# Precautionary Saving and Consumption Smoothing Across Time and Possibilities

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**Abstract**

This paper examines how aversion to risk and aversion to intertemporal substitution determine the strength of the precautionary saving motive in a two-period model with Selden/Kreps-Porteus preferences. For small risks, we derive a measure of the strength of the precautionary saving motive which generalizes the concept of "prudence" introduced by Kimball [12]. For large risks, we show that decreasing absolute risk aversion guarantees that the precautionary saving motive is stronger than risk aversion, regardless of the elasticity of intertemporal substitution. Holding risk preferences fixed, the extent to which the precautionary saving motive is stronger than risk aversion increases with the elasticity of intertemporal substitution. We derive sufficient conditions for a change in risk preferences alone to increase the strength of the precautionary saving motive and for the strength of the precautionary saving motive to decline with wealth. Within the class of constant elasticity of intertemporal substitution, constant-relative risk aversion utility functions, these conditions are also necessary.

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# 1 Introduction

Because it does not distinguish between aversion to risk and aversion to intertemporal substitution, the traditional theory of precautionary saving based on intertemporal expected utility maximization is a framework within which one cannot ask questions that are fundamental to the understanding of consumption in the face of labor income risk. For instance, how does the strength of the precautionary saving motive vary as the elasticity of intertemporal substitution changes, holding risk aversion constant? Or, how does the strength of the precautionary saving motive vary as risk aversion changes, holding the elasticity of intertemporal substitution constant?

Moreover, one might wonder whether some of the results from an intertemporal expected utility framework continue to hold once one distinguishes between aversion to risk and resistance to intertemporal substitution. For instance, does decreasing absolute risk aversion imply that the precautionary saving motive is stronger than risk aversion regardless of the elasticity of intertemporal substitution? And under what conditions does the precautionary saving motive decline with wealth?

To make sense of these questions, and to gain a better grasp on the channels through which precautionary saving may affect the economy,[1] we adopt a representation of preferences based on the Selden and Kreps-Porteus axiomatization, which separates attitudes towards risk and attitudes towards intertemporal substitution.[2]

[1] There has been a considerable resurgence of interest in precautionary saving in the last fifteen years. For some early papers in this period, see for instance Barsky et al. [3], Caballero [4], Kimball [12], Kimball and Mankiw [15], Skinner [21], Weil [24], and Zeldes [25]. The subsequent citation patterns of these early papers indicate continuing interest in precautionary saving.

[2] See, for instance, Selden [19], Selden [20], Kreps and Porteus [16], Hall [8], Farmer [7], Epstein and Zin [6], Weil [23], and Hansen, Sargent, and Tallarini [9].

The existing literature on the theory of precautionary saving under Selden/Kreps-Porteus preferences is not very extensive. Barsky [1] addresses some of the aspects of this theory in relation to rate-of-return risks in a two-period setup, Weil [24] analyzes a parametric infinite-horizon model with mixed isoelastic/constant absolute risk aversion preferences (which do not allow one to look at the consequences of decreasing absolute risk aversion), but there has not been as yet any systematic treatment of precautionary saving under Selden/Kreps-Porteus preferences.

Under intertemporal expected utility maximization, the strength of the precautionary saving motive is not an independent quantity, but is linked to other aspects of risk preferences. In that case, the absolute prudence $-v'''/v''$ of a von-Neumann Morgenstern second-period utility function $v$ measures the strength of the precautionary saving motive (see Kimball [12]]), and there is an identity linking prudence to risk aversion under additively time- and state-separable utility:

$$-\frac{v'''(x)}{v''(x)} = a(x) - \frac{a'(x)}{a(x)}, \tag{1.1}$$

where

$$a(x) = -\frac{v''(x)}{v'(x)} \tag{1.2}$$

is the Arrow-Pratt measure of absolute risk aversion. Similarly, the coefficient of relative prudence $-xv'''(x)/v''(x)$ satisfies

$$-\frac{xv'''(x)}{v''(x)} = \gamma(x) + \varepsilon(x), \tag{1.3}$$

where

$$\gamma(x) = -\frac{xv''(x)}{v'(x)} = xa(x) \tag{1.4}$$

is relative risk aversion, and

$$\varepsilon(x) = -\frac{xa'(x)}{a(x)} \tag{1.5}$$

is the elasticity of absolute risk tolerance,[3] which approximates the wealth elasticity of risky investment.

