New Methods in the Classical Economics of Uncertainty : Comparing Risks [1]

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Abstract

Second-order stochastic dominance answers the question "Under what conditions will all risk-averse agents prefer $\tilde{x}_2$ to $\tilde{x}_1$?" Consider the following related question: "Under what conditions will all risk-averse agents who prefer lottery $\tilde{x}_1$ to a reference lottery $\tilde{\omega}$ also prefer lottery $\tilde{x}_2$ to that reference lottery?" Each of these two questions is an example of a broad category of questions of great relevance for the economics of risk. The second question is an example of a *contingent* risk comparison, while the question behind second-order stochastic dominance is an example of a *noncontingent* risk comparison. The stochastic order arising from a contingent risk comparison is obviously weaker than that arising from the corresponding noncontingent risk comparison, but we show that the two stochastic orders are closely related, so that the answer to a noncontingent risk comparison problem always provides the answer to the corresponding contingent risk comparison problem.

In addition to showing the connection between parallel contingent and noncontingent risk comparison problems, we articulate a method for solving both kinds of problems using the "basis" approach. The basis approach has often been used implicitly, but we argue that there is value to making its use explicit, particularly in indicating which new, previously unsolved problems can readily be solved by the basis approach and which cannot.

1 Introduction

In even its classical, von-Neumann-Morgenstern formulation, let alone its more recent extensions into non-expected-utility theories, the economics of uncertainty encompasses a large number of seemingly disparate results. Because the economics of uncertainty is a technically difficult field, and its results are proved in a wide variety of ways, a substantial effort has been required to learn what is behind all of its major results. New results have been discovered and worked out in a way analogous to hand-crafting by a skilled artisan.

We propose a long-term project, of which this paper is a bare beginning, aimed at systematizing the economics of uncertainty. Systematizing involves clarifying connections among results and categories of results, organizing what is known into arrays that reveal by conspicuous gaps what is yet unknown, and standardizing methods of discovery and proof. Perfect order is neither attainable nor desirable, but organization and consolidation along one dimension can foster creativity along other dimensions.

This paper is the attempt to systematize the comparison of risks—especially results related to stochastic dominance. The problem of stochastic dominance is a standard one in the literature of statistics, operations research and economics. In economics, it was first examined by Hadar and Russel [1969], Hanoch and Levy [1969] and Rothschild and Stiglitz [1970,1971]. In many instances, making an appropriate ordering of uncertain prospects boils down to the problem of finding the restriction on the random variables $\tilde{x}_1$ and $\tilde{x}_2$ such that

$$Ef(\tilde{x}_2) \leq Ef(\tilde{x}_1) \qquad \text{(noncontingent comparison)} \tag{1}$$

for any function f belonging to some set P, where E denotes the expectation operator. For example, if P is the set of increasing and concave functions, we get the well-known second-order stochastic dominance order (SSD). If f represents a utility function, this order is useful to determine whether a change in risk is unanimously rejected in the population of risk-averse individuals. If f represents marginal utility, this order is relevant to determine whether risk-averse and prudent agents unanimously increase their precautionary saving in the face of a change in future income risks. Many other sets of functions P have been considered in the literature, leading to other stochastic orders such as first-order (FSD) or third-order (TSD) stochastic dominance.

In this paper, we investigate another type of stochastic dominance orders. Rather than considering the problem expressed by equation (1), we consider the following problem: under which condition on $\tilde{x}_1$ and $\tilde{x}_2$ can we guarantee that

$$Ef(\tilde{x}_1) \leq Ef(\tilde{\omega}) \quad \Longrightarrow \quad Ef(\tilde{x}_2) \leq Ef(\tilde{\omega}) \qquad \text{(contingent comparison)} \tag{2}$$

for a given reference lottery $\tilde{\omega}$ and for any f belonging to some set P. We include in this type all other problems in which the first and/or second inequality is replaced by an equality. Contingent risk comparisons have many useful applications. To start with, consider the contingent problem parallel to SSD. Instead of requiring unanimity of preferences for $\tilde{x}_1$ over $\tilde{x}_2$ in the whole population of risk-averters (a noncontingent risk comparison), we require that all those who prefer $\tilde{x}_1$ to the reference lottery $\tilde{\omega}$ prefer $\tilde{x}_2$ to $\tilde{\omega}$ (a contingent risk comparison). If $\tilde{x}_2$ is SSD-dominated by $\tilde{x}_1$, such a property is automatically satisfied. But it is easy to find a change in distribution that satisfies the latter property without being SSD. Thus, the contingent counterpart to SSD is weaker than SSD itself.

Contingent risk comparison problems also have applications in marketing, industrial organization and political economy. For example, consider the problem of a manager of an insurance company who would like to change the offered contract without losing customers. In this example, $\tilde{\omega}$ is the random wealth of an agent who does not purchase the contract, $\tilde{x}_1$ is the random wealth with the initial contract (possibly with partial insurance and an unfair premium), $\tilde{x}_2$ is the new contract and f is minus the utility function. Given risk aversion, does $\tilde{x}_2$ need to be an SSD-improvement over $\tilde{x}_1$? Not necessarily, because the purchase of the initial contract allows him to extract some information about the risk preferences of his customers. For example, if the initial policy has a premium much larger than the actuarial value, only highly risk-averse agents will have purchased it. The insurance company can design the new contract by using that information. The set of the company's customers is not a representative sample of the population of risk-averse agents. The contingent parallels to first- and third-order stochastic dominance can be interpreted in much the same way.

