Proof of Consumption Technology Neutrality

Miles Kimball
University of Michigan

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In this document I will try to back up the claim that multiplicative (Hicks-neutral) technology shocks to the production of nondurable consumption goods have very little effect on a perfectly competitive economy (more precisely, any economy that can be represented as solving a social planner's problem) beyond the direct effect of increasing the quantity of nondurable consumption and reducing the relative price of that consumption by an equal amount—which I will refer to as *rescaling consumption*.

The key assumption needed is that the utility function is of the King-Plosser-Rebelo type—nondurable consumption entering the utility function in a multiplicatively separable way—or, in the case where the elasticity of intertemporal subtitution for nondurable consumption is equal to one, entering in an additively separable way. The result for the logarithmic case of additive separability with intertemporal substitution equal to one is simpler in that the consumption technology shocks have *no* effect other than the rescaling of consumption. In the multiplicatively separable case, I will show that beyond the rescaling of consumption, the consumption technology shocks have an effect equivalent to the effect of multiplicative preference shocks, which effect the economy only to the extent that they are predictable. *Ex ante* expectations of future consumption technology shocks can have effects like variations in the utility discount rate, but consumption technology shocks have no other effect *ex post* except for the rescaling of consumption. In particular, if consumption technology shocks are *i.i.d.*, the only effect other than a rescaling of consumption is an effect equivalent to changing the utility discount rate by a constant.

The basic intuition for these results relies on the fact that the utility function is designed to match the stylized fact about trends that both consumption and the real wage are trending upward at about the same rate, while per capita labor hours has no large secular trend. The increase in consumption and in the real consumption wage that are part and parcel of the rescaling of consumption look qualitatively like the trend in consumption; therefore the household has no desire to change labor supply. Factor *demands* are unaffected because if the consumption technology gets twice as good, the same amount of capital and labor can produce twice as much. With the fall in the relative price of consumption, the marginal revenue products in terms of the investment good are the same as before.

Another way to state the intuition is that if a technology shock affects only consumption, the economy can immediately jump to the new steady state. By constrast, after a technology shock affecting the investment goods sector, the economy must accumulate capital over a long period to get to the new steady state. To show that this is the main thing at work, think of a small open economy in a world of perfectly mobile capital. On occasion by means of dramatic capital imports or exports, the small open economy could immediately jump to the new steady state in response to any kind of technology shock.[1] In a closed economy, any shock that has the power to change the steady state capital stock involves an adjustment process with many effects. Even the temporary version of such a shock must have many effects.

0.1 The Logarithmic Additively-Separable Case

Consider the following social planner's problem:

$$\max_{C,X} \; \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(C_t) + U(K_t, X_t, Z_t)] \middle| \, \Omega_0 \right\} \tag{1}$$

$$\text{subject to} \quad \begin{aligned} C_t &= A_t F(K_t, X_t, Z_t) \\ K_{t+1} &= H(K_t, X_t, Z_t) \\ C_t &\in \mathcal{R}^+ \\ (K_t, X_t) &\in \Psi(Z_t) \\ K_0 &= \bar{K}. \end{aligned}$$

C_t is consumption of nondurables and services or some component thereof. A_t is the (strictly positive) multiplicative level of the consumption goods producing technology. Ω_t is all the information available at time t. X_t is a vector of other control variables (besides C_t), such as the aggregate quantity of labor and the amounts of various factors used in each sector. K_t is a vector of state variables, such as the stocks of various types of capital (including sector specific capital), the stocks of various types of consumer durables, and variables depending on the past history of the other control variables in X, including the past history of labor, as in Kydland and Prescott (). Z_t is a vector of other exogenous variables (besides A_t), such as time t, technology shocks in the production of investment goods and consumer durables, the level and composition of government purchases, *etc.*

[1] Robert Barsky suggested this point about the small open economy.

