The Effect of Uncertainty on Optimal Control Models in the Neighborhood of a Steady State

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This paper derives the Taylor approximation to the effect of uncertainty on expected utility and optimal behavior in optimal stochastic control models when the uncertainty is small enough that one can focus on only the first term that involves uncertainty. This approximation is used to study the effect of uncertainty on welfare and behavior (1) in the Basic Real Business Cycle Model (a) with a choice between a risky and a safe technology, (b) with only a risky technology, and (c) with unpredictable permanent labor-augmenting productivity shocks and (2) in a growth model with investment adjustment costs with unpredictable productivity shocks. The last section extends the approach to Kreps-Porteus preferences.

Introduction

As a graduate student in 1983, after being initiated into the wonders of optimal control, phase diagrams and saddle paths, I was told that despite its limitations, linearizing the decision rules around the steady state was the "state of the art." Despite differences in notation and approach (primarily the difference between the use of continuous time and the use of discrete time), this linearizing of decision rules around the steady state essentially yields what in other circles has been called a linear-quadratic *approximation* to a problem that is not at its core linear-quadratic.

This paper goes one step beyond a linear-quadratic approximation that linearizes the decision rules around the steady state to include terms showing the effect of uncertainty on decision rules in the neighborhood of the steady state. To keep things simple, the analysis here will be confined to individual optimal control models—which are equivalent to macroeconomic models with a representative agent.

The method is known to mathematicians as the "perturbation method," taking the solution to the model in the absence of uncertainty as a benchmark and taking the derivative with respect to the quantity of risk. This is similar in spirit to taking the steady-state as a benchmark and then taking derivatives to find the behavior around the steady state. In fact, it is the combination of using the certain model as a benchmark and using the steady state as a benchmark, each at the appropriate points, that allows all the terms in the approximation to be expressed in terms of steady-state values of the decision variables and the first three derivatives of the objective and constraint functions.

The aim is to find out first the effect of uncertainty on the value function giving expected utility as a function of the state variable and the derivatives of the value function and then the total effect of uncertainty on the decision rules. Once the effect of uncertainty on the decision rules for the model are in hand, calculating approximate distributions is straightforward.

Kenneth Judd (1991, 1992) and Judd and Sy Ming Guu (1992 and forthcoming) are also currently in the business of importing perturbation methods into economics.[1] This paper differs from their work by treating in depth the effect of uncertainty on a general one-state-variable continuous-time optimal control problem in the neighborhood of the steady state, and by applying those results to the general equilibrium settings of (1) the Basic Real Business Cycle Model (a) with a choice between a risky and a safe technology, (b) with only a risky technology, and (c) with unpredictable permanent labor-augmenting productivity shocks, and (2) of a Q-theory growth model with unpredictable permanent labor-augmenting productivity shocks. Section 6 extends the approach to Kreps-Porteus preferences. Judd's and Judd and Guu's central application is to analyzing the effect of uncertainty on the basic optimal growth model with inelastic labor supply.

1. General Analysis for a One State Variable Optimizing Model

I will stay as close as possible to Kamien and Schwartz' notational conventions. The state variable is x, the vector of control variables is u, dz is a standard Wiener innovation, etc. Other than the constant utility

[1] Judd (1991) also discusses a wide variety of other useful approximation methods.

discount rate t, time does not explicitly enter the problem. Thus, the agent faces the stationary problem

$$\max_{\{x\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t}[U(k,x) + \omega v(k,x)]\, dt$$

$$s.t. \quad dk = [A(k,x) + \omega\alpha(k,x)]dt + \sqrt{\omega}\,\sigma(k,x)dz. \tag{1}$$

The parameter ω controls the presence or absence of uncertainty. The case of primary interest is $\omega = 1$ while $\omega = 0$ is the benchmark certainty case. I include a possible dependence of the objective function and the accumulation function on ω through v and α in order to ensure that the optimal value of control variables such as the degree of employment of the risky technology in Section 2 have a nontrivial limit as $\omega \to 0$.[2]

Following Kamien and Schwartz' recipe, the Bellman equation for the current value function can be concocted as

$$\begin{aligned}\rho V(k,\omega) &= \max_x F(k,x,\omega) \\ &= \max_x \left\{ U(k,x) + V_k(k,\omega)A(k,x) + \omega\left[v(k,x) + V_k(k,\omega)\alpha(k,x) + V_{kk}(k,\omega)\frac{\sigma^2(k,x)}{2}\right]\right\}.\end{aligned} \tag{2}$$

By the envelope theorem, the induced variation in x can be ignored taking the first derivatives of the Bellman equation.[3] Thus:

$$\begin{aligned}\rho V_k(k,\omega) &= F_k(k,x(k,\omega),\omega) \\ &= U_k + V_k A_k + V_{kk}A \\ &\quad + \omega\left[v_k + V_k\alpha_k + V_{kk}\sigma\sigma_k + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2}\right].\end{aligned} \tag{3}$$

and

$$\begin{aligned}\rho V_\omega(k,\omega) &= F_\omega(k,x(k,\omega),\omega) \\ &= V_{k\omega}A + v + V_k\alpha + V_{kk}\frac{\sigma^2}{2} + \omega\left[V_{k\omega}\alpha + V_{kk\omega}\frac{\sigma^2}{2}\right].\end{aligned} \tag{4}$$

The arguments of V and its derivatives are k and ω, while the arguments of all the other functions on the right-hand side of (3) and (4) are k and x. (Only the top equality in each of (3) and (4) depends on x being optimal given k and ω.) Note that at the certain steady state many terms drop out of (3) and (4) since there $\omega = 0$ *and* $A(k^*, 0) = 0$.

The Effect of Uncertainty on the Value at the Steady State. Equation (4) is the key to evaluating the effect of uncertainty on welfare. Solving (4) for V_ω at $\omega = 0$ and $k = k^*$, which with $\Delta\omega = 1$ is equal to the first term in the Taylor approximation for the effect of uncertainty on the level of the value function at the steady state yields

$$V_\omega(k^*,0) = \frac{v(k^*,x^*) + V_k(k^*,0)\alpha(k^*,x^*) + V_{kk}(k^*,0)\frac{\sigma^2(k^*,x^*)}{2}}{\rho}. \tag{5}$$

[2] Judd (1991) ch. 11 discusses from a technical angle the use of terms like those involving v and α under the heading "Bifurcates Methods."

[3] In this paper, I will assume at every turn as much differentiability as needed.

There is one important interpretive point to make here. $V_\omega(k^*, 0)$ gives the effect of uncertainty on welfare *at* k^*. Thus, it is *not* the effect of uncertainty on steady state welfare, but the effect of the introduction of uncertainty on an expected present value concept of welfare at time zero, *including the present value of all of the expected transitional behavior of the model.* Thus, in the analysis here, it is actually easier to include transitional effects on welfare than to neglect them.

The Effect of Uncertainty on the Marginal Value at the Steady State.

Taking the "total" derivative of (4) with respect to k—that is, the derivative with respect to k including the induced variation in x—yields the key equation for evaluating the effects of uncertainty on behavior:

$$\begin{aligned}\rho V_{k\omega}(k, \omega) &= \frac{d}{dk} F_\omega(k, x(k, \omega), \omega) \\ &= V_{kk\omega} A(k, x(k, \omega)) + V_{k\omega} \frac{d}{dk} A(k, x(k, \omega)) \\ &\quad + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2} + \upsilon_k + V_k \alpha_k + V_{kk}\sigma\sigma_k \\ &\quad + \left[\frac{dx(k^*, 0)}{dk}\right]^T [\upsilon_x + V_k \alpha_x + V_{kk}\sigma\sigma_x] \\ &\quad \omega \left[V_{kk\omega}\alpha + V_{kkk\omega}\frac{\sigma^2}{2} + V_{k\omega}\frac{d}{dk}\alpha(k, x(k, \omega)) + V_{kk\omega}\frac{d}{dk}\frac{\sigma^2(k, x(k, \omega))}{2}\right],\end{aligned} \tag{6}$$

where $\frac{d}{dk}A(k, x(k, \omega)) = A_k + \left[\frac{dx(k^*,0)}{dk}\right]^T A_x$ is the *total* derivative of the rate of change in k with respect to k itself under the optimal policy when $\omega = 0$. Note that under certainty, the rate of convergence to the steady state under certainty is

$$\begin{aligned}\kappa &= -\frac{d}{dk} A(k, x(k, 0)) \\ &= -A_k - [A_x]^T \frac{dx(k, \omega)}{dk}.\end{aligned} \tag{7}$$

Solving (6) for $V_{k\omega}$ at the certain steady state, where $\omega = 0$ and $g = 0$, one finds that

$$V_{k\omega}(k^*, 0) = \frac{\upsilon_k + V_k\alpha_k + V_{kk}\sigma\sigma_k + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2}}{\rho + \kappa} + \frac{\left[\frac{dx}{dk}\right]^T [\upsilon_x + V_k\alpha_x + V_{kk}\sigma\sigma_x]}{\rho + \kappa}. \tag{8}$$

Calculating $\frac{dx}{dk}$ and $\frac{dx}{d\omega}$ at the Certain Steady State. The factor $\frac{dx}{dk}$ in (8) represents what happens to the control variables as one traces the saddle path going to the right which is often known from one's analysis of the certainty model. Formally, to find $\frac{dx(k,\omega)}{dk}$, begin with the first order condition

$$F_x(k, x(k, \omega), \omega) \equiv 0. \tag{9}$$

Taking the total derivative of this first order condition with respect to k:

$$\frac{d}{dk} F_x(k, x(k, \omega), \omega) = F_{xk}(k, x(k, \omega), \omega) + F_{xx}(k, x(k, \omega), \omega)\frac{dx(k, \omega)}{dk} = 0, \tag{10}$$

so

$$\frac{dx(k,\omega)}{dk} = -[F_{xx}(k, x(k,\omega), \omega)]^{-1} F_{kx}(k, x(k,\omega), \omega). \tag{11}$$

At the certain steady state, (11) reduces to

$$\frac{dx(k^*,0)}{dk} = -[U_{xx} + V_k A_{xx}]^{-1}[U_{kx} + V_k A_{kx} + V_{kk} A_x]. \tag{12}$$

By a similar calculation, starting from the certain steady state, the effect of uncertainty on control variables is

$$\frac{dx(k^*,0)}{d\omega} = [U_{xx} + V_k A_{xx}]^{-1}[V_{k\omega} A_x + \upsilon_x + V_k \alpha_x + V_{kk} \sigma \sigma_x]. \tag{13}$$

The Case of Additive Separability between Two Sets of Control Variables, only one of which involves Uncertainty. There is a common special case in which the coefficient of $\frac{dx}{dk}$ on the second line of (8) is zero. If there is additive separability between two sets of control variables, with only one set entering into σ, then f and υ, g and α can be defined so that the set x_1 of control variables affecting f and g is separate from the set x_2 of control variables affecting υ, α and σ. This allows the first order condition—

$$F_x = U_x + V_k A_x + \omega \left[\upsilon_x + V_k \alpha_x + V_{kk} \sigma \sigma_x\right] = 0 \tag{14}$$

—to be analyzed into two separate (vector) first-order condition:

$$U_{x_1} + V_k A_{x_1} = 0 \tag{15}$$

and

$$\upsilon_{x_2} + V_k \alpha_{x_2} + V_{kk} \sigma \sigma_{x_2} = 0. \tag{16}$$

Equation (16) must hold even for ω small, and so in the limit must hold for $\omega = 0$ as well, given differentiability of V_k and V_{kk} with respect to ω. Since, by the assumption of separability, the control variables not explicitly mentioned in (15) and (16) do not even enter into the equations where they are omitted, it is also true that

$$U_x + V_k A_x = 0 \tag{17}$$

and

$$\upsilon_x + V_k \alpha_x + V_{kk} \sigma \sigma_x = 0. \tag{18}$$

Equation (18) allows one to simplify (8) to

$$V_{k\omega}(k^*,0) = \frac{\upsilon_k + V_k \alpha_k + V_{kk} \sigma \sigma_k + V_{kk} \alpha + V_{kkk} \frac{\sigma^2}{2}}{\rho + \kappa} \tag{19}$$

Separability between the two sets of control variables also simplifies $\frac{dx_1}{d\omega}$ as follows:

$$\frac{dx_1(k^*,0)}{d\omega} = -[U_{x_1 x_1} + V_k A_{x_1 x_1}]^{-1} A_{x_1} V_{k\omega}. \tag{20}$$

This is essentially the effect on the control variables in the model under certainty of raising the costate variable by $V_{k\omega}$ for a fixed k (in the neighborhood of the steady state).

On the phase diagram, the saddle path is the graph of $V_k(k,0)$. The derivative $V_{k\omega}$ can be thought of as the vertical shift of the saddle-path due to uncertainty. Because of the separability of the control variables, (20) then allows the subvector of control variables x_1 to be determined as a function of k and V_k in exactly the same way as under certainty.