There are many other examples in which a signal $(Ef(\tilde{x}_1) \leq Ef(\tilde{\omega}))$

can be used to infer the risk preferences of another agent. In industrial organization, there is the problem of choosing a pricing and production policy (in the face of random demand) to keep a potential competitor out of one's market, given only the information that the potential competitor rejected a particular risk in the past. In political economy, it could be quite important for a political entrepreneur to know which other proposals the electorate will reject if a majority of voters have been observed to vote against a particular risky policy. Any proposal that is dominated by the rejected policy *in a contingent sense* will be unappealing to at least that same (majority) group of voters, if not unappealing to an even larger majority.

Other illustrations of contingent risk comparison problems can be found in comparative statics analysis. Consider for example the problem of selecting the scalar α_1 that maximizes $H(\alpha) = Eu(z_0 + \alpha\tilde{x}_1)$. This is the standard portfolio problem where $\tilde{x}_1$ is the return of the risky asset and α is the demand for it. One can look for the condition under which all risk-averse investors reduce their optimal exposure to risk in the face of a change in risk from $\tilde{x}_1$ to $\tilde{x}_2$. Since H is a concave function, the problem simplifies to determining the condition under which

$$E\tilde{x}_1 u'(z_0 + \alpha_1\tilde{x}_1) = 0 \quad \Longrightarrow \quad E\tilde{x}_2 u'(z_0 + \alpha_1\tilde{x}_2) \leq 0, \qquad (3)$$

for any concave u. Eeckhoudt and Hansen [1980], Meyer and Ormiston [1985], Black and Bulkley [1989] and Dionne, Eeckhoudt and Gollier [1993] obtained sufficient conditions to this problem.

In this paper, we develop a general technique to solve all contingent risk comparison problems of the type given in (2) in terms of the solution to the corresponding noncontingent risk comparison problem of the type given in (1). We also spell out the technique that allows one to solve many of the most important *noncontingent* risk comparison problems—the basis approach. An important feature of expected utility is the linearity of the expectation operator. As a matter of fact, if functions f_1 and f_2 satisfy condition (1), so does $\lambda f_1 + (1-\lambda)f_2$ for any λ in $[0,1]$. The same property holds for contingent problems. Hence, if a simple basis can be found for the set of functions P under consideration, the problem can be much simplified. Such a basis exists for increasing functions, or for increasing and concave functions.

In section 2, we derive the general necessary and sufficient condition for contingent risk comparison problems and relate this condition to the parallel condition for noncontingent risk comparison problems. In section 3, we systematize the basis approach for noncontingent risk comparison problems. We show in Section 4 how our results can be applied to solve problems studied by Jewitt [1987, 1989] and Athey [1996].

2 Contingent Risk Comparison Problems

Let $F[a,b]$ denote the set of functions with domain in $[a,b]$. Consider a specific convex set C of functions in $F[a,b]$. We say that $\tilde{x}_2$ is C-dominated by $\tilde{x}_1$, iff

$$Ef(\tilde{x}_2) \leq Ef(\tilde{x}_1) \text{ for any } f \in C. \tag{4}$$

As mentioned in the introduction, to every noncontingent stochastic order, there exists an associated family of contingent stochastic orders indexed by a reference situation $\tilde{\omega}$. In many applications, $\tilde{\omega}$ will be degenerate. We say that $\tilde{x}_2$ is $C\tilde{\omega}$-dominated by $\tilde{x}_1$, viz. $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1$,[1] iff

$$Ef(\tilde{x}_1) \leq Ef(\tilde{\omega}) \quad \Longrightarrow \quad Ef(\tilde{x}_2) \leq Ef(\tilde{\omega}) \qquad \forall f \in C. \tag{5}$$

It is obvious that $\tilde{x}_2$ is $C\tilde{\omega}$-dominated by $\tilde{x}_1$ if $\tilde{x}_2$ is C-dominated by $\tilde{x}_1$. In words, the contingent stochastic order is weaker than the parallel noncontingent stochastic order, independent of the reference situation $\tilde{\omega}$.

To escape triviality, we hereafter assume that there is at least one function f in C such that the condition $Ef(\tilde{x}_1) \leq Ef(\tilde{\omega})$ is satisfied. The main tool for analyzing when condition (5) holds is presented in following Proposition, which is a rewriting of Farkas' Lemma. Farkas' Lemma has been very useful in game theory, operations research and general equilibrium analysis. The authors are not aware of the existence of any use of this mathematical tool in the economic theory of uncertainty, except in Jewitt (1986).

Proposition 1 *Take any set $C \in F[a,b]$ and any reference situation $\tilde{\omega}$. Condition (5) holds for any $f \in C$ if and only if there exists a positive scalar m such that*

$$Ef(\tilde{x}_2) - Ef(\tilde{\omega}) \leq m[Ef(\tilde{x}_1) - Ef(\tilde{\omega})] \;\forall f \in C. \tag{6}$$

[1] This notation is consistent with the fact that $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1 \quad \forall\tilde{\omega} \Longrightarrow \tilde{x}_2 \preceq_C \tilde{x}_1$.