"Now" is given the label of time zero. The function $\Lambda(Z_t)$ allows for discounting that may have an exogenous stochastic element. The function U gives the part of utility that does not involve C_t. In can depend on consumer durables, state variables that depend on past labor supply, *etc.*. The function F is the consumption goods producing technology when $A_t = 1$. It describes the substitution possibilities in producing consumption goods. The function H describes capital accumulation. The formulation is general enough to allow for investment adjustment costs or time-to-build. The set $\Psi(Z_t)$ delineates any restrictions on K_t and X_t. The vector $\bar{K}$ gives the initial values of all the state variables.

Proposition 1. If

$$\begin{aligned} C_t &= \Gamma^*(K_t, Z_t, \Omega_t) && (2) \\ X_t &= \mathcal{X}^*(K_t, Z_t, \Omega_t) && (3) \end{aligned}$$

and the implied evolution of the capital stock vector K_t solves the maximization problem

$$\max_{C,X} \; \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(C_t) + U(K_t, X_t, Z_t)] \,\middle|\, \Omega_0 \right\} \tag{4}$$

$$\text{subject to} \quad \begin{aligned} C_t &= F(K_t, X_t, Z_t) \\ K_{t+1} &= H(K_t, X_t, Z_t) \\ C_t &\in \mathcal{R}^+ \\ (K_t, X_t) &\in \Psi(Z_t) \\ K_0 &= \bar{K}. \end{aligned} \tag{5}$$

in which A_t has been replaced by 1 in 5, then

$$\begin{aligned} C_t &= A_t\Gamma^*(K_t, Z_t, \Omega_t) && (6) \\ X_t &= \mathcal{X}^*(K_t, Z_t, \Omega_t) && (7) \end{aligned}$$

and the same evolution of the capital stock vector K_t solves the original maximization problem 1.

(Note that Ω_t is assumed the same in both cases. Thus, to the extent that the history of A_t provides nonredundant information about other exogenous variables, one should continue to include the history of a purely informational duplicate of A_t as a part of Ω_t to make the theorem valid.)

Proof of Proposition 1. This is a special case of Boyd's () Symmetry Theorem. The proof works by showing (1) that each feasible program for one problem corresponds to a feasible program for the other problem and (2) that this correspondence maps a solution program into a solution program. The first step has to do with constraints, the second step with preferences.

To show the correspondence between feasible programs to one problem and feasible programs to another, I need to make sure that increasing consumption by same factor as the consumption technology leaves all of the constraints undisturbed. Clearly, if

$$\Gamma(K_t, Z_t) = F(K_t, \mathcal{X}_t(K_t, Z_t), Z_t)$$

then

$$A_t\Gamma(K_t, Z_t) = A_t F(K_t, \mathcal{X}_t(K_t, Z_t), Z_t),$$

positive consumption, multiplied by A_t, will still be positive, and the other constraints are unaffected because they involve neither A_t nor C_t. Thus, the same set of policy functions Γ and $\mathcal{X}$ are feasible in both cases.

To show that a solution to the simpler comparison problem maps into a solution to the original problem, define $S(\Gamma, \mathcal{X})$ as the value of the objective function in the comparison problem with A_t replace by 1 given the policy functions in 2 and 3. Define $J(\Gamma, \mathcal{X})$ as the value of the objective function in the original problem given policy functions as in 6 and 7. Suppose the policy functions Γ^* and $\mathcal{X}^*$ and the implied policy for the capital stock K_t^* maximize S—that is,

$$S(\Gamma^*, \mathcal{X}^*) \geq S(\Gamma, \mathcal{X})$$

for any other policy functions Γ and $\mathcal{X}$. We wish to prove that

$$J(\Gamma^*, \mathcal{X}^*) \geq J(\Gamma, \mathcal{X}),$$

as well. But we know that

$$
\begin{aligned}
J(\Gamma^*, \mathcal{X}^*) &= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(A_t \Gamma^*(K_t^*, Z_t)) + U(K_t^*, \mathcal{X}^*(K_t^*, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(\Gamma^*(K_t^*, Z_t)) + U(K_t^*, \mathcal{X}^*(K_t^*, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&\quad + \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t) \ln(A_t) \middle| \Omega_0 \right\} \\
&= S(\Gamma^*, \mathcal{X}^*) + \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t) \ln(A_t) \middle| \Omega_0 \right\} \\
&\geq S(\Gamma, \mathcal{X}) + \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t) \ln(A_t) \middle| \Omega_0 \right\} \\
&= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(\Gamma(K_t, Z_t)) + U(K_t, \mathcal{X}(K_t, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&\quad + \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t) \ln(A_t) \middle| \Omega_0 \right\} \\
&= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[\ln(A_t \Gamma(K_t, Z_t)) + U(K_t, \mathcal{X}(K_t, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= J(\Gamma, \mathcal{X}),
\end{aligned}
$$