The effect of uncertainty on the control variables having to do with uncertainty, $\frac{dx_2}{d\omega}$, is a higher-order issue that depends on $V_{kk\omega}$ as well as $V_{k\omega}$. I will not address that issue here.

Calculating V_k, V_{kk} and V_{kkk} at the Certain Steady-State. In order to make equations (5) and (8) or (19) operational, it is necessary to calculate the first three derivatives of the value function at the steady-state with $\omega = 0$. The marginal value $V_k(k^*,0)$ can be found as the steady-state value of the costate variable ("λ^*") using standard techniques. In brief, (3) implies that $\rho V_k = U_k + V_k A_k$ with all functions evaluated at the certain steady-state. This costate equation, the steady state first-order condition $F_x = U_x + V_k A_x = 0$, and the stationarity condition $A(k^*,x^*) = 0$ under certainty can be jointly solved for k^*, x^* and $V_k(k^*,0)$.

The most general way to calculate $V_{kk}(k^*,0)$ is actually to use standard techniques to find the slope of the saddle-path at the steady state (in (k,λ) space). In the notation here, *totally* differentiated (3) with respect to k, yields

$$\rho V_{kk} = F_{kk} + [F_{kx}]^T \frac{dx}{dk} = F_{kk} - [F_{kx}]^T [F_{xx}]^{-1} F_{kx}. \tag{21}$$

Calculating these derivatives at the certain steady state,

$$\begin{aligned} \rho V_{kk} &= U_{kk} + V_k A_{kk} + 2V_{kk} A_k \\ &\quad - [U_{kx} + V_k A_{kx} + V_{kk} A_x]^T [U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx} + V_{kk} A_x]. \end{aligned} \tag{22}$$

Equation (22) is a quadratic equation in $V_{kk}(k^*,0)$ that can be solved by the quadratic formula, taking $V_{kk}(k^*,0)$ as the negative root[4]:

$$\begin{aligned} &-\left([A_x]^T [U_{xx} + V_k A_{xx}]^{-1} A_x\right) V_{kk}^2 + \left(-\rho + 2A_k - 2A_x [U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx}]\right) V_{kk} \\ &\qquad + U_{kk} + V_k A_{kk} - [U_{kx} + V_k A_{kx}][U_{xx} + V_k A_{xx}]^{-1}[U_{kx} + V_k A_{kx}] = 0. \end{aligned} \tag{23}$$

(Every other quantity in (23) can be calculated prior to calculating V_{kk}.)

To calculate $V_{kkk}(k^*,0)$, it is easiest to start with the Bellman equation $\rho V = F$ and totally differentiate without using the envelope theorem:

$$\rho V_k = F_k + F_x \frac{dx}{dk} \tag{24}$$

[4] If there are two or zero negative roots, the methods of this paper are not applicable to that problem. When f and g are each jointly concave in k and x, the coefficient of V_{kk}^2 will be positive and the constant term will be negative, guaranteeing that there is one positive and one negative root. Note that only the submatrix of $U_{xx} + V_k A_{xx}$ for control variables relevant under certainty needs to be inverted. The contributions involving the control variables relevant only under uncertainty limit to zero as $\omega \to 0$.

$$\rho V_{kk} = F_{kk} + 2F_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk}\right]^T F_{xx} \frac{dx}{dk} + F_x \frac{d^2x}{dk^2} \tag{25}$$

$$\begin{aligned}\rho V_{kkk} = {} & F_{kkk} + 3[F_{kkx}]^T \frac{dx}{dk} + 3\left[\frac{dx}{dk}\right]^T [F_{kxx}] \frac{dx}{dk} \\ & + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T F_{xxx_i} \frac{dx}{dk} \right) \\ & + 3\left(F_{kx} + F_{xx} \frac{dx}{dk} \right) \frac{d^2x}{dk^2} + F_x \frac{d^3x}{dk^3}.\end{aligned} \tag{26}$$

Having held the envelope theorem in reserve, now it can be applied with a vengeance. The F_x that multiplies $\frac{d^3x}{dk^3}$ is zero by the first order condition. *Also*

$$\left(F_{xk} + F_{xx} \frac{dx}{dk} \right) = 0 \tag{27}$$

by the total derivative of the first order condition with respect to k. The quantity $\frac{d^2x}{dk^2}$ could be calculated from totally differentiating in turn with respect to k, but there is no need, since the coefficient of $\frac{d^2x}{dk^2}$ is zero. Deleting the last line of (26) in accordance with this curious higher-order envelope theorem and expanding at the certain steady state, (26) becomes

$$\begin{aligned}\rho V_{kkk} = {} & U_{kkk} + V_k A_{kkk} + 3V_{kk} A_{kk} + 3V_{kkk} A_k \\ & + 3[U_{kkx} + V_k A_{kkx} + 2V_{kk} A_{kx} + V_{kkk} A_x]^T \frac{dx}{dk} \\ & + 3\left[\frac{dx}{dk}\right]^T [U_{kxx} + V_k A_{kxx} + V_{kk} A_{xx}] \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T [U_{xxx_i} + V_k A_{xxx_i}] \frac{dx}{dk} \right)\end{aligned} \tag{28}$$

with, of course,

$$\frac{dx(k^*, 0)}{dk} = -[U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx} + V_{kk} A_x]. \tag{13}$$

Grouping terms in (28) according to the derivative of V in each,

$$\begin{aligned}\rho V_{kkk} = {} & U_{kkk} + 3U_{kkx}^T \frac{dx}{dk} + 3\left[\frac{dx}{dk}\right]^T U_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T U_{xxx_i} \frac{dx}{dk} \right) \\ & + V_k \left[A_{kkk} + 3A_{kkx}^T \frac{dx}{dk} + 3\left[\frac{dx}{dk}\right]^T A_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T A_{xxx_i} \frac{dx}{dk} \right) \right] \\ & + 3V_{kk} \left[A_{kk} + 2A_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk}\right]^T A_{xx} \frac{dx}{dk} \right] \\ & + 3V_{kkk} \left[A_k + A_x^T \frac{dx}{dk} \right].\end{aligned} \tag{29}$$

Since $\kappa = -A_k - [A_x]^T \frac{dx}{dk}$, the coefficient of V_{kkk} is $-3\kappa V_{kkk}$. Thus, solving (29) for V_{kkk} yields

$$\begin{aligned}V_{kkk}(k^*, 0) = (\rho + 3\kappa)^{-1} \Bigg\{ & U_{kkk} + 3U_{kkx}^T \frac{dx}{dk} + 3\left[\frac{dx}{dk}\right]^T U_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T U_{xxx_i} \frac{dx}{dk} \right) \\ & + V_k \left[A_{kkk} + 3A_{kkx}^T \frac{dx}{dk} + 3\left[\frac{dx}{dk}\right]^T A_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk}\right]^T A_{xxx_i} \frac{dx}{dk} \right) \right] \\ & + 3V_{kk} \left[A_{kk} + 2A_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk}\right]^T A_{xx} \frac{dx}{dk} \right] \Bigg\}.\end{aligned} \tag{30}$$

The nature of the numerator $\rho + 3\kappa$ on the right-hand side of (30) would be more evident if both sides of the equation were multiplied by $(k - k^*)^3$; it is exactly what one would expect for the present value of terms involving $(k - k^*)^3$, with k following a continuous-time AR(1) process that reverts to k^* at the rate κ. In the numerator that is within the curly brackets, the first line is the third derivative of the objective or "utility" function f with respect to the state variable k, including indirect effects through changes in the vector of control variables *evaluated as if the derivative of the control variables with respect to the state variable k were a constant.* The second line is the third derivative of the accumulation or "production" function g with respect to k (including indirect effects through changes in the control variables evaluated as if the derivative of the control variables with respect to k were a constant) *multiplied by the steady-state marginal value, which converts from units of production to units of utility.* The third line is three times the second derivative of the production function with respect to k (including indirect effects through changes in the control variables evaluated as if the derivative of the control variables with respect to k were a constant) multiplied by the rate at which marginal value declines with k.

Intuitively, in determining the degree of convexity of the marginal value function, the first line on the right-hand side of (30) (within the curly brackets) is the contribution of the convexity of the underlying marginal utility function. The second line is the contribution of the convexity of the marginal product function. The third line is the contribution of the interaction between declining marginal value and declining marginal product. With both the value function and accumulation function g concave, this third "covariance" term will be positive, since marginal value and marginal product will tend to be high together (when k is low) and low together (when k is high).

2. Choosing How Much to Use a Risky Technology in General Equilibrium

The Setup. Consider appending to the Basic Real Business Cycle Model (with endogenous labor supply and capital accumulation but no other complications) a choice between a risky technology and a safe technology. The risky technology produces $(1 + \mu)$ times as much on average, but also gives a standard deviation to the accumulation equation proportional to output. The stochastic optimization problem is then

$$\max_{\{c,n,\pi\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t}[x(c,n)]\, dt$$

$$s.t. \quad dk = [(1-\pi)nf(\frac{k}{n}) + \pi(1+\omega\mu)nf(\frac{k}{n}) - \delta k - c - g]dt + \sqrt{\omega}\pi Snf(\frac{k}{n})dz, \qquad (R.1)$$

where c is consumption, n is labor, π is the degree of employment of the risky technology, k is the capital stock and g is a constant level of government spending. To allow the problem to be the outcome of detrending a model with a steady growth path, the utility function has the most general form that is consistent with steady state growth[5]:

$$x(c,n) = [\frac{c^{1-\beta}}{1-\beta} e^{(\beta-1)v(n)}]. \qquad (R.2)$$

[5] See Robert King, Charles Plosser and Sergio Rebelo (1988).

with $v'(n) \geq 0$ and (assuming normality) $v''(n) \geq 0$. (The familiar logarithmic form is a limiting case as $\beta \to 1$.) Both production functions are constant returns to scale.

Defining

$$\vartheta = nf(\frac{k}{n})\pi, \tag{R.3}$$

this is the same problem as

$$\max_{\{c,n,\vartheta\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t}[\frac{c^{1-\beta}}{1-\beta}e^{(\beta-1)v(n)}]\,dt$$

$$s.t. \;\; dk = [nf(\frac{k}{n}) - \delta k - c - g + \omega\vartheta\mu]dt + \sqrt{\omega}\,\vartheta S dz. \tag{R.4}$$

The correspondence between (R.4) and the generic problem of section 1 is as follows:

$$\begin{aligned}
k &: \; k \\
x_1 &: \; \begin{bmatrix} c \\ n \end{bmatrix} \\
x_2 &: \; \vartheta \\
f &: \; \frac{c^{1-\beta}}{1-\beta}e^{(\beta-1)v(n)} = x(c,n) \\
g &: \; nf(\frac{k}{n}) - \delta k - c - g = A(k,n,c) \\
v &: \; 0 \\
\alpha &: \; \vartheta\mu \\
\sigma &: \; \vartheta S.
\end{aligned}$$

The Bellman equation for this problem is

$$\rho V(k,\omega) = \max_{c,n,\vartheta}\left\{x(c,n) + V_k[nf(\frac{k}{n}) - \delta k - c - g] + \omega\left[V_k\vartheta\mu + V_{kk}\frac{\vartheta^2 S^2}{2}\right]\right\}. \tag{R.5}$$

Note that functions of ϑ are always multiplied by ω, while functions of c and n are never multiplied by ω. This is the separability between two sets of control variables needed to use (32–36).

The first order conditions for c and n are

$$V_k(k,\omega) = x_c(c,n) = c^{-\beta}e^{(\beta-1)v(n)} \tag{R.6}$$

$$V_k(k,\omega)\left(f(\frac{k}{n}) - (\frac{k}{n})f'(\frac{k}{n})\right) = x_n(c,n) = -v'(n)c^{1-\beta}e^{(\beta-1)v(n)}. \tag{R.7}$$

Dividing (R.7) by (R.6), and defining the real wage w,

$$f(\frac{k}{n}) - (\frac{k}{n})f'(\frac{k}{n}) = -\frac{x_n(c,n)}{x_c(c,n)} = cv'(n) = w \tag{R.8}$$

The steady state condition for accumulation is

$$n^* f(k^*/n^*) = c^* + \delta k^* + g, \tag{R.9}$$

and the steady state Euler equation—divided through by $V_k(k^*, 0)$—is

$$\rho = A_k(k^*, n^*, c^*) = f'(k^*/n^*) - \delta. \tag{R.10}$$

The first order condition for ϑ is

$$V_k(k, \omega)\mu + V_{kk}(k, \omega)\vartheta S^2 = 0. \tag{R.11}$$

Solving (R.11) for ϑ,

$$\vartheta = \frac{-V_k(k, \omega)}{V_{kk}(k, \omega)} \frac{\mu}{S^2}. \tag{R.12}$$

Note that in (R.12), ϑ will have the well-defined limit $\frac{-V_k(k,0)}{V_{kk}(k,0)} \frac{\mu}{S^2}$ as $\omega \to 0$. In the limit as $\omega \to 0$, the "risky" technology is in use, but becomes identical to the safe technology so that the certain steady state is indistinguishable from the certain steady state of a model with only the safe technology.