The sufficiency of (6) is trivial. Necessity is more complex. In section 3, we will present an informal way of proving necessity that relies on the basis approach.

One key contribution of Rothschild and Stiglitz [1970] is to associate with the SSD-integral condition a generating process that transforms the initial distribution into the SSD-dominated one—the concept of a mean-preserving spread. Can we find such a generating process for *contingent* stochastic orders? To answer this question, suppose first that m is in $[0,1]$. In that case, condition (6) can be rewritten as:

$$Ef(\tilde{x}_2) \leq mEf(\tilde{x}_1) + (1-m)Ef(\tilde{\omega}) \;\forall f \in C. \tag{7}$$

Consider the function $\hat{G} = mG_1 + (1-m)W$, where W is the cumulative distribution function of $\tilde{\omega}$. $\hat{G}$ is the cumulative distribution function for random variable $\tilde{z} = (m, \tilde{x}_1; 1-m, \tilde{\omega})$. Condition (7) means that $\tilde{x}_2 \preceq_C \tilde{z}$. If m is larger than 1, condition (6) is equivalent to

$$\frac{1}{m}Ef(\tilde{x}_2) + (1-\frac{1}{m})Ef(\tilde{\omega}) \leq Ef(\tilde{x}_1) \;\forall f \in C. \tag{8}$$

Condition (8) means that $\tilde{y} \preceq_C \tilde{x}_1$, where $\tilde{y}$ is distributed as $(\frac{1}{m}, \tilde{x}_2; 1-\frac{1}{m}, \tilde{\omega})$. We summarize our findings for characterizing contingent stochastic orders in the following Proposition.

Proposition 2 *Take any convex set $C \in F[a,b]$ and any reference situation $\tilde{\omega}$. $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1$ if and only if one of the following two conditions is satisfied:*

(a) *there exists $p \in [0,1]$ such that $\tilde{x}_2 \preceq_C \tilde{z}$, with $\tilde{z}$ distributed as $(p, \tilde{x}_1; 1-p, \tilde{\omega})$;*

(b) *there exists $p \in [0,1]$ such that $\tilde{y} \preceq_C \tilde{x}_1$, with $\tilde{y}$ distributed as $(p, \tilde{x}_2; 1-p, \tilde{\omega})$.*

This result provides a generating process for any contingent stochastic order of the type given in (2). Namely, it is obtained by combining three operations: first, replace the initial distribution by a mixture of it and the reference lottery (that is, construct a gamble over the gambles $\tilde{x}_1$ and $\tilde{\omega}$). Second, do any change to this mixture that is compatible with the generating process of the parallel noncontingent problem. Third, if the obtained distribution is a compound lottery with $\tilde{\omega}$ and something else, remove a portion of the $\tilde{\omega}$ component from it.

For example, take $\tilde{x}_1$ distributed as $(-2, 1/2; 4, 1/2)$ and $\tilde{x}_2$ distributed as $(-2, 1/3; -1, 1/6; 1, 1/6; 4, 1/3)$. The reference lottery is degenerate at 0. Consider the contingent dominance for the set of increasing and concave function, i.e. the contingent dominance order parallel to SSD. It appears that $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1$. Indeed, $\tilde{x}_2$ is obtained by first gambling between $\tilde{x}_1$ and $\tilde{\omega}$, with a probability 1/3 on $\tilde{\omega}$. Then, the occurrence of $\tilde{\omega}$ is replaced by the gamble $(-1, 1/2; 1, 1/2)$ which is a mean-preserving spread of $\tilde{\omega} \equiv 0$. Notice that this is an example in which not all risk-averse agents prefer $\tilde{x}_1$ to $\tilde{x}_2$, but all those who dislike $\tilde{x}_1$ (i.e. who prefer $\tilde{\omega} \equiv 0$ to $\tilde{x}_1$), also dislike $\tilde{x}_2$.

Other forms of contingent stochastic dominance could be considered. For example, if both inequalities in (2) are reversed, the integral condition is condition (6) with the inequality reversed. If we consider the condition under which

$$Ef(\tilde{x}_1) = Ef(\tilde{\omega}) \quad \Longrightarrow \quad Ef(\tilde{x}_2) \leq Ef(\tilde{\omega}), \tag{9}$$

the necessary and sufficient condition is again condition (6) with m being free to be negative or positive. So in addition to the three manipulations of the original probability distribution that have been presented above for the case $m > 0$, one can also make changes such that $\tilde{x} = (p, \tilde{x}_1; 1-p, \tilde{x}_2)$ is C-dominated by $\tilde{\omega}$. This is due to the fact that, when m is negative, condition (6) can be rewritten

$$\frac{1}{1+\mid m \mid} Ef(\tilde{x}_2) + \frac{\mid m \mid}{1+\mid m \mid} Ef(\tilde{x}_1) \leq Ef(\tilde{\omega}). \tag{10}$$

To sum up, we have been able in this section to transform the contingent risk comparison problem into a more standard noncontingent risk comparison problem where $\tilde{x}_1$, $\tilde{x}_2$ and the reference lottery $\tilde{\omega}$ are compounded before being compared.