where Γ and $\mathcal{X}$ are any other feasible policy and K_t follows the implied policy for the capital stock.

0.2 The Multiplicatively-Separable Case

For this case the social planner's problem is

$$
\max_{C,X} \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t) C_t^{1-\beta} U(K_t, X_t, Z_t)] \middle| \Omega_0 \right\} \tag{8}
$$

$$
\begin{aligned}
\text{subject to} \qquad C_t &= A_t F(K_t, X_t, Z_t) \\
K_{t+1} &= H(K_t, X_t, Z_t) \\
C_t &\in \mathcal{R}^+ \\
(K_t, X_t) &\in \Psi(Z_t) \\
K_0 &= \bar{K}.
\end{aligned}
$$

The most general result for this case is that, except for the rescaling of consumption, the optimal policies are the same for the problem 8 and the following problem, where A_t has no effect on the production function but enters in as an argument of a multiplicative preference shock:

$$\max_{C,X} \mathrm{E}\left\{ \sum_{t=0}^{\infty} \left(\frac{A_t}{A_0}\right)^{1-\beta} \Lambda(Z_t) C_t^{1-\beta} U(K_t, X_t, Z_t)] \middle| \Omega_0 \right\} \tag{9}$$

$$\begin{aligned} \text{subject to} \qquad C_t &= F(K_t, X_t, Z_t) \\ K_{t+1} &= H(K_t, X_t, Z_t) \\ C_t &\in \mathcal{R}^+ \\ (K_t, X_t) &\in \Psi(Z_t) \\ K_0 &= \bar{K}. \end{aligned}$$

More formally,

Proposition 2. If

$$C_t = \Gamma^*(K_t, Z_t, \Omega_t) \tag{10}$$
$$X_t = \mathcal{X}^*(K_t, Z_t, \Omega_t) \tag{11}$$

and the implied evolution of the capital stock vector K_t solves the maximization problem 9, then

$$C_t = A_t \Gamma^*(K_t, Z_t, \Omega_t) \tag{12}$$
$$X_t = \mathcal{X}^*(K_t, Z_t, \Omega_t) \tag{13}$$

and the same evolution of the capital stock vector K_t solves the maximization problem 8.

(Again, note that Ω_t is assumed the same in both cases. Thus, to the extent that the history of A_t provides nonredundant information about other exogenous variables, one should continue to include the history of a purely informational duplicate of A_t as a part of Ω_t to make the theorem valid.)

Proof of Proposition 2. The constraints for the original problem and the comparison problem are exactly the same as the corresponding optimization problems in the additively separable case. Therefore a feasible program for the reference problem again corresponds to a feasible program for the original problem.

To show that a solution to the comparison problem maps into a solution to the original problem, define $S(\Gamma, \mathcal{X})$ as the value of the objective function in the comparison problem 9 given policy functions as in 10 and 11. Define $J(\Gamma, \mathcal{X})$ as the value of the objective function in the original problem 8 given policy functions as in 12 and 13. Suppose the policy functions Γ^* and $\mathcal{X}^*$ and the implied policy for the capital stock K_t^* maximize S—that is,

