as an exercice to the reader. Many other examples can be found by combining different types of function f_i that we considered in the previous section.

5 Other extensions

5.1 Allowing for an arbitrary reference situation

All applications of the Diffidence Theorem relied on a prespecified reference random variable $\tilde{\omega}$ that was degenerated at 0 in all applications. But one can imagine applications in which the initial distribution of wealth is not specified. What would happen to the characterization of the problem if the distribution of $\tilde{\omega}$ be arbitrary? In other words, what are the restrictions on functions f_1, f_2 to get

$$\forall \tilde{x}, \tilde{\omega} : \quad Ef_1(\tilde{x}) \leq Ef_1(\tilde{\omega}) \Longrightarrow Ef_2(\tilde{x}) \leq Ef_2(\tilde{\omega}), \tag{64}$$

To illustrate, if these functions are utility functions, under which conditions on preferences is it true that if agent u_1 prefers a lottery to another, then agent u_2 also prefers the former to the latter? We know the answer to this simple problem: the two functions must represent the same preferences, i.e. $\exists m > 0 : \ f_2(.) \equiv mf_1(.) + b$. We generalize this result in the following Proposition.

Proposition 6 *Suppose that f_1 and f_2 are twice differentiable. Then, condition (64) holds if and only if there exists a positive scalar m such that $f_2(x) - f_2(0) = m[f_1(x) - f_1(0)]$.*

Proof: Applying the Diffidence Theorem for $\tilde{\omega}$ degenerated at arbitrary value x_0, we get the necessary condition

$$f_2''(x_0) \leq \frac{f_2'(x_0)}{f_1'(x_0)} f_1''(x_0).$$

Notice that condition (64) is equivalent to

$$\forall \tilde{x}, \tilde{\omega} : \quad Ef_1(\tilde{x}) \geq Ef_1(\tilde{\omega}) \Longrightarrow Ef_2(\tilde{x}) \geq Ef_2(\tilde{\omega}),$$

for which a necessary condition when $\tilde{\omega} = x_0$ is

$$f_2''(x_0) \geq \frac{f_2'(x_0)}{f_1'(x_0)} f_1''(x_0).$$

Thus, $\frac{f_2''(x_0)}{f_2'(x_0)} = \frac{f_1''(x_0)}{f_1'(x_0)}$, for all x_0. This is equivalent to f_2'/f_1' being constant.
■

5.2 Multi-antecedent Diffidence

Another possible extension is to consider more than one condition on $\tilde{x}$. Namely, consider an index set $\Theta \subset R$ that can be either finite or infinite, and a set of functions $\{f_\theta(x),\ \theta \in \Theta\}$. The multi-antecedent problem is written as:

$$\forall \tilde{x}: \quad Ef_\theta(\tilde{x}) \leq Ef_\theta(\tilde{\omega}_\theta)\ \forall \theta \in \Theta \implies Eg(\tilde{x}) \leq Eg(\tilde{\omega}), \tag{65}$$

where $\{\tilde{\omega}_\theta | \theta \in \Theta\}$ is a prespecified set of random variables. The Multi-antecedent Diffidence Theorem provides a tool to solve this kind of problem.

Theorem 3 *Suppose that functions f_θ and g are in $L_2[a,b]$. Then condition (65) holds if and only if there exists a nonnegative Lebesgues-integrable function $m : \Theta \to R$ such that*

$$g(x) - Eg(\tilde{\omega}) \leq \int_\Theta m(\theta)[f_\theta(x) - Ef_\theta(\tilde{\omega}_\theta)]d\theta, \tag{66}$$

for all x in $[a,b]$.

Proof: See the Appendix.