3 The Basis Approach for Comparing Risks

We have seen in the previous section that a full characterization of a contingent stochastic order can be obtained by a standardized method if the corresponding noncontingent stochastic order is itself characterized. In this section, we will survey from a unified perspective existing results about noncontingent risk comparisons—each of which implies a parallel result for *contingent risk comparisons.*

Consider a subset S of C such that $C = conv(S)$, i.e. C can be expressed as convex combinations of elements in S. Elements in S are extreme elements of the convex set C.[2] We hereafter call S a "basis" for set C.

As is well-known[3], since stochastic orders are additive in the function f, a necessary and sufficient condition for (4) to hold is

$$Ef(\tilde{x}_2) \leq Ef(\tilde{x}_1) \qquad \forall f \in S. \tag{11}$$

To verify that condition (4) holds for any f in the large set C, it is sufficient to verify that this condition holds for any function in the generating basis S. This simple property is implicitly or explicitly used in all proofs of stochastic dominance integral properties that exist in the literature. Athey [1995] extends this idea to other stochastic optimization problems.

It is noteworthy that the necessity of Proposition 1 can be obtained by using the basis approach for comparing risks. Indeed, we obtained the following Proposition.

Proposition 3 *Suppose that S is a basis for the convex set C. Condition (5) holds for any function in C if and only if it holds for any convex combination of any two functions in S.*

Proof: Define $S \equiv \{f_\theta(.) \mid \theta \in \Theta\}$. Thus, any $f \in C$ can be represented in set S by its transform H such that $f(x) = \int_\Theta f_\theta(x) dH(\theta)$ for all x. Note that $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1$ is equivalent to the negativity of the following program:

$$\max_{H(.)\mid\ dH \geq 0,\ \int_\Theta dH(\theta)=1} \quad \int_\Theta [E[f_\theta(\tilde{x}_2)] - w(\theta)] dH(\theta)$$

$$\text{s.t.} \quad \int_\Theta [E[f_\theta(\tilde{x}_1)] - w(\theta)] dH(\theta) \leq 0,$$

with $w(\theta) = Ef_\theta(\tilde{\omega})$. Because the set of H that satisfy the constraint is bounded and nonempty, a bounded solution exists to this problem. Both

[2] An extreme element of a convex set is an element that cannot be obtained by a convex combination of other elements in C.

[3] See for example Vickson [1975].

the objective function and the constraint are linear in dH. Thus, this is a linear programming problem on the unit simplex. The maximum value of the objective must be achieved by one of the two types of solution: (A) a one-step function H with all the weight on one value of θ that satisfies the constraint with strict inequality, or (B) a two-step function H that satisfies the constraint with equality. ■

We show in the Appendix that condition (6) is precisely equivalent to requiring that condition (5) holds for any convex combination of any couple in the basis. Thus, if condition $\tilde{x}_2 \preceq_{C\tilde{\omega}} \tilde{x}_1$ is violated, there always exists a pair (f_1, f_2) of functions in S and a weighting λ such that condition (2) is violated by $\lambda f_1 + (1-\lambda) f_2$. In other words, in order to verify dominance of a contingent type, it is sufficient to look at all functions in C that are the combination of only two functions in the basis. For comparison, to verify dominance of a noncontingent type it is sufficient to look at all individual functions in the basis. Thus, it is more complex to check a contingent stochastic property than the corresponding noncontingent stochastic property. This additional complexity is kept to a minimum by using Proposition 2, which replaces the contingent comparison by an noncontingent one.

The smaller or more special S is, the more significant the reduction in the complexity of the verification procedure (11) for noncontingent comparisons. In some cases, the minimal basis is simple, while in other cases it is complex. Consider the following geometric analogy. The extreme points of a cube are a set of 8 points, whereas the extreme points of a sphere are the entire (surface of the) sphere. This distinction between simple and complex minimal bases will be an important theme in our brief survey of noncontingent stochastic orders. Let us begin with the well-known nth-degree stochastic dominance orders.

3.1 Nth-degree stochastic dominance orders

Consider first the convex set of all increasing utility functions that leads to first-order stochastic dominance. As is well-known, the basis for such a set is the set of nondecreasing step functions: $f_\theta(x)$ equals 0 if x is less than θ and 1 otherwise. Condition (11) applied to step functions leads to the familiar condition for FSD: $\int_\theta^b dG_2(x) \leq \int_\theta^b dG_1(x)$, or $G_2(\theta) \geq G_1(\theta)$ for all θ in $[a, b]$, with G_i denoting the cumulative distribution function of $\tilde{x}_i$.

In the same vein, the convex set of increasing and concave functions has the set of "angles" – or "min" functions – as a simple basis: $b_\theta(x) = \min(x-\theta, 0)$.[4] The necessary and sufficient condition here becomes

$$\int_a^\theta (x-\theta)dG_2(x) \leq \int_a^\theta (x-\theta)dG_1(x),$$

Integrating by parts, this is equivalent to the well-known SSD integral condition

$$\int_a^\theta (G_2(x) - G_1(x))dx \geq 0. \tag{12}$$

Notice that because an increasing and concave function is characterized by a (positive) decreasing derivative, the basis for increasing concave functions is obtained by integrating the functions of the basis for positive decreasing functions, i.e. the positive decreasing step functions. The same integration technique can be used to build the basis for functions with $f' \geq 0$, $f'' \leq 0$ and $f''' \geq 0$. It yields the integral condition for third-order stochastic dominance.