$$S(\Gamma^*, \mathcal{X}^*) \geq S(\Gamma, \mathcal{X})$$

for any other policy functions Γ and $\mathcal{X}$. To see that

$$J(\Gamma^*, \mathcal{X}^*) \geq J(\Gamma, \mathcal{X}),$$

as well, consider that

$$\begin{aligned}
J(\Gamma^*, \mathcal{X}^*) &= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[A_t\Gamma^*(K_t^*, Z_t))]^{1-\beta} U(K_t^*, \mathcal{X}^*(K_t^*, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= A_0^{1-\beta}\mathrm{E}\left\{ \sum_{t=0}^{\infty} \left(\frac{A_t}{A_0}\right)^{1-\beta} \Lambda(Z_t)[\Gamma^*(K_t^*, Z_t))]^{1-\beta} U(K_t^*, \mathcal{X}^*(K_t^*, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= A_0^{1-\beta} S(\Gamma^*, \mathcal{X}^*) \\
&\geq A_0^{1-\beta} S(\Gamma, \mathcal{X}) \\
&= A_0^{1-\beta}\mathrm{E}\left\{ \sum_{t=0}^{\infty} \left(\frac{A_t}{A_0}\right)^{1-\beta} \Lambda(Z_t)[\Gamma(K_t, Z_t))]^{1-\beta} U(K_t, \mathcal{X}(K_t, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= \mathrm{E}\left\{ \sum_{t=0}^{\infty} \Lambda(Z_t)[A_t\Gamma(K_t, Z_t))]^{1-\beta} U(K_t, \mathcal{X}(K_t, Z_t), Z_t)] \middle| \Omega_0 \right\} \\
&= J(\Gamma, \mathcal{X}).
\end{aligned}$$

where Γ and $\mathcal{X}$ are any other feasible policy and K_t follows the implied policy for the capital stock.

Interpreting Proposition 2. In interpreting Proposition 2, it is good to remember that what applies to time zero applies to any time t. At any time t, only the ratio of the level of consumption technology in the future to the consumption technology has any effect other than rescaling consumption. Since a technology shock is defined as a change in the log-level of technology, this can be restated as only predicted future consumption technology shocks have any effect other than rescaling consumption. *Ex post*, the current consumption technology shock has only the rescaling effect, plus any purely informational role it plays.

In order to better understand the effect of expected future consumption technology shocks, consider the Bellman equation for the comparison problem in which the effects of A_t look like the effects of multiplicative preference shocks. The Bellman equation is

$$V(K_t, Z_t, \Omega_t) = \max_{C_t, X_t} \left[\Lambda(Z_t) C_t^{1-\beta} U(K_t, X_t, Z_t) + \mathrm{E} \left\{ \left(\frac{A_{t+1}}{A_t} \right)^{1-\beta} V(G(K_t, X_t, Z_t), Z_{t+1}, \Omega_{t+1}) \middle| \Omega_t \right\} \right], \tag{14}$$

where $V(K_t, \Omega_t)$ is the maximized objective function for 9 from time t on, given the vector of state variables K_t and the information Ω_t, with A_t divided out as A_0 is in the statement of the comparison problem. If, conditional on the information available at time t, the A_{t+1} is independent of Z_{t+1}, then

$$\begin{aligned} V(K_t, \Omega_t) &= \max_{C_t, X_t} \Big[\Lambda(Z_t) C_t^{1-\beta} U(K_t, X_t, Z_t) \\ &\quad + \mathrm{E} \left\{ \left(\frac{A_{t+1}}{A_t} \right)^{1-\beta} \middle| \Omega_t \right\} \mathrm{E} \left\{ V(G(K_t, X_t, Z_t), Z_{t+1}, \Omega_{t+1}) | \Omega_t \right\} \Big]. \end{aligned}$$

Thus, the anticipation of changes in A acts like a change in the effective utility discount rate.

If the consumption technology shocks (changes in the log-level of A) are *i.i.d* (independent of the process of Z_t as well), then the effective change in the utility discount rate is constant. With consumption technology shocks *i.i.d.*, A is also essentially redundant informationally and so there is no additional effect of a purely informational duplicate of A_t in Ω_t.

If consumption technology shocks are more predictable, with $\beta > 1$, an anticipated improvement in consumption technology is like a decrease in patience, as the effective weight on the future falls. A temporary movement in the consumption technology that leads to an anticipated worsening in the consumption technology as it returns to normal is like an increase in patience, as the effective weight on the future increases.