Following (5) and using (R.12), the effect of the option of using the risky technology on *welfare* with ω going from zero to one is approximately given by

$$\begin{aligned} V_\omega(k^*, 0) &= \rho^{-1} \left[V_k(k^*, 0)\vartheta\mu + V_{kk}(k^*, 0)\frac{\vartheta^2 S^2}{2} \right] \\ &= \rho^{-1} \left[V_k \left(\frac{-V_k}{V_{kk}} \frac{\mu}{S^2} \right) \mu + V_{kk} \frac{\left(\frac{-V_k}{V_{kk}} \frac{\mu}{S^2} \right)^2 S^2}{2} \right] \\ &= \frac{-V_k^2}{V_{kk}} \frac{\mu^2}{2\rho S^2} \end{aligned} \tag{R.13}$$

Following (19) and using (R.12), the effect of the option of using the risky technology on the *marginal value* with ω going from zero to one is approximately

$$\begin{aligned} V_{k\omega}(k^*, 0) &= (\rho + \kappa)^{-1} \left[V_{kk}\vartheta\mu + V_{kkk}\frac{\vartheta^2 S^2}{2} \right] \\ &= \frac{V_k}{\rho + \kappa} \frac{\mu^2}{2S^2} \left[\frac{V_{kkk} V_k}{V_{kk}^2} - 2 \right], \end{aligned} \tag{R.14}$$

since neither $\upsilon := 0$ nor $\alpha := \vartheta\mu$ nor $\sigma := \vartheta S$ is a function of the state variable k.

The Certainty Model (Second Order). The determination of κ, $V_{kk}(k^*, 0)$ and $V_{kkk}(k^*, 0)$ can piggyback on treatments of the model under certainty. In this case, Kimball (1991) gives a detailed analysis of the Basic Real Business Model (log-)linearized around the steady state under certainty.

The convergence rate κ depends only on the zeroth, first and second derivatives of the functions of the model which appear in linearizing the model around the steady state under certainty. The convergence rate under certainty is shown to be

$$\kappa = \frac{1}{2}\sqrt{\rho^2 + 4\frac{s(1-\theta)\nu(\rho+\delta)^2}{(1-h)\theta[\sigma + \beta s\nu\theta]} \left[1 - \theta\left(h + (1-h)\frac{\delta}{\rho+\delta} \right) \right]} - \frac{\rho}{2} \tag{R.15}$$

where σ is here the steady-state elasticity of substitution between capital and labor,

$$\sigma = \frac{-f'\left(\frac{k^*}{n^*}\right)\left[h\left(\frac{k^*}{n^*}\right) - \left(\frac{k^*}{n^*}\right)f'\left(\frac{k^*}{n^*}\right)\right]}{\frac{k^*}{n^*}f''\left(\frac{k^*}{n^*}\right)h\left(\frac{k^*}{n^*}\right)}, \tag{R.16}$$

θ is capital's share in production and $1-\theta$ is labor's share in production,

$$\theta = \frac{\frac{k^*}{n^*}f'\left(\frac{k^*}{n^*}\right)}{h\left(\frac{k^*}{n^*}\right)} = \frac{(\rho+\delta)k^*}{y^*} = 1 - \frac{w^*n^*}{y^*} \tag{R.17}$$

where

$$y = nf\left(\frac{k}{n}\right) \tag{R.18}$$

is output, ν is the consumption-constant labor supply elasticity

$$\nu = \frac{v'(n^*)}{n^*v''(n^*)}, \tag{R.19}$$

h is the marginal expenditure share of consumption when moving along the income expansion path,

$$h = \frac{v''(n^*)}{v''(n^*) + v'(n^*)^2} = \frac{c^*}{c^* + \nu w^* n^*}, \tag{R.20}$$

$1-h$ is the marginal expenditure share of leisure when moving along the income expansion path, and

$$s = \frac{h}{h+\beta-1} \tag{R.21}$$

is the elasticity of intertemporal substitution for consumption and leisure combined[6] when "leisure" is defined as

$$\ell = (1+\nu)n^* - n. \tag{R.22}$$

When $\nu = 0$, $h = 1$, $1-h = 0$ and $s = \beta$. In (R.15),

$$\frac{\nu}{1-h} = \frac{c^* + \nu w^* n^*}{w^* n^*} \to \frac{c^*}{w^* n^*} \tag{R.23}$$

as $\nu \to 0$. Thus, when $\nu = 0$,

$$\kappa = \frac{1}{2}\sqrt{\rho^2 + 4\frac{(1-\theta)(\rho+\delta)}{\beta\sigma}\frac{c^*}{k^*}} - \frac{\rho}{2}. \tag{R.24}$$

Defining the steady state consumption share of output

$$\zeta_c = \frac{c^*}{y^*}, \tag{R.25}$$

the ratio $\frac{c^*}{k^*}$ is

$$\frac{c^*}{k^*} = (\rho+\delta)\frac{c^*}{y^*}\frac{y^*}{\rho+\delta} = (\rho+\delta)\frac{\zeta_c}{\theta}, \tag{R.26}$$

[6] Equation (R.21) can be turned inside out to get $\beta = 1 - h + \frac{h}{s}$. Both this equation and (R.21) point to the fact that $\beta > 1-h$ is necessary for concavity of $x(c,n)$.

so that

$$\kappa = \frac{1}{2}\sqrt{\rho^2 + 4\frac{(1-\theta)\zeta_c}{\beta\theta\sigma}(\rho+\delta)^2} - \frac{\rho}{2} \tag{R.27}$$

when $\nu = 0$.

In this model, there is a very useful relationship between the convergence rate κ and the effect of k on c and n along the saddle path. By definition,

$$\kappa = A_k + A_n\frac{dn}{dk} + A_c\frac{dc}{dk} = \rho + w^*\frac{dn}{dk} - \frac{dc}{dk}. \tag{R.28}$$

Thus,

$$\rho + \kappa = \frac{dc}{dk} - w^*\frac{dn}{dk}. \tag{R.29}$$

In words, having one extra unit of capital adds $\rho+\kappa$ to the expenditure on consumption and leisure combined.

Define ξ as the marginal expenditure share of leisure *when moving along the saddle path*:

$$\xi = \frac{-w^*\frac{dn}{dk}}{\frac{dc}{dk} - w^*\frac{dn}{dk}}, \tag{R.30}$$

so that

$$\frac{dc}{dk} = (1-\xi)(\rho+\kappa) \tag{R.31}$$

and

$$\frac{dn}{dk} = -\frac{\xi}{w^*}(\rho+\kappa) \tag{R.32}$$

Unlike the marginal expenditure share of leisure *when moving along the income expansion path* $1-h$, ξ includes the effect of additional capital on labor supply through higher wages as well as through higher wealth. To find ξ, take a logarithmic derivative of (R.8) with respect to k:

$$\frac{\theta}{\sigma}\left(1 - \frac{dn}{dk}\frac{k^*}{n^*}\right) = \frac{dw}{dk}\frac{k^*}{w^*} = \frac{dc}{dk}\frac{k^*}{c^*} + \frac{n^*v''(n)}{v'(n)}\frac{dn}{dk}\frac{k^*}{n^*}. \tag{R.33}$$

Using (R.31) and (R.32),

$$\frac{\theta}{\sigma}\left(1 + \frac{(\rho+\kappa)k^*}{w^*n^*}\xi\right) = \frac{dw}{dk}\frac{k^*}{w^*} = \frac{(\rho+\kappa)k^*}{c^*}(1-\xi) - \frac{1}{\nu}\frac{(\rho+\kappa)k^*}{w^*n^*}\xi. \tag{R.34}$$

Using (R.25) and (R.17),

$$\begin{aligned}\frac{\theta}{\sigma}\left(1 + \frac{(\rho+\kappa)\theta}{(\rho+\delta)(1-\theta)}\xi\right) = \frac{dw}{dk}\frac{k^*}{w^*} &= \frac{(\rho+\kappa)\theta}{(\rho+\delta)\zeta_c}(1-\xi) - \frac{1}{\nu}\frac{(\rho+\kappa)\theta}{(\rho+\delta)(1-\theta)}\xi \\ &= \frac{(\rho+\kappa)\theta}{(\rho+\delta)\zeta_c}\left[1 - \xi - \frac{\zeta_c}{\nu(1-\theta)}\xi\right]\end{aligned} \tag{R.35}$$

Solving (R.35) for ξ,

$$\xi = \frac{1-h}{1+\nu h\frac{\theta}{\sigma}}\left[1 - \frac{\zeta_c}{\sigma}\frac{\rho+\delta}{\rho+\kappa}\right] \le 1-h. \tag{R.36}$$

For reasonable parameters, ξ can take on either sign or be zero. The upper limit of $1-h$ is reached when the wage effect is eliminated by setting $\sigma = \infty$. The derivative $\frac{dn}{dk}$ has the sign opposite to ξ; from (R.36) and (R.32),

$$\frac{dn}{dk} = \frac{1-h}{1+\nu h \frac{\theta}{\sigma}} w^* \left[\frac{\zeta_c}{\sigma}(\rho+\delta) - (\rho+\kappa) \right]. \qquad (R.37)$$

Roughly speaking, within the square brackets $\frac{\zeta_c}{\sigma}(\rho+\delta)$ represents the *wage* effect of having more capital, while $-(\rho+\kappa)$ represents the *wealth* effect of having more capital.

Putting Things Together. The easiest way to find $V_{kk}(k^*, 0)$ is to differentiate the first-order condition for consumption (R.6) with respect to k:

$$\begin{aligned} V_{kk}(k^*, 0) &= x_{cc}(c^*, n^*)\frac{dc}{dk} + x_{cn}(c^*, n^*)\frac{dn}{dk} \\ &= (c^*)^{-\beta-1} e^{(\beta-1)v(n^*)} \left[-\beta \frac{dc}{dk} + (\beta-1)c^* v'(n^*)\frac{dn}{dk} \right] \\ &= \frac{V_k(k^*, 0)}{c^*}(\rho+\kappa)\left[-(1-\xi)\beta - \xi(\beta-1)\right] \\ &= \frac{-(\beta-\xi)(\rho+\kappa)}{c^*} V_k(k^*, 0), \end{aligned} \qquad (R.38)$$

using the fact that $w^* = c^* v'(n^*)$. Thus, absolute risk aversion in general equilibrium is

$$\frac{-V_{kk}(k^*, 0)}{V_k(k^*, 0)} = \frac{(\beta-\xi)(\rho+\kappa)}{c^*}. \qquad (R.39)$$

In comparison to the partial equilibrium absolute risk aversion[7] $\frac{\beta-(1-h)}{c^*}$, absolute risk aversion in general equilibrium goes up dramatically with a higher convergence rate κ. Absolute risk aversion in general equilibrium also tends to be higher than the partial equilibrium absolute risk aversion because the wage effect on labor reduces the extent to which variation in labor hours absorbs risk; when times are bad and the capital stock is low, the wage is also low, so that consumption, which has a greater effect on marginal utility, bears the brunt of the reduction in expenditure.

By (R.13) and (R.39), the effect of the option to invest in the risky technology on welfare is approximately

$$V_\omega(k^*, 0) = \frac{V_k(k^*, 0)c^*}{\rho} \frac{\mu^2}{2(\beta-\xi)(\rho+\kappa)S^2}. \qquad (R.40)$$

The factor $\frac{V_k(k^*, 0)c^*}{\rho}$ establishes a welfare unit equal to the additional welfare that would result from a certain percentage increase in consumption in perpetuity. Thus, the additional welfare from the option to invest in the risky technology is approximately equal to the additional welfare that would result from a permanent increase in consumption by the fraction $\frac{\mu^2}{2(\beta-\xi)(\rho+\kappa)S^2}$.