3.2 The stochastic order associated to the set of completely monotone utility functions

Pratt and Zeckhauser [1987] and Caballe and Pomansky [1996] consider the set of "completely monotone" utility functions that are functions with all positive even derivatives and all negative odd derivatives. Alternatively, the set of completely monotone functions is the set of Laplace transform of probability distributions (Feller [1971], p. 439). Thus, the basis for this set is the set of (CARA) exponential functions. This set contains most familiar utility functions, like exponential functions, power functions, logarithmic functions and all combinations of them. Since the set of exponential utility functions is the simple basis for completely monotone utility functions, all "completely monotone" agents prefer $\tilde{x}_1$ to $\tilde{x}_2$ iff all exponential agents do. If we limit the analysis to mean-preserving changes in distribution, integrating this condition by parts twice yields

[4] In a totally different framework, Leland [1980] uses the parallel property that "max" functions form a basis for convex functions, showing that any convex (portfolio insurance) payoff function for an investor can be achieved by purchasing put options in a complete set of option markets.

$$\int_a^b e^{-\theta t} \left[\int_a^t (G_2(x) - G_1(x)) dx \right] dt \geq 0 \qquad \forall \theta \geq 0, \tag{13}$$

which is a weaker condition than the SSD condition (12). It was discovered independently by Caballe and Pomansky [1996].

3.3 The stochastic order associated with decreasing absolute risk aversion

Another set of utility functions that is often used is the convex set of increasing and concave utility functions with nonincreasing absolute risk aversion (DARA).[5] Despite the fact that DARA is a widely accepted assumption, there has been no satisfactory analysis of the stochastic dominance order for this set of functions. Vickson [1975] provided a dynamic programming algorithm to check whether two random variables can be ordered according to this stochastic dominance property. The difficulty is that the basis approach does not help with the set of DARA utility functions since there is no simple basis for this set. In the proof of Proposition 2, we show that the minimal basis is the set of piecewise CARA functions.

Proposition 4 *The basis for the set of functions exhibiting nonincreasing absolute risk aversion is dense in this set, i.e. for any DARA function there is a function in the basis that is arbitrarily pointwise close to it. The same property holds for functions with decreasing absolute prudence.*

Proof: As in Pratt [1964], let us consider three utility functions u_1, u_2 and $u = u_1 + u_2$. Let $r_i(x) = -u_i''(x)/u_i'(x)$ be the index of absolute risk aversion of u_i, and

$$r(x) = \frac{-u''(x)}{u'(x)} = \frac{u_1'(x)}{u_1'(x) + u_2'(x)} r_1(x) + \frac{u_2'(x)}{u_1'(x) + u_2'(x)} r_2(x). \tag{14}$$

Then

[5] This set contains completely monotone functions as a subset. The corresponding stochastic order is thus stronger than (13), but weaker than (12).

$$r'(x) = \frac{u_1'(x)}{u_1'(x)+u_2'(x)} r_1'(x) + \frac{u_2'(x)}{u_1'(x)+u_2'(x)} r_2'(x) + \frac{-u_1'(x)u_2'(x)}{u_1'(x)+u_2'(x)} (r_1(x) - r_2(x))^2. \tag{15}$$

Assuming that u_1 and u_2 exhibit DARA, the left-hand side of this equation is the sum of three negative terms, so u is DARA. This proves the convexity of the set of DARA functions. This equation also shows that a function u which exhibits constant absolute risk aversion locally at x cannot be obtained by combining DARA functions with different levels of absolute risk aversion. Therefore, any piecewise exponential function with a decreasing stepwise absolute risk aversion profile is an extreme element—i.e., it must be in the basis. Any DARA function can be approximated by a sequence of such piecewise exponential functions, so proving something for all DARA piecewise exponential functions is not much easier than proving it directly for all DARA utility functions. ■

3.4 The stochastic order with respect to a function

Meyer [1977] examined the stochastic order for the set of increasing utility functions u which are more concave than a given utility function u_1. Specifically, take

$$C \equiv \{u : [0,1[\rightarrow \Re \ \mid \ -\frac{u''(x)}{u'(x)} \geq -\frac{u_1''(x)}{u_1'(x)} \ \forall x\}.$$

It is easy to check that P is convex. Define

$$S \equiv \{f_\theta : [0,1[\rightarrow \Re \ \mid \ f_\theta(x) = \min(u_1(x), u_1(\theta)) \ \forall x, \theta\} \in: [0,1[\}.$$

It is easy to check that S is a basis for the set C of utility functions that are more concave than u_1. This immediately yields the following condition for the stochastic order associated with the function u_1: (Meyer [1977], Proposition 3)

$$\int^\theta u_1(x)dG_2(x) + u_1(\theta)(1 - G_2(\theta)) \leq \int^\theta u_1(x)dG_1(x) + u_1(\theta)(1 - G_1(\theta)), \tag{16}$$

or, after integrating by parts,

$$\int^{\theta}(G_2(x)-G_1(x))u_1'(x)dx \geq 0,\ \forall\theta. \tag{17}$$

When u_1 is the identity function, we get back the first-degree stochastic dominance order.