The purpose of this paper is to determine what links exist, in the more general Selden/Kreps-Porteus framework that allows for risk preferences and intertemporal substitution to be varied independently from each other, between the strength of the precautionary saving motive, risk aversion, and intertemporal substitution.

In Section 2, we set up the model. In Section 3 we derive a local measure of the precautionary saving motive for small risks. Section 4 deals with large risks: it performs various comparative statics experiments to provide answers to the questions we asked in our first two paragraphs. The conclusion discusses the implications of our main results.

## 2 Setup

This section describes our basic model, and characterizes optimal consumption and saving decisions.

[3] Absolute risk tolerance is defined as the reciprocal of absolute risk aversion: $1/a(x)$.

## 2.1 The model

We use essentially the same two-period model of the consumption-saving decision as in Kimball [12], except for departing from the assumption of intertemporal expected utility maximization.[4] We assume that the agent can freely borrow and lend at a fixed risk-free rate, and that the constraint that an agent cannot borrow against more than the minimum value of human wealth is not binding at the end of the first period. Since the interest rate is exogenously given, all magnitudes can be represented in present-value terms, so that, without loss of generality, the real risk-free rate can be assumed to be zero. We also assume that labor supply is inelastic, so that labor income can be treated like manna from heaven. Finally, we assume that preferences are additively time-separable.

The preferences of our agent can be represented in two equivalent ways. *First*, there is the Selden OCE representation,

$$u(c_1) + U(v^{-1}(\mathbf{E}\, v(\tilde{c}_2))) = u(c_1) + U(M(\tilde{c}_2)),$$

where $c_1$ and $c_2$ are first- and second-period consumption, $u$ is the first-period utility function, $U$ is the second-period utility function for the certainty equivalent of random second-period consumption (computed according to the atemporal von Neumann-Morgenstern utility function $v$),[5] $\mathbf{E}$ is an expectation conditional on all

[4] Because there is enough to be said about the two-period case, we leave comprehensive treatment of the multi-period case to future research. The results derived by Weil [24] and van der Ploeg [22] for particular multi-period parametric versions of Kreps-Porteus preferences suggest that the two-period results of this paper have natural generalizations to multi-period settings. In the case of intertemporal expected utility maximization, Kimball [11] extends two-period results about precautionary saving effects decreasing with wealth to multi-period settings. As in that paper, characterization of the multiperiod case must rely on the tools developed in the two-period case. In addition, the two-period model highlights what cannot possibly be proven for the multiperiod case.

[5] Since $U$ is a general function, any time discounting can be embedded in the functional form of

information available during the first period, and $M$ is the certainty equivalent operator associated with $v$:

$$M(\tilde{c}_2) = v^{-1}(\mathbf{E}\, v(\tilde{c}_2)).$$

According to this representation, the utility our consumer derives from the consumption lottery $(c_1, \tilde{c}_2)$ is the sum of the felicity provided by $c_1$ and the felicity provided by the certainty equivalent $M(\tilde{c}_2)$ of $\tilde{c}_2$. Obviously, $v$ plays no role, and $M$ is an identity, under certainty. These functions thus capture pure risk preferences.

*Second*, there is the Kreps-Porteus representation,

$$u(c_1) + \phi(\mathbf{E}\, v(\tilde{c}_2)),$$

where nonlinearity of the function $\phi$ indicates departure from intertemporal expected utility maximization. This formulation expresses total utility as the nonlinear aggregate of current felicity and expected future felicity.[6]

These two representations are equivalent as long as $v$ is a continuous, monotonically increasing function, so that $M(\tilde{c}_2)$ is well defined whenever $\mathbf{E}\, v(\tilde{c}_2)$ is well defined. The link between the two representations is that

$$\phi(\nu) = U(v^{-1}(\nu)). \tag{2.1}$$

Thus, as can be seen by straightforward differentiation, $\phi$ is concave if $-U''/U' \geq$

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$U$.

[6] Thus, when we refer in this paper to the Kreps-Porteus representation or to Kreps-Porteus preferences, we point to the *specific form of the aggregator function* between current and future consumption proposed by Kreps and Porteus [16] in their multi-period axiomatization.

$-v''/v'$, while $\phi$ is convex if $-U''/U' \leq -v''/v'$. We will mainly use the Selden representation—which is more intuitive—but, when more convenient mathematically, we use the Kreps-Porteus representation.

## 2.2 Optimal consumption and saving

Our consumer solves the following problem:

$$\max_x \; u(w-x) + U(M(x+\tilde{y})), \tag{2.2}$$

where $w$ is the sum of initial wealth, first-period income, and the mean of second-period income, $x$ is "saving" out of this whole sum, and $\tilde{y}$ is the deviation of second-period income from its mean.