All that remains is to calculate $V_{kkk}(k^*, 0)$ and $V_{k\omega}(k^*, 0)$. Following (30), simplified greatly by the fact that $x(c, n)$ is not a function of k, so that terms in (30) like U_{kkk}, U_{kkx} and U_{kxx} are not relevant—and that

[7] The general equilibrium effects operating through changes in factor prices can be eliminated by setting $\sigma = \infty$ in (R.36) and setting $\kappa = 0$ in (R.39).

c occurs only linearly in the accumulation function $A(k,n,c)$—

$$(\rho+3\kappa)\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} = \frac{x_{ccc}\left(\frac{dc}{dk}\right)^3 + 3x_{ccn}\left(\frac{dc}{dk}\right)^2\left(\frac{dn}{dk}\right) + 3x_{cnn}\left(\frac{dc}{dk}\right)\left(\frac{dn}{dk}\right)^2 + x_{nnn}\left(\frac{dn}{dk}\right)^3}{x_c} \tag{R.41}$$
$$+ A_{kkk} + 3A_{kkn}\left(\frac{dn}{dk}\right) + 3A_{knn}\left(\frac{dn}{dk}\right)^2 + A_{nnn}\left(\frac{dn}{dk}\right)^3$$
$$+ 3\frac{V_{kk}}{V_k}\left[A_{kk} + 2A_{kn}\left(\frac{dn}{dk}\right) + A_{nn}\left(\frac{dn}{dk}\right)^2\right],$$

since $V_k = x_c$. Using (R.31), (R.32) and (R.39) and expanding the powers of $1-\xi$,

$$(\rho+3\kappa)\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} = \tag{R.42}$$
$$(\rho+\kappa)^3\left[\frac{x_{ccc} - 3\left(\frac{x_{ccn}}{w^*} + x_{ccc}\right)\xi + 3\left(\frac{x_{cnn}}{(w^*)^2} + 2\frac{x_{ccn}}{w^*} + x_{ccc}\right)\xi^2 - \left(\frac{x_{nnn}}{(w^*)^3} + 3\frac{x_{cnn}}{(w^*)^2} + 3\frac{x_{ccn}}{w^*} + x_{ccc}\right)\xi^3}{x_c}\right]$$
$$+ A_{kkk} - 3\frac{A_{kkn}}{w^*}(\rho+\kappa)\xi + 3\frac{A_{knn}}{(w^*)^2}(\rho+\kappa)^2\xi^2 - \frac{A_{nnn}}{(w^*)^3}(\rho+\kappa)^3\xi^3$$
$$- 3\left((\rho+\kappa)\frac{\beta-\xi}{c^*}\right)\left[A_{kk} - 2\frac{A_{kn}}{w^*}(\rho+\kappa)\xi + \frac{A_{nn}}{(w^*)^2}(\rho+\kappa)^2\xi^2\right].$$

The Certainty Model (Third Order). As an aid to expressing x_{cnn} in an interpretable way, note that by (R.8), (R.17–R.19) and (R.25),

$$\frac{v''(n^*)}{v'(n^*)^2} = \frac{1}{\nu n^* v'(n^*)} = \frac{c^*}{\nu n^* w^*} = \frac{\zeta_c}{\nu(1-\theta)}. \tag{R.43}$$

As an aid to expressing x_{nnn} in an interpretable way, note in addition that the income expansion path through the steady state is defined by

$$c = \frac{w^*}{v'(n)} \tag{R.44}$$

with w^* treated as a constant. Taking the derivative (R.44) twice with respect to n yields the equation

$$\left.\frac{d^2c}{dn^2}\right|_{x_n/x_c=-w^*} = w^*\left[-\frac{v'''(n)}{v'(n)^2} + 2\frac{v''(n)^2}{v'(n)^3}\right]. \tag{R.45}$$

Multiplying by $\frac{c^*}{(w^*)^2} = \frac{1}{w^* v'(n^*)}$ and evaluating at the steady state yields a unit-free parameter for the curvature of the income expansion path—ψ:

$$\psi = \frac{c^*}{(w^*)^2}\left(\left.\frac{d^2c}{dn^2}\right|_{x_n/x_c=-w^*}\right) = -\frac{v'''(n^*)}{v'(n^*)^3} + 2\frac{v''(n^*)^2}{v'(n^*)^4} = -\frac{v'''(n^*)}{v'(n^*)^3} + 2\left(\frac{\zeta_c}{\nu(1-\theta)}\right)^2. \tag{R.46}$$

Now, calculate as follows.

$$\frac{x_{ccc}}{x_c} = \frac{\beta^2+\beta}{(c^*)^2} \tag{R.47}$$

$$\frac{x_{ccn}}{x_c w^*} = \frac{-\beta^2+\beta}{(c^*)^2} \tag{R.48}$$

$$\frac{x_{cnn}}{x_c(w^*)^2} = \frac{\beta^2 - 2\beta + 1 + (\beta - 1)\frac{\zeta_c}{\nu(1-\theta)}}{(c^*)^2} \tag{R.49}$$

$$\frac{x_{nnn}}{x_c(w^*)^3} = \frac{-\beta^2 + 2\beta - 1 - 3(\beta - 1)\frac{\zeta_c}{\nu(1-\theta)} - 2\left(\frac{\zeta_c}{\nu(1-\theta)}\right)^2 + \psi}{(c^*)^2}. \tag{R.50}$$

The derivatives of the accumulation function $A(k, n, c)$ needed to evaluate (R.42) are as follows:

$$A_{kk} = \frac{1}{n} f''\left(\frac{k}{n}\right) \tag{R.51}$$

$$A_{kn} = -\frac{k}{n^2} f''\left(\frac{k}{n}\right) \tag{R.52}$$

$$A_{nn} = \frac{k^2}{n^3} f''\left(\frac{k}{n}\right) \tag{R.53}$$

$$A_{kkk} = \frac{1}{n^2} f'''\left(\frac{k}{n}\right) \tag{R.54}$$

$$A_{kkn} = -\frac{k}{n^3} f'''\left(\frac{k}{n}\right) - \frac{1}{n^2} f''\left(\frac{k}{n}\right) \tag{R.55}$$

$$A_{knn} = \frac{k^2}{n^4} f'''\left(\frac{k}{n}\right) + 2\frac{k}{n^3} f''\left(\frac{k}{n}\right) \tag{R.56}$$

$$A_{nnn} = -\frac{k^3}{n^5} f'''\left(\frac{k}{n}\right) - 3\frac{k^2}{n^4} f''\left(\frac{k}{n}\right). \tag{R.57}$$

From (R.16), using the notation $\chi = \frac{k}{n}$ for convenience, and allowing the elasticity of capital-labor substitution σ and capital's share θ to vary with χ,

$$-\frac{\chi f''(\chi)}{f'(\chi)} = \frac{f(\chi) - \chi f'(\chi)}{f(\chi)\sigma(\chi)} = \frac{1 - \theta(\chi)}{\sigma(\chi)}. \tag{R.58}$$

Defining

$$\epsilon = \frac{\chi^* \sigma'(\chi^*)}{\sigma(\chi^*)}, \tag{R.59}$$

(R.16) can be logarithmically differentiated with respect to χ at χ^*, to get

$$\begin{aligned} -\frac{\chi^* f'''(\chi^*)}{f''(\chi^*)} &= 1 + \frac{\chi^* f'(\chi^*)}{f(\chi^*)} + \frac{(\chi^*)^2 f''(\chi^*)}{f(\chi^*) - \chi^* f'(\chi^*)} - \frac{\chi^* f''(\chi^*)}{f'(\chi^*)} + \frac{\chi^* \sigma'(\chi^*)}{\sigma(\chi^*)} \\ &= 1 + \theta(\chi^*) - \frac{\theta(\chi^*)}{\sigma(\chi^*)} + \frac{1 - \theta(\chi^*)}{\sigma(\chi^*)} + \epsilon \\ &= 1 + \theta + \frac{1 - 2\theta}{\sigma} + \epsilon \end{aligned} \tag{R.60}$$

One last fact that is helpful in dealing with (R.42) is that since $\frac{(\rho+\delta)k^*}{w^* n^*} = \frac{\theta}{1-\theta}$,

$$\frac{(\rho + \kappa)k^*}{w^* n^*} = \frac{\rho + \kappa}{\rho + \delta} \frac{\theta}{1 - \theta}. \tag{R.61}$$

Substituting (R.47–R.61) into (R.42) and grouping together all the terms that originally involving $f''\left(\frac{k^*}{n^*}\right)$ yields

$$\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} = \frac{1}{(\rho+3\kappa)(c^*)^2}\left\{(\rho+\kappa)^3\left[\beta(\beta+1)-6\beta\xi+3\left(1+\frac{(\beta-1)\zeta_c}{\nu(1-\theta)}\right)\xi^2-\left(2-\frac{2\zeta_c^2}{\nu^2(1-\theta)^2}+\psi\right)\xi^3\right]\right.$$
$$+\frac{(\rho+\delta)^3\zeta_c^2}{\theta^2}\left(\frac{1-\theta}{\sigma}\right)\left(1+\theta+\frac{1-2\theta}{\sigma}+\epsilon\right)\left(1+\frac{(\rho+\kappa)\theta}{(\rho+\delta)(1-\theta)}\xi\right)^3$$
$$\left.+3\frac{(\rho+\kappa)(\rho+\delta)^2\zeta_c}{\theta}\frac{(1-\theta)}{\sigma}\left(\beta-\left(1+\frac{\zeta_c}{1-\theta}\right)\xi\right)\left(1+\frac{(\rho+\kappa)\theta}{(\rho+\delta)(1-\theta)}\xi\right)^2\right\}. \quad (R.62)$$

Now, note that by (R.35)

$$1+\frac{(\rho+\kappa)\theta}{(\rho+\delta)(1-\theta)}\xi = \frac{(\rho+\kappa)\sigma}{(\rho+\delta)\zeta_c}\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right), \quad (R.63)$$

so that (R.62) can be simplified to

$$\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} = \frac{(\rho+\kappa)^3}{(\rho+3\kappa)(c^*)^2}\left\{\beta(\beta+1)-6\beta\xi+3\left(1+\frac{(\beta-1)\zeta_c}{\nu(1-\theta)}\right)\xi^2-\left(2-\frac{2\zeta_c^2}{\nu^2(1-\theta)^2}+\psi\right)\xi^3\right.$$
$$+\frac{\sigma(1-\theta)}{\zeta_c\theta}\left[\left(\frac{1}{\theta}-2+\frac{\sigma}{\theta}(1+\theta+\epsilon)\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^3\right.$$
$$\left.\left.+3\left(\beta-\xi-\frac{\zeta_c}{1-\theta}\xi\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^2\right]\right\}. \quad (R.64)$$

Putting Everything Together. Finally, (R.14), (R.39) and (R.64) can be used to find the effect of uncertainty on the marginal value. In proportion to the initial value of the marginal value, this is

$$\frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} = -\frac{\mu^2}{(\rho+\kappa)S^2} \quad (R.65)$$
$$+\frac{\mu^2}{2(\beta-\xi)^2(\rho+3\kappa)S^2}\left\{\beta(\beta+1)-6\beta\xi+3\left(1+\frac{(\beta-1)\zeta_c}{\nu(1-\theta)}\right)\xi^2-\left(2-\frac{2\zeta_c^2}{\nu^2(1-\theta)^2}+\psi\right)\xi^3\right.$$
$$+\frac{\sigma(1-\theta)}{\zeta_c\theta}\left[\left(\frac{1}{\theta}-2+\frac{\sigma}{\theta}(1+\theta+\epsilon)\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^3\right.$$
$$\left.\left.+3\left(\beta-\xi-\frac{\zeta_c}{1-\theta}\xi\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^2\right]\right\}.$$

The effects of this change in the marginal value on the control variables c and n are indicating by the logarithmic derivatives of c and n with respect to the costate variable "λ" in Kimball (1991). An increase in the costate variable always raises the accumulation rate of the state variable, so a positive $V_{k\omega}$ means that uncertainty tends to make the mean growth rate of capital will be positive at the certain steady state level of capital k^*.

3. When the Only Technology Available is Risky

There are economies of scale in examining several models that all reduce to the same model under certainty. One model the has an especially low marginal cost given what has gone before is one in which, instead of having a choice between a risky and a safe technology, the economy is stuck with only a risky technology. To model this, modify (R.1) only by setting $\pi = 1$ and $\mu = 0$:

$$\max_{\{c,n\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t}[x(c,n)]\, dt$$

$$s.t. \;\; dk = [nf(\frac{k}{n}) - \delta k - c - g]dt + \sqrt{\omega}\, Snf(\frac{k}{n})dz. \qquad (S.1)$$

The control variables in this model cannot be neatly separated, since n affects both the standard deviation and utility. The only correspondences that differ from the previous section are as follows:

$$x: \begin{bmatrix} c \\ n \end{bmatrix}$$

$$\upsilon: \; 0$$

$$\alpha: \; 0$$

$$\sigma: \; Snf\left(\frac{k}{n}\right) = Sy.$$

Substituting these into (5) and (8) and using the expressions for the certainty derivatives of V from the previous section yields

$$\begin{aligned} V_\omega(k^*,0) &= \frac{V_{kk}S^2(y^*)^2}{2\rho} \\ &= -\left[\frac{V_k c^*}{\rho}\right]\left(\frac{(\beta-\xi)(\rho+\kappa)}{(c^*)^2}\right)\frac{S^2(y^*)^2}{2} \\ &= -(\beta-\xi)(\rho+\kappa)\frac{S^2}{2\zeta_c^2}\left[\frac{V_k c^*}{\rho}\right] \end{aligned} \qquad (S.2)$$

and

$$\begin{aligned} \frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} &= \frac{V_{kk}\frac{\partial}{\partial k}\frac{(Snf(k/n))^2}{2} + V_{kk}\frac{dn}{dk}\frac{\partial}{\partial n}\frac{(Snf(k/n))^2}{2} + V_{kkk}\frac{(Snf(k/n))^2}{2}}{(\rho+\kappa)V_k} \\ &= \frac{V_{kk}S^2y^*\left[\rho+\delta+w^*\frac{dn}{dk}\right] + V_{kkk}\frac{S^2(y^*)^2}{2}}{(\rho+\kappa)V_k} \\ &= -\frac{(\beta-\xi)S^2}{\zeta_c}[\rho+\delta-(\rho+\kappa)\xi] + \frac{S^2(y^*)^2}{2(\rho+\kappa)}\frac{V_{kkk}}{V_k} \end{aligned} \qquad (S.3)$$