Meyer [1977] obtained a symmetric result for the set of utility functions that are less concave than u_1. Finally, he considered the set of utility functions that are more concave than u_1 and less concave than u_2. Like Vickson [1977], he provides a dynamic programming algorithm to check whether two random variables satisfy the corresponding stochastic order. Again, the outcome of Meyer's algorithm may be understood by realizing that the minimal basis for this convex set is the set of functions that *piecewise* have the same degree of concavity as either u_1 or u_2.

Proposition 5 *The basis for the set of functions which are more risk-averse than u_1 and less risk-averse than u_2 is dense in this set.*

Proof: The proof is parallel to the proof of Proposition 4. Using condition (14), it is apparent that any function that piecewise has the same degree of concavity as either u_1 or u_2 must be in the basis. Also any convex combination of such functions has a degree of risk aversion between $-\frac{u_1''}{u_1'}$ and $-\frac{-u_2''}{u_2'}$. Finally, any function that is in this convex set is arbitrarily pointwise close to a function in the basis. ■

4 Applications to Comparative Statics Under Uncertainty

A standard problem in the economics of uncertainty is the effect of a change in risk on the optimal exposure to it. Rothschild and Stiglitz [1971], Kraus [1979], Meyer and Ormiston [1983,1985], Black and Bulkley [1989] and Gollier [1995] analyzed the following problem:

$$\alpha_i \in \arg\max_{\alpha}\ H(\alpha) = Eu(z(\tilde{x}_i,\alpha))$$

where z is a payoff function that depends upon the realization of random variable $\tilde{x}_1$ and the decision variable α. The portfolio problem is a special case of this problem, with $z(x,\alpha) = z_0 + \alpha(x-r)$, $\tilde{x}_1$ the return of the risky asset, r the riskfree rate and α the demand for it. In many instances, α can be viewed as the level of exposure to risk $\tilde{x}_i$. One can look for the condition under which all risk-averse investors reduce their optimal exposure to risk in the face of a change in risk from $\tilde{x}_1$ to $\tilde{x}_2$. Assuming the concavity of the objective function and denoting the optimal solution under $\tilde{x}_1$ by α_1 with first-order condition $E\frac{\partial z}{\partial \alpha}(\tilde{x}_1,\alpha_1)u'(z(\tilde{x}_1,\alpha_1)) = 0$, the problem simplifies to determining the conditions under which

$$E\frac{\partial z}{\partial \alpha}(\tilde{x}_2,\alpha_1)u'(z(\tilde{x}_2,\alpha_1)) \leq 0$$

is negative, whenever α_1 satisfies the first-order condition

$$E\frac{\partial z}{\partial \alpha}(\tilde{x}_1,\alpha_1)u'(z(\tilde{x}_1,\alpha_1)) = 0.$$

This is a contingent risk comparison problem with $f_u(x) = \frac{\partial z}{\partial \alpha}(x,\alpha_1)u'(z(x,\alpha_1))$. In many applications, α_1 can be normalized to 1 because of the power of the universal quantifiers over $\tilde{x}_1$ and $\tilde{x}_2$. Eeckhoudt and Hansen [1980], Meyer and Ormiston [1985], Black and Bulkley [1989] and Dionne, Eeckhoudt and Gollier [1993] obtain sufficient conditions for this problem.

To solve the standard portfolio problem, Rothschild and Stiglitz [1971] looked for the necessary and sufficient condition for

$$E(\tilde{x}_1 - r)u'(z_0 + (\tilde{x}_1 - r)) = 0 \quad \Longrightarrow \quad E(\tilde{x}_2 - r)u'(z_0 + (\tilde{x}_2 - r)) \leq 0, \quad (18)$$

for any concave u. They ended up with the condition that

$$\int_a^\theta (x-r)dG_2(x) \leq \int_a^\theta (x-r)dG_1(x), \quad (19)$$

for any θ in $[a,b]$. But as it appears from the use of the basis technique, this is the necessary and sufficient condition for $E(\tilde{x}_2 - r)u'(z_0 + (\tilde{x}_2 - r)) \leq E(\tilde{x}_1 - r)u'(z_0 + (\tilde{x}_1 - r))$ for all concave u, i.e. this is the necessary and sufficient condition for the parallel noncontingent problem. Gollier [1995] provides some counterexamples in which condition (19) is violated, but condition (18) is satisfied. He did not explained the origin of the error made by Rothschild and Stiglitz. This is by now clear: they considered contingent and noncontingent problems equivalent. For a long time, all sufficient conditions to problem (18) that were proposed in the literature, like those by Eeckhoudt and Hansen [1980] and Meyer and Ormiston [1985], satisfied the overly restrictive condition (19). The first authors to provide a sufficient condition that violated condition (19) were Black and Bulkley [1989], followed by Dionne, Eeckhoudt and Gollier [1993]. But these authors did not realize the incompatibility of their sufficient condition with the Rothschild and Stiglitz condition (19) that Rothschild and Stiglitz claimed to be necessary.