Notice that condition (66) is itself equivalent to the condition that

$$\forall \tilde{x}: \quad E\int_\Theta m(\theta)f_\theta(\tilde{x})d\theta \leq E\int_\Theta m(\theta)f_\theta(\tilde{\omega})d\theta \Longrightarrow Eg(\tilde{x}) \leq Eg(\tilde{\omega}). \tag{67}$$

This means that any problem with more than one antecedent is equivalent to another problem with only one antecedent on which the single-antecedent Diffidence Theorem can be applied. To illustrate, consider the problem of an agent v rejecting any lottery that both agents u_1 and u_2 reject:

$$\forall \tilde{x}: \quad Eu_i(w+\tilde{x}) \leq u_i(w),\ i = 1,2 \Longrightarrow Ev(w+\tilde{x}) \leq v(w). \tag{68}$$

From our results, this is possible only if there exists two nonnegative scalars m_1, m_2 such that v is more diffident around w than $u_3 = m_1u_1 + m_2u_2$.

5.3 Multi-antecedent multi-reference Diffidence

Of course, one can easily combine the two previous results to analyze problems with more than one antecedent and with an arbitrary reference point:

$$\forall \tilde{x}, \tilde{\omega} : \quad Ef_\theta(\tilde{x}) \leq Ef_\theta(\tilde{\omega}_\theta) \ \forall \theta \in \Theta \implies Eg(\tilde{x}) \leq Eg(\tilde{\omega}). \tag{69}$$

This is possible only if there exists a nonnegative function $m : \Theta \to R$ such that

$$g(x) = \int_\Theta m(\theta) f_\theta(x) d\theta. \tag{70}$$

Jewitt [1987] obtained this condition. He applied it to the Ross' concept of "more risk aversion". A simpler illustration is to determine under which condition agent v dislikes any change in risk that agents both u_1 and u_2 dislike. From the discussion above, this is possible iff v is a convex combination of u_1 and u_2. More generally, if an agent v dislikes any change in risk that any agent with constant absolute risk aversion dislikes, it must be the case that v is completely monotone, i.e. all odd derivatives of v must be positive and all even derivatives of v must be negative: $v(w) = -\int_0^{+\infty} m(\theta) e^{-\theta w} d\theta$.

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APPENDIX

Proof of Proposition 3

The two necessary conditions are simple rewritings of conditions (46) and (47). The sufficiency of $u_2''(w) - \frac{u_2'(w)}{u_1'(w)} u_1''(w)$ increasing is obtained in the following way. Take any positive scalar ξ. The above condition means that

$$u_2''(w+\xi) - \frac{u_2'(w+\xi)}{u_1'(w+\xi)} u_1''(w+\xi) - u_2''(w) + \frac{u_2'(w)}{u_1'(w)} u_1''(w) \geq 0. \tag{71}$$

Dividing by $u_2'(w)$ and integrating the corresponding condition on $[0, x]$, $x > 0$, yields

$$\int_0^x \frac{u_2''(w+\xi)}{u_2'(w)} - \frac{u_1''(w+\xi)}{u_1'(w)} \frac{\frac{u_2'(w+\xi)}{u_2'(w)}}{\frac{u_1'(w+\xi)}{u_1'(w)}} d\xi \geq x \left[\frac{u_2''(w)}{u_2'(w)} - \frac{u_1''(w)}{u_1'(w)} \right].$$

The same inequality holds for $x < 0$ since the change of the direction of inequality (71) is compensated by the direction of the integration. So, for any x, we obtain

$$\int_0^x \frac{u_2''(w+\xi)}{u_2'(w)} - \frac{u_1''(w+\xi)}{u_1'(w)} d\xi + g(x) \geq x \left[\frac{u_2''(w)}{u_2'(w)} - \frac{u_1''(w)}{u_1'(w)} \right],$$

or, equivalently,

$$\left[\frac{u_2'(w+x)}{u_2'(w)} - \frac{u_1'(w+x)}{u_1'(w)} \right] + g(x) \geq x \left[\frac{u_2''(w)}{u_2'(w)} - \frac{u_1''(w)}{u_1'(w)} \right], \tag{72}$$

where

$$g(x) = \int_0^x \frac{u_1''(w+\xi)}{u_1'(w)} \left[\frac{\frac{u_1'(w+\xi)}{u_1'(w)} - \frac{u_2'(w+\xi)}{u_2'(w)}}{\frac{u_1'(w+\xi)}{u_1'(w)}} \right] d\xi.$$