—that is,

$$\begin{aligned} \frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} &= -\frac{(\beta-\xi)S^2}{\zeta_c}[\rho+\delta-(\rho+\kappa)\xi] \\ &+ \frac{(\rho+\kappa)^2S^2}{2(\rho+3\kappa)\zeta_c^2}\Bigg\{\beta(\beta+1) - 6\beta\xi + 3\left(1+\frac{(\beta-1)\zeta_c}{\nu(1-\theta)}\right)\xi^2 - \left(2-\frac{2\zeta_c^2}{\nu^2(1-\theta)^2}+\psi\right)\xi^3 \\ &\quad + \frac{\sigma(1-\theta)}{\zeta_c\theta}\Bigg[\left(\frac{1}{\theta}-2+\frac{\sigma}{\theta}(1+\theta+\epsilon)\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^3 \\ &\quad + 3\left(\beta-\xi-\frac{\zeta_c}{1-\theta}\xi\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^2\Bigg]\Bigg\}. \end{aligned} \qquad (S.4)$$

4. The Effect of the Variance of Permanent Technology Shocks

Using the Scale Symmetry to Simplify the Problem. The ultimate problem of interest in this section is

$$\max_{\{C,n\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t} \frac{C^{1-\beta}}{1-\beta} e^{(\beta-1)v(n)}\, dt$$

$$s.t.\quad dK = [Znf(\frac{K}{Zn}) - \delta K - C - gZ]dt. \tag{T.1}$$

$$s.t.\quad dZ = \omega Z\pi\mu dt + \sqrt{\omega}\, Z\pi S dz.$$

The Bellman equation for this problem is

$$\rho J(K,Z,\omega) = \max_{C,n} \left\{ \frac{C^{1-\beta}}{1-\beta} e^{(\beta-1)v(n)} + J_K(K,Z,\omega)\left[Znf\left(\frac{K}{Zn}\right) - \delta K - C \right] \right. \tag{T.2}$$
$$\left. + \,\omega \left[J_Z(K,Z,\omega) Z\pi\mu + J_{ZZ}(K,Z,\omega)\frac{Z^2\pi^2S^2}{2} \right] \right\},$$

where $J(K,Z,\omega)$ is the (current value) value function.

John Boyd (1990) has an excellent treatment of how to systematically use the symmetry of a problem like this to reduce it to a problem with only one stationary state variable. In particular, since doubling both the initial K and Z makes a doubling of C feasible, the value function must be homogenous of degree $1-\beta$:

$$J(K,Z,\omega) = Z^{1-\beta} V\left(\frac{K}{Z},\omega\right) = Z^{1-\beta}V(k,\omega), \tag{T.3}$$

where V is a "reduced" value function, and

$$k = \frac{K}{Z}. \tag{T.4}$$

The derivatives of J needed for the Bellman equation can be calculated as follows:

$$J_K(K,Z,\omega) = Z^{-\beta} V_k\left(\frac{K}{Z},\omega\right) \tag{T.5}$$

$$J_Z(K,Z,\omega) = (1-\beta)Z^{-\beta} V\left(\frac{K}{Z},\omega\right) - KZ^{-(\beta+1)}V_k\left(\frac{K}{Z},\omega\right) \tag{T.6}$$

$$J_{ZZ}(K,Z,\omega) = Z^{-(\beta+1)}\left[\beta(\beta-1)V\left(\frac{K}{Z},\omega\right) + 2\beta\frac{K}{Z}V_k\left(\frac{K}{Z},\omega\right) + \frac{K^2}{Z^2}V_{kk}\left(\frac{K}{Z},\omega\right)\right]. \tag{T.7}$$

Defining

$$c = \frac{C}{Z}, \tag{T.8}$$

then substituting from (T.3)–(T.8) into the Bellman equation (T.2) and dividing by $Z^{1-\beta}$ yields the reduced Bellman equation

$$\rho V(k,\omega) = \max_{c,n}\left\{ \frac{c^{1-\beta}}{1-\beta}e^{(\beta-1)v(n)} + V_k(k,\omega)\left[nf\left(\frac{k}{n}\right) - \delta k - c\right] \right. \tag{T.9}$$
$$\left. + \,\omega\left[\pi\mu\left(V(k,\omega)(1-\beta) - kV_k(k,\omega)\right) + \frac{\pi^2S^2}{2}[\beta(\beta-1)V(k,\omega) + 2\beta k V_k(k,\omega) + k^2V_{kk}(k,\omega)]\right]\right\}.$$

Allowing the "Utility Discount Rate" to Depend on Uncertainty. In order to analyze the effect of the variance of permanent technology shocks, it is necessary to make a modest generalization of they equations in Section 1: allowing the discount rate to be of the form $\rho - \omega p(k,x)$. Consider the following Bellman equation:

$$\begin{aligned}\rho V(k,\omega) &= \max_x F(k,x,\omega) \\ &= \max_x \{U(k,x) + V_k(k,\omega)A(k,x) \\ &\qquad + \omega \left[\upsilon(k,x) + V(k,\omega)p(k,x) + V_k(k,\omega)\alpha(k,x) + V_{kk}(k,\omega)\frac{\sigma^2(k,x)}{2} \right] \right\}.\end{aligned} \tag{T.10}$$

Despite the addition of the term involving $p(k,x)$, all derivatives of V or F that do not involve ω and are evaluated at $\omega = 0$ can be calculated exactly as detailed in Section 1. Using the envelope theorem,

$$\begin{aligned}\rho V_\omega(k,\omega) &= F_\omega(k, x(k,\omega), \omega) \\ &= \upsilon + Vp + V_k\alpha + V_{kk}\frac{\sigma^2}{2} + V_{k\omega}A + \omega\left[V_\omega p + V_{k\omega}\alpha + V_{kk\omega}\frac{\sigma^2}{2}\right].\end{aligned} \tag{T.11}$$

Setting k to k^*, ω to zero and dividing by ρ, this implies

$$V_\omega(k^*,0) = \frac{\upsilon + Vp + V_k\alpha + V_{kk}\frac{\sigma^2}{2}}{\rho}. \tag{T.12}$$

"Totally" differentiating (T.11) with respect to k and *then* setting k to k^* and ω to zero yields

$$\begin{aligned}\rho V_{k\omega}(k^*,0) = V_{k\omega}\frac{dA}{dk} + V_k p + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2} + \upsilon_k + Vp_k + V_k\alpha_k + V_{kk}\sigma\sigma_k \\ + \left[\frac{dx}{dk}\right]^T [\upsilon_x + Vp_x + V_k\alpha_x + V_{kk}\sigma\sigma_x].\end{aligned} \tag{T.13}$$

Recognizing $\frac{dA}{dk}$ as $-\kappa$, and solving for $V_{k\omega}$,

$$\begin{aligned}V_{k\omega}(k^*,0) = \frac{\upsilon_k + Vp_k + V_k(p+\alpha_k) + V_{kk}(\alpha + \sigma\sigma_k) + V_{kkk}\frac{\sigma^2}{2}}{\rho+\kappa} \\ + \frac{\left[\frac{dx}{dk}\right]^T [\upsilon_x + Vp_x + V_k\alpha_x + V_{kk}\sigma\sigma_x]}{\rho+\kappa}.\end{aligned} \tag{T.14}$$

Separability between control variables involved in f and g on one hand and control variables involved in υ, p, α and σ on the other hand would make the second line on the right-hand side of (T.14) zero by the first order condition for the maximization indicated in (T.10). In the absence of such separability, it is helps one's interpretation to combine partial derivatives with respect to x with the corresponding partial derivatives with respect to k (as is done in Section 4). The derivative of the control variables with respect to the state variable, $\frac{dx}{dk}$ is determined by the certainty model and so can be calculated exactly as before.

The Final Analysis. Equations (T.10)–(T.14) can now be used to analyze the reduced Bellman equation in the neighborhood of $k = k^*$ and $\omega = 0$. The correspondence is as follows.

$$k: \; k$$

$$x_1: \; \begin{bmatrix} c \\ n \end{bmatrix}$$

$$x_2: \; \emptyset$$

$$f: \; \frac{c^{1-\beta}}{1-\beta}e^{(\beta-1)v(n)} = x(c,n)$$

$$g: \; nf(\frac{k}{n}) - \delta k - c - g = A(k,n,c)$$

$$v: \; 0$$

$$p: \; (1-\beta)\left[\mu\pi - \beta\frac{S^2\pi^2}{2}\right]$$

$$\alpha: \; -k[\mu\pi - \beta S^2\pi^2]$$

$$\sigma: \; kS\pi.$$

The value at the certain steady state—which is needed to apply (T.12) and (T.14)—is

$$V(k^*,0) = \frac{x(c^*,n^*)}{\rho} = \frac{(c^*)^{1-\beta}}{(1-\beta)\rho}e^{(\beta-1)v(n)} = \frac{c^*}{(1-\beta)\rho}x_c(c^*,n^*) = \frac{c^*}{(1-\beta)\rho}V_k(k^*,0). \tag{T.15}$$

Applying (T.12),

$$\begin{aligned} V_\omega(k^*,0) &= \rho^{-1}\left\{\left[\mu\pi - \beta\frac{S^2\pi^2}{2}\right](1-\beta)V + \left[\beta S^2\pi^2 - \mu\pi\right]kV_k + \frac{S^2\pi^2}{2}k^2V_{kk}\right\} \qquad (T.16) \\ &= \frac{c^*V_k}{\rho}\left\{\frac{\mu\pi}{\rho} - \beta\frac{S^2\pi^2}{2\rho} + \frac{k^*}{c^*}[\beta S^2\pi^2 - \mu\pi] - \left(\frac{k^*}{c^*}\right)^2(\beta-\xi)(\rho+\kappa)\frac{S^2\pi^2}{2}\right\} \\ &= \frac{c^*V_k}{\rho}\left\{\left[\frac{1}{\rho} - \frac{\theta}{(\rho+\delta)\zeta_c}\right]\mu\pi - \left[\frac{\beta}{\rho} + (\beta-\xi)\frac{\theta^2(\rho+\kappa)}{\zeta_c^2(\rho+\delta)^2} - 2\frac{\beta\theta}{(\rho+\delta)\zeta_c}\right]\frac{S^2\pi^2}{2}\right\}. \end{aligned}$$

One odd but correct fact hinted at by (T.16) is that for values of capital's share θ very close to 1 and other cooperative parameter values such as capital-labor substitution σ large (and only for such parameter values), the technology shock variance S^2 might have a positive effect on welfare near $\omega = 0$. This is because J_{ZZ} does not have to be negative. The first derivative J_Z, which is the effect of labor-augmenting productivity on welfare, is positively related to the level of labor effort n, as can be seen from (T.1). Since an increase in Z can lead to an increase in n through the wage effect (and a very low labor's share $1-\theta$ would make the wealth effect of Z small), an increase in Z can lead to an increase in J_Z, if this change in n overwhelms the direct decline in marginal utility with Z. To bolster this intuitive argument, note that in (T.16), the mechanical reason that the coefficient of S^2 would be a positive value for ξ. If ξ is positive, then the capital starvation caused by an increase in Z is associated with lower leisure and higher labor supply.[8]

Applying (T.14), with the control variables c and n not arguments in any of the uncertainty coefficient functions (the counterparts to v, p, α and σ),

$$\tag{T.17}$$

[8] This effect that tends to make J_{ZZ} positive is analogous to the partial equilibrium fact that variance in factor prices increases expected profit, since the firm can adjust factor quantities to take advantage of the fluctuating factor prices.

$$
\begin{aligned}
\frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} &= \frac{1}{(\rho+\kappa)V_k}\left\{V_k\beta\left[(\beta+1)\frac{S^2\pi^2}{2}-\mu\pi\right]+k^*V_{kk}[(\beta+1)S^2\pi^2-\mu\pi]+(k^*)^2V_{kkk}\frac{S^2\pi^2}{2}\right\} \\
&= \frac{\beta}{\rho+\kappa}\left[(\beta+1)\frac{S^2\pi^2}{2}-\mu\pi\right]-\frac{\theta(\beta-\xi)}{(\rho+\delta)\zeta_c}[(\beta+1)S^2\pi^2-\mu\pi] \\
&+\frac{(\rho+\kappa)^2\theta^2S^2\pi^2}{2(\rho+3\kappa)(\rho+\delta)^2\zeta_c^2}\left\{\beta(\beta+1)-6\beta\xi+3\left(1+\frac{(\beta-1)\zeta_c}{\nu(1-\theta)}\right)\xi^2-\left(2-\frac{2\zeta_c^2}{\nu^2(1-\theta)^2}+\psi\right)\xi^3\right. \\
&\qquad+\frac{\sigma(1-\theta)}{\zeta_c\theta}\left[\left(\frac{1}{\theta}-2+\frac{\sigma}{\theta}(1+\theta+\epsilon)\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^3\right. \\
&\qquad\left.\left.+3\left(\beta-\xi-\frac{\zeta_c}{1-\theta}\xi\right)\left(1-\xi-\frac{\zeta_c}{\nu(1-\theta)}\xi\right)^2\right]\right\}.
\end{aligned}
$$