Our results can be directly applied to this problem, where u is in the monotonic set, monotonic and concave set, DARA set, etc. This is left as an exercise for the reader.[6]

5 Patently Greater Risk

The technique presented in this paper provides a simple way to solve problems raised by Jewitt [1987,1989] and to make progress toward solving an important unsolved problem proposed by Kimball [1993]. Consider any two random variables $\tilde{x}_1$ and $\tilde{x}_2$, with cdf F_1 and F_2 respectively. Let G denote $F_2 - F_1$. Under what conditions can we guarantee that if an individual $u \in C$ prefers $\tilde{x}_1$ to $\tilde{x}_2$, so does any individual v who is more risk-averse than u? This problem is technically equivalent to determining whether, for any $u \in C$,

$$Eu(\tilde{x}_2) \leq Eu(\tilde{x}_1) \implies E\phi(u(\tilde{x}_2)) \leq E\phi(u(\tilde{x}_1)) \tag{20}$$

for any function ϕ which is increasing and concave. Jewitt [1987,1989] solves this characterization problem when C is the set of increasing functions, and when C is the set of increasing and concave functions. Kimball's definition of "patently greater risk" is equivalent to this property when C is the set of increasing, concave functions with decreasing absolute risk aversion. Although the lack of a simple basis for the set of increasing, concave utility functions

[6] Landsberger and Meilijson [1993] considered the problem of determining the stochastic dominance condition under which all risk-averse investors reduce their demand for the risky asset when its returns undergo a change in distribution, *independent of the riskfree rate.* Gollier [1997] recently obtained the necessary and sufficient condition by means of the basis approach advocated here.

with decreasing absolute risk aversion makes full characterization difficult, increasing, concave utility functions with decreasing absolute risk aversion are a *subset* of increase, concave, prudent functions (those with $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$) and a *superset* of completely monotonic utility functions. Therefore, the results for C equal to the set of *increasing, concave, prudent* utility functions provide *sufficient* conditions for patently greater risk, while the results for C equal to the set of *completely monotonic* utility functions provide *necessary* conditions for patently greater risk.

Despite the fact that this problem is not in the form (5), the basis technique can be used. Indeed, using angle functions as the basis for the increasing, concave function ϕ[7], Proposition 3 allows problem (20) to be rewritten as

$$\int_a^b u(t)dG(t) \leq 0 \implies \int_a^\theta [u(t) - u(\theta)]dG(t) \leq 0 \tag{21}$$

for any $\theta \in [a, b]$.

5.1 C is the set of increasing functions

Suppose first that C is the set of increasing functions. Using the same proof as for Proposition 3, condition (21) holds for any $u \in C$ if and only if it holds for any one-step utility function, i.e. for

$$u(t) = \begin{cases} 0, & \text{if } t < c; \\ z, & \text{if } c \leq t < d; \\ 1, & \text{if } d \leq t. \end{cases} \tag{22}$$

If c and d are on the same side with respect to θ, condition (21) is trivially satisfied. If $c \leq \theta \leq d$, this condition can be rewritten as

$$zG(c) + (1-z)G(d) \geq 0 \implies zG(c) \geq 0. \tag{23}$$

By contraposition, this is equivalent to

$$G(c) \leq 0 \implies zG(c) + (1-z)G(d) \leq 0. \tag{24}$$

Since z can be made arbitrarily close to zero, this is equivalent to the condition that

[7]Note that increasing linear transformations of ϕ are irrelevant.

$$G(c) \leq 0 \implies G(d) \leq 0, \tag{25}$$

for all $c \leq d$. This is the weak single-crossing condition obtained by Jewitt [1987] that has been refined by Athey [1996]: the preference ordering between F_1 and F_2 is preserved by more risk aversion if and only if F_2 crosses F_1 only once, from below. The "only if" means that if F_2 crosses F_1 more than once, then one can obtain an increasing utility function u and a concave transformation function ϕ such that F_1 is prefered to F_2 by u, but F_2 is prefered to F_1 by the more risk-averse $v = \phi(u)$. Athey [1996] makes this statement more precise by introducing the concept of an "identified pair of sufficient conditions relative to a conclusion". Notice that our technique of proof shows us how to build a counter-example. Just take a two-step utility function for u and a one-step utility function for v.

5.2 C is the set of increasing and concave functions

Suppose now that C is the set of increasing and concave functions. Integrating by parts, condition (21) can be rewritten as

$$\int_a^b u'(t)G(t)dt \geq 0 \implies \int_a^\theta u'(t)G(t)dt \geq 0 \tag{26}$$

for all $\theta \in [a,b]$. Using the same proof as for Proposition 3, condition (26) holds for any $u \in C$ if and only if it holds for any piecewise linear function with three segments, i.e. for

$$u'(z) = \begin{cases} u'_c, & \text{if } z \leq c; \\ u'_d, & \text{if } c < z \leq d; \\ 0, & \text{if } d > z. \end{cases} \tag{27}$$

with $u'_c \geq u'_d \geq 0$. Define function ψ as $\psi(z) = \int_a^z G(t)dt$. Again, if c and d are on the same side with respect to θ, condition (26) is trivially satisfied. If $c \leq \theta \leq d$, this condition can be rewritten as