Because u_2 is more risk-averse than u_1, u_2 is centrally more risk-averse than u_1 around w. It implies that the bracketed term in the integrant of g has the same sign as x. It implies that $g(x)$ is always negative. Combining this fact with condition (72) yields the sufficient condition (45).

The sufficiency of condition (ii) comes from the observation that:

$$\frac{1}{u_2'}\frac{d}{dw}\left[u_2'' - \frac{u_2'}{u_1'}u_1''\right] = [A_2(P_2 - 0.5A_2) - A_1(P_1 - 0.5A_1)] + 0.5[A_2 - A_1]^2.$$

If (ii) is satisfied, then (i) is satisfied. ■

Proof of Theorem 3

Consider the set of Lebesgue-integrable real functions in $L_2[a,b]$, with the associated product scalar $< f, g >= \int fg$. If h is a density function for $\tilde{x}$, then, $< f, h >$ represents $Ef(\tilde{x})$.

Consider any subset $D \subset L_2[a,b]$ and define :

- $\hat{D} = \{G \in L_2 | \forall h \in L_2 \;\; : < G, h > \leq 0$ whenever $< F, h > \leq 0 \quad \forall f \in D\}$
- $C(D) =$ the smallest closed convex cone containing D.

To prove Theorem 3, we need first to prove the following Lemma, which is also used by Jewitt [1987].

Lemma : $\hat{D} = C(D)$ **Proof :**

1. $C(D) \subset \hat{D}$ is obvious.
2. $\hat{D} \subset C(D)$: by contradiction, suppose that $\exists \bar{G} \in \hat{D}$, but $\bar{G} \notin C(D)$. Since $C(D)$ is convex, one may apply the standard separation theorem : $\exists m \in L_2[a,b] \;\; : < \bar{G}, m > \geq 0$ and $< F, m > \leq 0 \; \forall F \in C(D)$. It implies that $\bar{G} \notin \hat{D}$, a contradiction.

■

We are now in a position to prove Theorem 3. Take $D = \{F_\theta = f_\theta - Ef_\theta(\tilde{\omega}_\theta) | \theta \in \Theta\} \cup D_-$, with $D_- = \{k | k(x) \leq 0 \;\; \forall x\}$, the set of non positive functions. Thus,

$$\hat{D} = \{G | \forall h \in L_2 \;\; : < G, h > \leq 0 \text{ whenever}$$

$$< F_\theta, h > \leq 0 \; \forall \theta \in \Theta \text{ and } < k, h > \leq 0 \;\; \forall k \in D_-\}$$

We introduced the set of conditions $< k, h > \leq 0 \;\; \forall k \in D_-$ to ensure that h may be interpreted as a density function. Thus, one can interpret $\hat{D}$ as

$$\hat{D} = \{G | EG(\tilde{x}) \leq 0 \text{ whenever } EF_\theta(\tilde{x}) \leq 0 \; \forall \theta \in \Theta\}$$

Applying the lemma yields that $G \in \hat{D}$ if and only if there exist a function $k \in D_-$ and a nonnegative function m, such that

$$G(x) \leq \int_\Theta m(\theta) F_\theta(x) d\theta + k(x)$$

for all x. Define $G = g - Eg(\tilde{\omega})$. The above condition is thus equivalent to

$$g(x) - Eg(\tilde{\omega}) \leq \int_\Theta m(\theta)[f_\theta(x) - Ef_\theta(\tilde{\omega}_\theta)] d\theta + k(x)$$

for all x. This is equivalent to (66), since $k(x) \leq 0$. ■