5. The Interaction of Investment Adjustment Costs with Uncertainty

Let me begin with the certainty model of investment adjustment costs. In order to keep things simple, I will make labor supply exogenous as I introduce investment adjustment costs. Following Hayashi (), the optimization problem is then

$$\max_{c,x}\int_0^\infty u(c)dt,$$

$$s.t.\quad f(k)=c+xk+g \tag{Q.1}$$

$$s.t.\quad \dot{k}=k\phi(x), \tag{Q.2}$$

where

$$x=\frac{i}{k} \tag{Q.3}$$

is the investment rate. The current-value Hamiltonian is

$$\mathcal{H}=u(c)+\mu[f(k)-c-xk-g]+\lambda k\phi(x). \tag{Q.4}$$

The first-order conditions are

$$u'(c)=\mu \tag{Q.5}$$

and $k\mu=k\lambda\phi'(x)$ or

$$\mu=\lambda\phi'(x). \tag{Q.6}$$

The multiplier μ represents the marginal value of *investment*, while λ represents the marginal value of *capital.* The Euler equation is

$$\dot{\lambda}=\rho\lambda-\mathcal{H}_k=\rho\lambda-\mu[f'(k)-x]-\lambda\phi(x). \tag{Q.7}$$

The Steady State. The steady state investment rate can be found by combining the steady state condition $\dot{k}=0$ with (Q.2) to get

$$\phi(x^*)=0 \tag{Q.8}$$

The closest counterpart to the depreciation rate in a model with adjustment costs is the steady-state investment rate. Thus, define

$$\delta = x^*. \tag{Q.9}$$

(When there is steady growth, the investment rate on the steady growth path is more central to the model than the "depreciation rate." See Kimball (1991) for a discussion.)

The normalization implicit in the national income accounts is in normal times (translate "at the steady state") one unit of investment expenditure yields one unit of capital per unit time. Formally, this normalization is

$$\phi'(x^*) = \phi'(\delta) = 1. \tag{Q.10}$$

Combining (Q.10) with the first-order conditions (Q.5) and (Q.6),

$$u'(c^*) = \mu^* = \lambda^*. \tag{Q.11}$$

At the steady state, the Euler equation (Q.6) becomes

$$\lambda^* \rho - \lambda^*(f'(k^*) - x^*) - \lambda\phi(x^*), \tag{Q.12}$$

or

$$f'(k^*) = \rho + \delta. \tag{Q.13}$$

Finally, the material balance condition at the steady state is

$$f(k^*) = c^* + \delta k^* + g^*. \tag{Q.14}$$

Remembering that capital's share is $\theta = \frac{(\rho+\delta)k^*}{y^*}$, dividing (Q.14) through by $f(k^*) = y^*$ yields a relationship between the GDP shares of consumption and government spending:

$$1 = \zeta_c + \frac{\delta\theta}{\rho+\delta} + \zeta_g. \tag{Q.15}$$

Log-Linearizing Around the Steady State.

Let me use a tilde (˜) to denote a deviation from a steady state value and a hacek (ˇ) to denote a logarithmic deviation. Log-linearizing (Q.1) around the steady state yields

$$(\rho + \delta)k^* \check{k} = c^* \check{c} + \delta k^*(\check{k} + \check{x}). \tag{Q.16}$$

Dividing both sides of (Q.16) by $f(k^*) = y^*$ to express things in terms of GDP shares, and collecting the terms involving $\check{k}$,

$$\frac{\rho\theta}{\rho+\delta}\check{k} = \zeta_c \check{c} + \frac{\delta\theta}{\rho+\delta}\check{x}. \tag{Q.17}$$

Log-linearizing the first-order conditions (Q.5) and (Q.6) yields

$$\check{\mu} = -\beta \check{c} \tag{Q.18}$$

with

$$\beta = -\frac{c^* u''(c^*)}{u'(c^*)} \tag{Q.19}$$

and

$$\check{\lambda} - \check{\mu} = j \check{x} \tag{Q.20}$$

with

$$j = -\frac{\delta \phi''(\delta)}{\phi'(\delta)} = -\delta \phi''(\delta). \tag{Q.21}$$

That is, β is the reciprocal of intertemporal substitution (and also relative risk aversion) and j is the adjustment cost elasticity (and also the reciprocal of the elasticity of investment with respect to $q = \frac{\lambda}{\mu}$). Log-linearizing the accumulation equation (Q.2),

$$\begin{aligned} \dot{\check{k}} &= \frac{\dot{\tilde{k}}}{k^*} \\ &= \frac{k^* \phi'(x^*) \tilde{x} + \phi(x^*) \tilde{k}}{k^*} \\ &= \tilde{x} \frac{x^*}{x^*} \\ &= \delta \check{x}. \end{aligned} \tag{Q.22}$$

Finally, log-linearizing the Euler equation yields

$$\begin{aligned} \dot{\check{\lambda}} &= \frac{\dot{\tilde{\lambda}}}{\lambda^*} \\ &= \frac{\rho \tilde{\lambda} - \rho \tilde{\mu} - \mu^* f''(k^*) \tilde{k} + \mu^* \tilde{x} - \phi(\delta) \tilde{\lambda} - \lambda^* \phi'(\delta) \tilde{x}}{\lambda^*} \\ &= \rho(\check{\lambda} - \check{\mu}) - k^* f''(k^*) \check{k} \\ &= \rho j \check{x} + \frac{(1-\theta)(\rho+\delta)}{\sigma} \check{k} \end{aligned} \tag{Q.23}$$

Solving for the Dynamic Equations. The investment rate x is the key to the dynamic equations. Combining (Q.18) and (Q.20) indicates that

$$j \check{x} - \check{\lambda} = \beta \check{c}. \tag{Q.24}$$

Multiplying (Q.17) by β and using (Q.24) to substitute out $\beta \check{c}$,

$$\beta \frac{\rho \theta}{\rho + \delta} \check{k} = \zeta_c (j \check{x} - \check{\lambda}) + \frac{\beta \delta \theta}{\rho + \delta} \check{x}. \tag{Q.25}$$

Solving for $\check{x}$,

$$\check{x} = \frac{\frac{\beta\theta\rho}{\rho+\delta}\check{k} + \zeta_c\check{\lambda}}{j\zeta_c + \frac{\beta\theta\delta}{\rho+\delta}} \tag{Q.25}$$
$$= \frac{\rho}{\delta}(1-\aleph)\check{k} + \frac{\aleph}{j}\check{\lambda},$$

where

$$\aleph = \frac{j\zeta_c}{j\zeta_c + \frac{\beta\theta\delta}{\rho+\delta}} \tag{Q.26}$$

and

$$1-\aleph = \frac{\frac{\beta\theta\delta}{\rho+\delta}}{j\zeta_c + \frac{\beta\theta\delta}{\rho+\delta}}. \tag{Q.27}$$

With (Q.25) in hand, (Q.22) and (Q.23) can be used directly to find the dynamic equations:

$$\dot{\check{k}} = \delta\check{x} = \rho(1-\aleph)\check{k} + \frac{\aleph\delta}{j}\check{\lambda} \tag{Q.28}$$

$$\dot{\check{\lambda}} = \rho j\check{x} + \frac{1-\theta}{\sigma}(\rho+\delta)\check{k} = \left[\frac{\rho^2}{\delta}j(1-\aleph) + \frac{1-\theta}{\sigma}(\rho+\delta)\right]\check{k} + \rho\aleph\check{\lambda}. \tag{Q.29}$$

In matrix form,

$$\begin{bmatrix}\dot{\check{k}}\\ \dot{\check{\lambda}}\end{bmatrix} = \begin{bmatrix}\rho(1-\aleph) & \frac{\delta}{j}\aleph\\ \frac{\rho^2}{\delta}j(1-\aleph) + \frac{1-\theta}{\sigma}(\rho+\delta) & \rho\aleph\end{bmatrix}\begin{bmatrix}\check{k}\\ \check{\lambda}\end{bmatrix}. \tag{Q.30}$$

The trace of this matrix is ρ—as it is for all undistorted one-state variable stationary optimal control models. The determinant is

$$det = \rho^2\aleph(1-\aleph) - \rho^2\aleph(1-\aleph) - \frac{(1-\theta)}{\sigma}(\rho+\delta)\delta\frac{\aleph}{j} \tag{Q.31}$$
$$= -\frac{(1-\theta)}{\sigma}\frac{\delta(\rho+\delta)\zeta_c}{j\zeta_c + \frac{\beta\theta\delta}{\rho+\delta}}$$
$$= -\frac{(1-\theta)}{\sigma}\frac{(\rho+\delta)^2}{j\left(\frac{\rho+\delta}{\delta}\right) + \frac{\beta\theta}{\zeta_c}}.$$

The convergence rate κ is

$$\kappa = \frac{\sqrt{tr^2 - 4\,det} - tr}{2} \tag{Q.32}$$
$$= \frac{1}{2}\sqrt{\rho^2 + 4\frac{(1-\theta)}{\sigma}\frac{(\rho+\delta)^2}{j\left(\frac{\rho+\delta}{\delta}\right) + \frac{\beta\theta}{\zeta_c}}} - \frac{\rho}{2}.$$

Note how (Q.32) reduces to (R.27) when $j = 0$.

Finding $\frac{dx}{dk}$, $\frac{dc}{dk}$ and V_{kk}. By definition, minus the convergence rate is the total derivative at the steady state of the accumulation rate with respect to the state variable:

$$-\kappa = \frac{d}{dk}k\phi(x)|_{k^*} = \phi(x^*) + k^*\phi'(x^*)\frac{dx}{dk} = k^*\frac{dx}{dk}. \tag{Q.33}$$

Thus,

$$\frac{dx}{dk} = -\frac{\kappa}{k^*}. \tag{Q.34}$$

Also,

$$\begin{aligned}\frac{dc}{dk} &= \frac{d}{dk}[f(k) - xk - g]\bigg|_{k^*} \\ &= f'(k^*) - k^*\frac{dx}{dk} - x^* \\ &= \rho + \delta + \kappa - \delta \\ &= \rho + \kappa.\end{aligned} \tag{Q.35}$$

Finally, given the fact that

$$\lambda = \frac{\mu}{\phi'(x)} = \frac{u'(c)}{\phi'(x)}, \tag{Q.36}$$

one can calculate

$$\frac{V_{kk}(k^*,0)}{V_k(k^*,0)} = \frac{d\lambda}{dk}\frac{1}{\lambda}\bigg|_{k^*} = \frac{u''(c^*)}{u'(c^*)}\frac{dc}{dk} - \frac{\phi''(x^*)}{\phi'(x^*)}\frac{dx}{dk} = -\frac{\beta(\rho+\kappa)}{c^*} - \frac{j\kappa}{\delta k^*}. \tag{Q.37}$$

Calculating V_{kkk}. The envelope theorem and its higher order counterpart that I used in deriving (30) has a price (a price well worth paying). It requires that all the control variables be locally unconstrained. To apply (30) directly, one cannot have two control variables such as c and x subject to the constraint (Q.1). Instead, recast the certainty problem in the equivalent form

$$\max_x \int_0^\infty u\left(f(k) - xk - g\right) dt,$$

$$s.t. \quad \dot{k} = k\phi(x),$$

and define

$$U(k,x) = u\left(f(k) - xk - g\right) \tag{Q.38}$$

$$A(k,x) = k\phi(x). \tag{Q.39}$$

Then

$$\begin{aligned}(\rho+3\kappa)\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} &= \frac{U_{kkk} - 3U_{kkx}\frac{\kappa}{k^*} + 3U_{kxx}\frac{\kappa^2}{(k^*)^2} - U_{xxx}\frac{\kappa^3}{(k^*)^3}}{V_k} \\ &\quad + A_{kkk} - 3A_{kkx}\frac{\kappa}{k^*} + 3A_{kxx}\frac{\kappa^2}{(k^*)^2} - A_{xxx}\frac{\kappa^3}{(k^*)^3} \\ &\quad + 3\left(\frac{V_{kk}}{V_k}\right)\left[A_{kk} - 2A_{kx}\frac{\kappa}{k^*} + A_{xx}\frac{\kappa^2}{(k^*)^2}\right].\end{aligned} \tag{Q.40}$$