$$\psi(c)u'_c + (\psi(d) - \psi(c))u'_d \geq 0 \implies \psi(c)u'_c + (\psi(\theta) - \psi(c))u'_d \geq 0, \tag{28}$$

or equivalently,

$$\psi(d) \geq -\frac{u'_c - u'_d}{u'_d}\psi(c) \implies \psi(\theta) \geq -\frac{u'_c - u'_d}{u'_d}\psi(c), \tag{29}$$

for any $c \leq \theta \leq d$, and any $u'_c \geq u'_d \geq 0$. Because $(u'_c - u'_d)/u'_d$ can be made as large as one wants, this condition is equivalent to the conditions that [a] ψ crosses the horizontal axe only once, from above, and that [b] ψ is non-increasing when it is negative. This is the result obtained by Jewitt [1989]. Notice that Jewitt uses a piecewise linear function with three segments to prove the necessity of the condition, as we do here. Our technique gives an intuition for why such an exotic concave function must be used.

6 Concluding remarks

In this paper, we have relaxed the standard stochastic order relations by requiring a preference condition to be satisfied only for a subset of the population that satisfies another preference condition. The technique we have developed is particularly useful in comparative statics analyses. We have shown that these new stochastic orders can easily be derived from the standard noncontingent ones by considering mixtures of distributions.

We have also explained why the literature on stochastic dominance is lacking when considering sets of utility functions other than those defined by the signs of the first n derivatives. It is because almost all of the results in this literature rely on the "basis approach". While simple bases exist for sets of utility functions defined by the signs of the first n derivatives, no simple basis exists for convex sets like the intuitively appealing set of decreasing absolute risk aversion (or the set of decreasing absolute prudence) utility functions.

Finally, notice that, using the basis approach, all contingent problems of the type given in (2) can be rewritten as

$$Ef(\tilde{x}_1, \tilde{\theta}_1) \leq Ef(\tilde{\omega}, \tilde{\theta}_1) \Longrightarrow Ef(\tilde{x}_2, \tilde{\theta}_1) \leq Ef(\tilde{\omega}, \tilde{\theta}_1), \tag{30}$$

for any distribution H of random variable $\tilde{\theta}_1$, with $f(.,\theta) \equiv f_\theta(.)$. A dual problem is the following:

$$Ef(\tilde{x}_1, \tilde{\theta}_1) \leq Ef(\tilde{\omega}, \tilde{\theta}_1) \Longrightarrow Ef(\tilde{x}_1, \tilde{\theta}_2) \leq Ef(\tilde{\omega}, \tilde{\theta}_2), \tag{31}$$

for any random variable $\tilde{x}_1$. This dual problem—comparing utility functions instead of random variables—is analyzed in a closely related paper (Kimball and Gollier [1995]).

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APPENDIX

Proof of Proposition 3

Take any f_1 and f_2 in S and denote f_{ij} for $E[f_i(\tilde{x}_j) - f_i(\tilde{\omega})]$. We would like to prove that condition (6) is equivalent to the fact that

$$\lambda f_{11} + (1-\lambda) f_{21} = 0 \tag{32}$$

implies

$$\lambda f_{12} + (1-\lambda) f_{22} \leq 0., \tag{33}$$

for any $\lambda \in [0,1]$ and any (f_1, f_2) in S^2.

Without loss of generality, we can assume that $f_{21} < 0 < f_{11}$.[8] Eliminating λ from (33) by using equality (32) makes the above equation equivalent to $f_{12}f_{21} \leq f_{22}f_{11}$, or $f_{12}/f_{11} \geq f_{22}/f_{21}$ for any $(\theta_1, \theta_2) \in \Theta^2$ such that $f_{21} < 0 < f_{11}$. This is equivalent to the property that

$$\min_{f_1 \in S | f_{11} > 0} \frac{f_{12}}{f_{11}} \quad \geq \quad \max_{f_2 \in S | f_{21} < 0} \frac{f_{22}}{f_{21}}. \tag{34}$$

It must be the case that the right-hand side of the above inequality be nonnegative (otherwise $\tilde{x}_2 \not\preceq_{C\tilde{\omega}} \tilde{x}_1$). It implies that there is a nonnegative m such that

$$\frac{f_{12}}{f_{11}} \quad \geq \quad m \quad \geq \quad \frac{f_{22}}{f_{21}}$$

for any f_1 and f_2 such that $f_{21} < 0 < f_{11}$. It yields

$$f_{12} \leq m f_{11} \text{ and } f_{22} \leq m f_{21}. \tag{35}$$

It means that

$$Ef(\tilde{x}_2) - Ef(\tilde{\omega}) \leq m[Ef(\tilde{x}_1) - Ef(\tilde{\omega})] \; \forall f \in S. \tag{36}$$

Using the basis approach, this is equivalent to requiring that

$$Ef(\tilde{x}_2) - Ef(\tilde{\omega}) \leq m[Ef(\tilde{x}_1) - Ef(\tilde{\omega})] \; \forall f \in C. \tag{37}$$

This is condition (6). Proposition 1 concludes the proof. ■

[8] The case $f_{21} = f_{11} = 0$ leads to the necessary conditions $f_{22} \leq 0$ and $f_{12} \leq 0$. This is the same necessary condition as (35).