In order to express the third derivatives in an interpretable way, extend the definitions of β and j to more general arguments and define

$$\upsilon = -\frac{c^*\beta'(c^*)}{\beta(c^*)}. \tag{Q.41}$$

and

$$j = -\frac{x^* j'(x^*)}{j(x^*)}. \tag{Q.42}$$

Then it is not hard to show that

$$u'''(c^*) = \frac{\beta(\beta+1+\upsilon)}{(c^*)^2} u'(c^*) \tag{Q.43}$$

and

$$\phi'''(x^*) = \phi'''(\delta) = \frac{j(j+1+\eta)}{\delta^2}. \tag{Q.44}$$

Calculating at the steady state, and using (Q.19), (Q.21), (Q.43), (Q.44) and (R.60),

$$\begin{aligned} U_{kkk} &= \rho^3 u'''(c^*) + 3\rho f''(k^*) u''(c^*) + f'''(k^*) u'(c^*) \\ U_{kkx} &= -\rho^2 k^* u'''(c^*) - 2\rho u''(c^*) - k^* f''(k^*) u''(c^*) \\ U_{kxx} &= 2k^* u''(c^*) + (k^*)^2 \rho u'''(c^*) \\ U_{xxx} &= -(k^*)^3 u'''(c^*) \end{aligned} \tag{Q.45}$$

$$\begin{aligned} A_{kk} &= 0 \\ A_{kx} &= \phi'(x^*) = 1 \\ A_{xx} &= k^* \phi''(x^*) = -\frac{jk^*}{\delta} \end{aligned} \tag{Q.46}$$

$$\begin{aligned} A_{kkk} &= 0 \\ A_{kkx} &= 0 \\ A_{kxx} &= \phi''(x^*) = -\frac{j}{\delta} \\ A_{xxx} &= k^* \phi'''(x^*) = \frac{k^* j(j+1+\eta)}{\delta^2}. \end{aligned} \tag{Q.47}$$

Subtituting from (Q.45)–(Q.47) into (Q.40),

$$\begin{aligned} (\rho+3\kappa)\frac{V_{kkk}(k^*,0)}{V_k(k^*,0)} &= (\rho+\kappa)^3 \frac{u'''(c^*)}{u'(c^*)} + 3(\rho+\kappa)\left[f''(k^*) + 2\frac{\kappa}{k^*}\right]\frac{u''(c^*)}{u'(c^*)} + f'''(k^*) \\ &\quad + 3\frac{\kappa^2 \phi''(x^*)}{(\rho+3\kappa)(k^*)^2} - \frac{\kappa^3 \phi'''(x^*)}{(k^*)^2} + 3\left(\frac{V_{kk}}{(\rho+3\kappa)V_k}\right)\left[-2\frac{\kappa}{k^*} + \frac{\kappa^2 \phi''(x^*)}{k^*}\right] \\ &= \frac{(\rho+\kappa)^3 \beta(\beta+1+\upsilon)}{(c^*)^2} + 3\frac{\beta(\rho+\kappa)}{c^* k^*}\left[(\rho+\delta)\frac{(1-\theta)}{\sigma} - 2\kappa\right] \\ &\quad + \frac{(\rho+\delta)}{(k^*)^2}\frac{(1-\theta)}{\sigma}\left[1+\theta+\frac{1-2\theta}{\sigma}+\epsilon\right] \\ &\quad - 3\frac{\kappa^2 j}{\delta(k^*)^2} - \frac{\kappa^3 j(j+1+\eta)}{\delta^2 (k^*)^2} + 3\frac{\kappa}{c^* k^*}\left(\beta(\rho+\kappa) + \frac{j\kappa c^*}{\delta k^*}\right)\left(2 + j\frac{\kappa}{\delta}\right) \\ &= (k^*)^{-2}\left\{\frac{(\rho+\kappa)^3 \theta^2}{(\rho+\delta)^2 \zeta_c^2}\beta(\beta+1+\upsilon) + \frac{(1-\theta)}{\sigma}\left[3(\rho+\kappa)\frac{\beta\theta}{\zeta_c} + (\rho+\delta)\left(1+\theta+\frac{1-2\theta}{\sigma}+\epsilon\right)\right]\right. \\ &\quad \left. + \frac{j\kappa^2}{\delta^2}\left[3\delta\left(1+\beta\frac{(\rho+\kappa)\theta}{(\rho+\delta)\zeta_c}\right) + \kappa(2j-1-\eta)\right]\right\}, \end{aligned} \tag{Q.48}$$

since, as before,

$$\frac{c^*}{k^*} = \frac{(\rho+\delta)\zeta_c}{\theta}. \tag{R.26}$$

The Effect of the Variance of Permanent Technology Shocks in the Presence of Investment Adjustment Costs. With

$$u(c) = \frac{c^{1-\beta}}{1-\beta}, \tag{Q.49}$$

(making υ zero) the same symmetry as in Section 4 applies, and the results of Section 4 can be applied without modification except for the expressions for $V_{kk}(k^*,0)$ and $V_{kkk}(k^*,0)$ and the value of the convergence rate κ.

Equations (T.1)–(T.9) remain true. The reduced Bellman equation is

$$\rho V(k,\omega) = \max_x \left\{ \frac{(f(k)-xk-g)^{1-\beta}}{1-\beta} + V_k(k,\omega)[k\phi(x)] \right. \tag{Q.50}$$
$$\left. + \omega \left[\pi\mu\left(V(k,\omega)(1-\beta) - kV_k(k,\omega)\right) + \frac{\pi^2 S^2}{2}[\beta(\beta-1)V(k,\omega) + 2\beta k V_k(k,\omega) + k^2 V_{kk}(k,\omega)]\right]\right\}.$$

Equation (T.15), relating $V(k^*,0)$ to $V_k(k^*,0)$, remains true, since the investment adjustment costs do not affect the steady state and totally inelastic labor supply is allowed as a possibility in Section 4. The first lines of (T.16) and (T.17) also remain true, since none of the uncertainty coefficient functions depend on any of the control variables or involve the adjustment cost in any way. Thus,

$$V_\omega(k^*,0) = \rho^{-1}\left\{\left[\mu\pi - \beta\frac{S^2\pi^2}{2}\right](1-\beta)V + \left[\beta S^2\pi^2 - \mu\pi\right]kV_k + \frac{S^2\pi^2}{2}k^2V_{kk}\right\} \tag{Q.51}$$
$$= \frac{c^*V_k}{\rho}\left\{\left[1-\frac{\rho k^*}{c^*}\right]\frac{\mu\pi}{\rho} - \left[\frac{\beta}{\rho}\left(1-\frac{\rho k^*}{c^*}\right)^2\frac{\kappa k^*}{c^*} + \left(\beta\frac{k^*}{c^*}+\frac{j}{\delta}\right)\right]\frac{S^2\pi^2}{2}\right\}.$$

Here, the variance of the technology shocks definitely has an adverse effect on welfare. Any ambiguity was removed by making labor supply totally inelastic. The investment adjustment cost, as parametrized by j makes technology shock variance more costly. Using the first line of (T.17), but then substituting in from (Q.37) and (Q.48),

$$\frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} = \frac{1}{(\rho+\kappa)V_k}\left\{V_k\beta\left[(\beta+1)\frac{S^2\pi^2}{2} - \mu\pi\right] + k^*V_{kk}[(\beta+1)S^2\pi^2 - \mu\pi] + (k^*)^2V_{kkk}\frac{S^2\pi^2}{2}\right\} \tag{Q.52}$$
$$= \frac{\beta}{\rho+\kappa}\left[(\beta+1)\frac{S^2\pi^2}{2} - \mu\pi\right] - \left(\beta\frac{k^*}{c^*} + \frac{j\kappa}{\delta(\rho+\kappa)}\right)[(\beta+1)S^2\pi^2 - \mu\pi]$$
$$+ \frac{S^2\pi^2}{2(\rho+3\kappa)}\left\{\frac{(\rho+\kappa)^3\theta^2}{(\rho+\delta)^2\zeta_c^2}\beta(\beta+1+\upsilon) + \frac{(1-\theta)}{\sigma}\left[3(\rho+\kappa)\frac{\beta\theta}{\zeta_c} + (\rho+\delta)\left(1+\theta+\frac{1-2\theta}{\sigma}+\epsilon\right)\right]\right.$$
$$\left. + \frac{j\kappa^2}{\delta^2}\left[3\delta\left(1+\beta\frac{(\rho+\kappa)\theta}{(\rho+\delta)\zeta_c}\right) + \kappa(2j-1-\eta)\right]\right\}.$$

Surprisingly, adjustment costs as parametrized by j appear at least as likely interact with the variance of technology shocks in a way that increases investment as one that lowers investment, though a third derivative larger than given by a constant elasticity accumulation function has a positive ψ, which interacts with the variance of technology shocks in a way that lowers investment.

6. Kreps-Porteus Preferences

The Bellman Equation with Kreps-Porteus Preferences. The definitive treatment of Kreps-Porteus preferences in continuous time can be found in Duffie and Epstein (1992). They show that there are an infinite number of ways of expressing the same continuous-time Kreps-Porteus preferences in differential form. I give below an elementary treatment of one way to get a continuous-time Bellman equation for Kreps-Porteus preferences in the form that leaves the expression of the *certainty model* unaltered by the shift from intertemporal von Neumann-Morgenstern to Kreps-Porteus preferences.

In the last few years, there has been considerable interest in preferences over temporal prospects that allow for a clean distinction between risk aversion and the resistance to intertemporal substitution. This is a distinction that is absent for the usual case of additively time-separable expected utility maximization, for which the objective function is

$$\mathcal{V}_t = \mathrm{E}_t \left[\sum_{j=0}^{\infty} e^{-\rho t} U_t(x_t) \right]. \tag{A.1}$$

Generalizing Selden's (1978) specification of two-period preferences, Weil (1988) and Epstein and Zin (1987) independently arrived at a convenient specification of multiperiod preferences that has exactly this property. A reasonably general representation of these preferences can be given by the recursive relationship

$$\mathcal{V}_t = U_t(x_t) + e^{-\rho} \Phi_t^{-1} \big(\mathrm{E}_t \left[\Phi_t(\mathcal{V}_{t+1}) \right] \big), \tag{A.2}$$

where $\mathcal{V}_t$ is the agent's objective function at time t, U_t is a standard single-period utility function, ρ is a discount rate, E_t is an expectation conditional on information available to the agent at time t and Φ_t is a twice-differentiable function with a strictly positive first derivative. If $\Phi_t(\xi) = \xi$ for all t and x, (A.2) reduces to (A.1), but if Φ_t is a nonlinear function for some present or future values of t, then Φ_t causes a divergence between risk aversion and the resistance to intertemporal substitution. In particular, if Φ_t is concave, it tends to increase risk aversion without affecting intertemporal substitution in the absence of risk, while if Φ_t is convex, it tends to reduce risk aversion without affecting intertemporal substitution in the absence of risk. In the last term in (A.2), there is a bending before taking the expectation conditional on information at time t and an unbending afterward that leaves certain quantities unaffected but in general returns a number other than the conditional mean for a random variable—less than the conditional mean if Φ_t is concave, but more than the conditional mean if Φ_t is convex.

Together with an appropriate endpoint condition preventing $\mathcal{V}_t$ from going wild as $t \to \infty$ (or pinning down $\mathcal{V}_T$ in the corresponding finite horizon problem), Equation (A.2) defines the agent's utility function over uncertain prospects. It will be true whether or not the agent optimizes. If the agent optimizes, we can add a bit more structure to this recursion. Recognizing the dependence of $\mathcal{V}_{t+1}$ on x_t through the effect of x_t on the state variables at time $t+1$, we have

$$\mathcal{V}_t = \max_{x_t} \left\{ U_t(x_t) + e^{-\rho} \Phi_t^{-1} \big(\mathrm{E}_t \left[\Phi_t \left(\mathcal{V}_{t+1}(x_t) \right) \right] \big) \right\}, \tag{A.3}$$

In order to find the equation corresponding to (A.3) in continuous time, we can make the length of a period equal to h, instead of equal to 1, and take the limit as the length of a period goes to zero. With the length of a period equal to h, (A.3) becomes

$$\mathcal{V}_t = \max_{x_t} \left\{ hU_t(x_t) + e^{-\rho h} \Phi_t^{-1} \big(\mathrm{E}_t \left[\Phi_t \left(\mathcal{V}_{t+h}(x_t) \right) \right] \big) \right\}, \tag{A.4}$$

To obtain a non-trivial equation with a well defined, non-zero limit as h goes to zero, we can move $\mathcal{V}_t$ over to the right-hand side of this equation and divide by h, obtaining the following:

$$\begin{aligned} 0 &= \frac{1}{h} \max_{x_t} \left\{ hU_t(x_t) + e^{-\rho h} \Phi_t^{-1} \big(\mathrm{E}_t \left[\Phi_t \left(\mathcal{V}_{t+h}(x_t) \right) \right] \big) \right\} - \frac{\mathcal{V}_t}{h} \\ &= \max_{x_t} \left\{ U_t(x_t) + e^{-\rho h} \frac{\Phi_t^{-1} \big(\mathrm{E}_t \left[\Phi_t \left(\mathcal{V}_{t+h}(x_t) \right) \right] \big) - \Phi_t^{-1} \left(\Phi_t(\mathcal{V}_t) \right)}{h} - \frac{1 - e^{-\rho h}}{h} \mathcal{V}_t \right\}. \end{aligned} \tag{A.5}$$

Bringing $\mathcal{V}_t$ under the maximization sign will not affect the validity of the equation as long as $\mathcal{V}_t$ is treated as a constant for the purposes of the maximization. Assuming that the expression in brackets in (A.5) is uniformly continuous in h, the order of limiting and maximization operations can be interchanged, so that by L'Hôpital's rule, the limit of (A.5) as h goes to zero is

$$0 = \max_{x_t} \left\{ U_t(x_t) - \rho \mathcal{V}_t + \frac{1}{\Phi_t'(\mathcal{V}_t)} \frac{\partial}{\partial h} \mathrm{E}_t \, \Phi_t \left(\mathcal{V}_{t+h}(x_t) \right) \Big|_{h=0} \right\} = \max_{x_t} \left\{ U_t(x_t) - \rho \mathcal{V}_t + \frac{1}{\Phi_t'(\mathcal{V}_t)} \mathrm{E}_t \; \frac{d}{dt} \Phi_\tau(\mathcal{V}_t) \Big|_{\tau=t} \right\}, \tag{A.6}$$

Note that the expression $\frac{d}{dt}$ on the right-hand side must not be applied to changes in the quantity $\Phi_t(\mathcal{V}_t)$ due to changes in the function $\Phi_t(\cdot)$ itself. Rearranging,

$$\rho \mathcal{V}_t = \max_{x_t} \left\{ U_t(x_t) + \frac{1}{\Phi_t'(\mathcal{V}_t)} \mathrm{E}_t \; \frac{d}{dt} \Phi_\tau(\mathcal{V}_t) \Big|_{\tau=t} \right\}. \tag{A.7}$$

By retracing the steps taken to get to (A.7) from (A.3), it is easy to verify that simply by deleting the maximization sign from (A.7) yields the continuous-time counterpart to (A.2), which is valid whether or not the agent optimizes. With the maximization sign, (A.7) is a type of Bellman equation, without the maximization sign, it merely specifies the agent's preferences.

Now, to be more specific, suppose that there is just one state variable, k, so that we can write $\mathcal{V}_t = V(k_t, t)$ and $U_t(x) = U(k_t, x, t)$. (Hereafter, I will suppress time subscripts.) Let the evolution of k be described by

$$dk = A(k, x, t)dt + \sigma(k, x, t)dz, \tag{A.8}$$

where dz is a standard Brownian-motion increment. Then by Itô's lemma and the rearranging of one term, (A.7) becomes

$$(A.9)$$

$$\begin{aligned}\rho V(k,t) - V_t(k,t) &= \max_x \left\{ U(t,k,x) + \frac{1}{\Phi_t'(V(k,t))} \mathrm{E}_t \left[\Phi_t'(V(k,t)) dV + \Phi_t''(V(k,t)) \frac{dV^2}{2} \right] \right\} \\ &= \max_x \left\{ U(t,k,x) + \frac{1}{\Phi_t'(V(k,t))} \left[\Phi_t'(V(k,t)) \left(V_k(k,t) A(k,x,t) + V_{xx}(x,t) \frac{\sigma^2(t,x,u)}{2} \right) \right.\right. \\ &\qquad \left.\left. + \Phi_t''(V(k,t)) \frac{V_k(k,t)^2 \sigma^2(t,k,x)}{2} \right] \right\} \\ &= \max_x \left\{ U(t,k,x) + V_k(k,t) A(t,k,x) + \left(V_{kk}(k,t) + \frac{\Phi_t''(V(k,t))}{\Phi_t'(V(k,t))} V_k(k,t)^2 \right) \frac{\sigma^2(t,k,x)}{2} \right\},\end{aligned}$$

where all of the subscripts on V (including the subscript on V_t) denote partial derivatives.

Similarly, if k does not follow a pure diffusion process, but has a Poisson jump component as well, then (A.7) becomes

$$(A.10)$$

$$\begin{aligned}\rho V(k,t) - V_t(k,t) &= \max_x \left\{ U(t,k,x) + \frac{1}{\Phi_t'(V(k,t))} \left[\Phi_t'(V(k,t)) \left(V_k(k,t) A(k,x,t) + V_{kk}(k,t) \frac{\sigma^2(t,k,x)}{2} \right) \right.\right. \\ &\qquad \left.\left. + \ \Phi_t''(V(k,t)) \frac{V_k(k,t)^2 \sigma^2(t,k,x)}{2} + \lambda(t,k,x) \int_\xi \left[\Phi_t(V(\xi,t)) - \Phi_t(V(k,t) \right] dF(\xi,k,x,t) \right] \right\} \\ &= \max_x \left\{ U(t,k,x) + V_k(k,t) A(t,k,x) + \left[V_{kk}(k,t) + \frac{\Phi_t''(V(k,t)}{\Phi_t'(V(k,t))} V_k(k,t)^2 \right] \frac{\sigma^2(t,k,x)}{2} \right. \\ &\qquad \left. + \ \frac{\lambda(t,k,x)}{\Phi_t'(V(k,t))} \int_\xi \left[\Phi_t(V(\xi,t)) - \Phi_t(V(k,t)) \right] dF(\xi,k,x,t) \right\},\end{aligned}$$

where $\lambda(t,k,u)$ is the instantaneous probability of a Poisson jump and $F(\xi,k,x,t)$ is the probability distribution for the new value of k in case of a jump (with ξ as the dummy variable representing the new value of k).

Finally, (A.10) and (A.11) can easily be generalized to allow for many state variables and many control variables, along the lines of Malliaris and Brock's (1982) equations for the many-state-variable, many-control-variable case.

The Effect of Uncertainty in the Neighborhood of a Certain Steady State with Kreps-Porteus Preferences. Based on (A.10), without the jump term, (T.10) can be modified to accomodate Kreps-Porteus preferences as follows:

$$\begin{aligned}\rho V(k,\omega) &= \max_x F(k,x,\omega) \qquad (A.11) \\ &= \max_x \{ U(k,x) + V_k(k,\omega) A(k,x) + \omega \left[v(k,x) + V(k,\omega) p(k,x) + V_k(k,\omega) \alpha(k,x) \right. \\ &\qquad \left. + \ \left(V_{kk}(k,\omega) + \beta(V(k,\omega)) V_k(k,\omega)^2 \right) \frac{\sigma^2(k,u)}{2} \right] \right\},\end{aligned}$$

where

$$\beta(V) = \frac{\Phi''(V)}{\Phi'(V)}. \tag{A.12}$$

Equation (T.12) can be modified to

$$V_\omega(k^*, 0) = \frac{\upsilon + Vp + V_k\alpha + (V_{kk} + \beta(V)V_x^2)\frac{\sigma^2}{2}}{\rho} \tag{A.13}$$

and equation (T.14) can be modified to

$$\begin{aligned} V_{k\omega}(k^*, 0) = {} & \frac{\upsilon_k + Vp_k + V_k(p + \alpha_k) + V_{kk}\alpha + [V_{kk} + \beta(V)V_k^2]\sigma\sigma_k + \left(V_{kkk} + 2\beta(V)V_{kk}V_k + \beta'(V)V_k^3\right)\frac{\sigma^2}{2}}{\rho + \kappa} \\ & + \frac{\left[\frac{dx}{dk}\right]^T [\upsilon_x + Vp_x + V_k\alpha_x + (V_{kk} + \beta(V)V_k^2)\sigma\sigma_x]}{\rho + \kappa}. \end{aligned} \tag{A.14}$$

Again, the second line will be zero if the control variables can be separated into those involved in the certainty coefficient functions and those involved in the uncertainty coefficient functions.

Applications

A Power Φ. Consider

$$\Phi(V) = \frac{[(1-\beta)V]^{\frac{1-r}{1-\beta}}}{1-r}. \tag{A.15}$$

Then

$$\alpha(V) = \frac{\Phi''(V)}{\Phi'(V)} = \frac{\beta - r}{(1-\beta)V} \tag{A.16}$$

and

$$\alpha'(V) = \frac{r - \beta}{(1-\beta)V^2}. \tag{A.17}$$

In all of the models above, in the main body of the paper,

$$V(k^*, 0) = \frac{c^*}{(1-\beta)\rho} V_k(k^*, 0). \tag{T.15}$$

In that case

$$\alpha(V(k^*, 0)) = \frac{(\beta - r)\rho}{c^* V_k(k^*, 0)} \tag{A.18}$$

and

$$\alpha'(V(k^*, 0)) = \frac{(\beta - 1)(\beta - r)\rho^2}{(c^*)^2 V_k(k^*, 0)^2}. \tag{A.19}$$

Thus, at the steady state,

$$\frac{V_{kk} + \alpha(V)V_k^2}{V_k} = \frac{V_{kk}}{V_k} + \frac{\rho(\beta - r)}{c^*} \tag{A.20}$$

and

$$\frac{V_{kkk} + 2\alpha(V)V_{kk}V_k + \alpha'(V)V_k^3}{V_k} = \frac{V_{kkk}}{V_k} + \frac{2\rho(\beta - r)}{c^*}\frac{V_{kk}}{V_k} + \frac{\rho^2(\beta - 1)(\beta - r)}{(c^*)^2}. \tag{A.21}$$

Choosing How Much to Use a Risky Technology in General Equilibrium. The certainty model in Section 2 is totally unaffected by going to Kreps-Porteus preferences. The first-order condition for ϑ, (R.12), needs to be modified to

$$\vartheta = \frac{-V_k(k,\omega)}{V_{kk}(k,\omega) + \alpha(V)V_k^2}\frac{\mu}{S^2}. \tag{A.20}$$

Using (A.18), and the fact that $V_{kk}(k^*,0)$ is the same as in Section 2, the general equilibrium absolute risk aversion is

$$-\frac{V_{kk} + \alpha(V)V_k^2}{V_k} = \frac{(\rho+\kappa)(\beta-\xi) + \rho(r-\beta)}{c^*} = \frac{\rho(r-\xi) + \kappa(\beta-\xi)}{c^*}. \tag{A.21}$$

Applying (A.13), with an eye on (R.13),

$$V_\omega(k^*,0) = \frac{-V_k^2}{(V_{kk} + \alpha V_k^2)}\frac{\mu^2}{2\rho S^2} = \left(\frac{c^* V_k}{\rho}\right)\frac{\mu^2}{2[\rho(r-\xi) + \kappa(\beta-\xi)]S^2}. \tag{A.22}$$

Applying (A.14),

$$\begin{aligned}\frac{(\rho+\kappa)V_{k\omega}(k^*,0)}{V_k(k^*,0)} &= \frac{V_{kk}}{V_k}\vartheta\mu + \frac{V_{kkk} + 2\alpha V_{kk}V_k + \alpha' V_k^3}{V_k}\frac{\vartheta^2\mu^2}{2} \\ &= \frac{\mu^2}{2S^2[V_{kk} + \alpha V_k^2]^2}\left\{-2V_{kk}[V_{kk} + \alpha V_k^2] + V_{kkk}V_k + 2\alpha V_{kk}V_k^2 + \alpha' V_k^4\right\} \\ &= \frac{\mu^2}{2S^2}\left(\frac{V_k}{V_{kk} + \alpha V_k^2}\right)^2\left\{-2\frac{V_{kk}^2}{V_k^2} + \alpha' V_k^2 + \frac{V_{kkk}}{V_k}\right\} \\ &= \frac{\mu^2}{2S^2[\rho(r-\xi) + \kappa(\beta-\xi)]^2}\left\{-2(\rho+\kappa)^2(\beta-\xi)^2 + \rho^2(\beta-1)(\beta-r) + \frac{(c^*)^2 V_{kkk}}{V_k}\right\},\end{aligned} \tag{A.23}$$

where $\frac{V_{kkk}}{V_k}$ can be taken from (R.62).

When the Only Technology Available is Risky. For this case,

$$\begin{aligned}V_\omega &= \frac{(V_{kk} + \alpha V_k^2)(y^*)^2 S^2}{2} \\ &= -[\rho(r-\xi) + \kappa(\beta-\xi)]\frac{S^2}{2\zeta_c^2}\left[\frac{V_k c^*}{\rho}\right].\end{aligned} \tag{A.24}$$

and

$$\begin{aligned}\frac{V_{k\omega}(k^*,0)}{V_k(k^*,0)} &= \frac{V_{kk} + \alpha V_k^2}{(\rho+\kappa)V_k}S^2 y^*\left[\rho + \delta + w^*\frac{dn}{dk}\right] + \frac{V_{kkk} + 2\alpha V_{kk}V_k + \alpha' V_k^3}{(\rho+\kappa)V_k}\frac{S^2(y^*)^2}{2} \\ &= -[\rho(r-\xi) + \kappa(\beta-\xi)]\frac{S^2}{\zeta_c}\left[\frac{\rho+\delta}{\rho+\kappa} - \xi\right] \\ &\quad + \frac{S^2}{2(\rho+\kappa)\zeta_c^2}\left[\frac{(c^*)^2 V_{kkk}}{V_k} - 2\rho(\beta-r)(\beta-\xi)(\rho+\kappa) + \rho^2(\beta-1)(\beta-r)\right].\end{aligned} \tag{A.25}$$

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