The first-order condition for the optimal level of saving $x$ is

$$u'(w-x) = U'(M(x+\tilde{y}))M'(x+\tilde{y}), \tag{2.3}$$

where $M'$ is defined by

$$M'(x+\tilde{y}) = \frac{\partial}{\partial x} M(x+\tilde{y}). \tag{2.4}$$

To guarantee that the solution to (2.3) is uniquely determined, we would like the marginal utility of saving,

$$U'(M(x+\tilde{y}))M'(x+\tilde{y}),$$

to be a decreasing function of $x$. In the main body of the paper, we will simply assume that this condition holds; it is equivalent to the reasonable assumption that first-period consumption is a normal good. Appendix A gives a deeper treatment, proving that the marginal utility of saving decreases under plausible restrictions on preferences.

Equation (2.3) and the assumption of a decreasing marginal utility of saving imply that the uncertainty represented by $\tilde{y}$ will cause additional saving if

$$U'(M(x+\tilde{y}))M'(x+\tilde{y}) > U'(x), \tag{2.5}$$

that is, if the risk $\tilde{y}$ raises the marginal utility of saving.

As in Kimball [12], we can study the strength of precautionary saving effects by looking at the size of the precautionary premium $\theta^*$ needed to compensate for the effect of the risk $\tilde{y}$ on the marginal utility of saving. The precautionary premium $\theta^*$ is the solution to the equation

$$U'(M(x+\theta^*+\tilde{y}))M'(x+\theta^*+\tilde{y}) = U'(x). \tag{2.6}$$

We call $\theta^*$ the *compensating Kreps-Porteus precautionary premium*. It is analogous to the compensating von Neumann-Morgenstern precautionary premium $\psi^*$ that is defined[7] as the solution to

$$\mathbf{E}\, v'(x+\psi^*+\tilde{y}) = v'(x). \tag{2.7}$$

As illustrated in Figure 1, the Kreps-Porteus precautionary premium $\theta^*$, which obviously depends (among other things) on $x$, measures at each point the right-

[7] See Kimball [12].

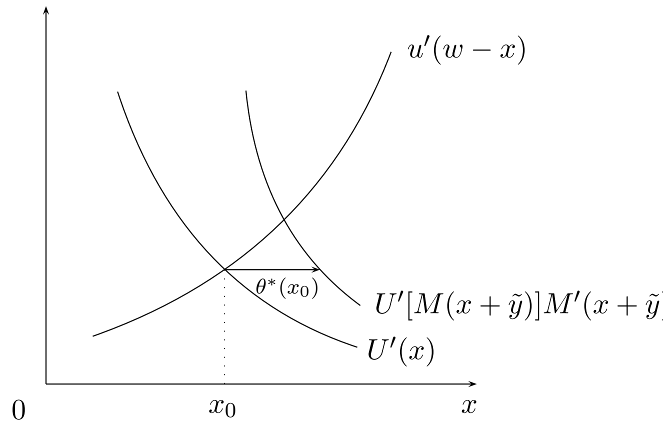


Figure 1: Precautionary saving

ward shift due to the risk $\tilde{y}$ of the marginal utility of saving curve. It enables us to isolate the effect of uncertainty from the other (purely intertemporal) factors that determine optimal savings under certainty.

# 3 Small risks: a local measure of the strength of the precautionary saving motive

In appendix B, we prove that, for a small risk $\tilde{y}$ with mean zero and variance $\sigma^2$,

$$\theta^*(x) = a(x)[1 + s(x)\varepsilon(x)]\frac{\sigma^2}{2} + o(\sigma^2), \tag{3.1}$$

where

$$s(x) = \frac{-U'(x)}{xU''(x)} \tag{3.2}$$

denotes the elasticity of intertemporal substitution for the second period utility function $U(x)$, $o(\sigma^2)$ collects terms going to zero faster than $\sigma^2$, $a(x)$ is the absolute risk aversion of $v$ defined in (1.2), and $\varepsilon(x)$ denotes the elasticity of absolute risk tolerance defined in (1.5).

Therefore, the local counterpart for Kreps-Porteus preferences to the concept of absolute prudence defined by Kimball [12] for intertemporal expected utility preferences is

$$a(x)[1 + s(x)\varepsilon(x)].$$

Similarly, the local counterpart for Kreps-Porteus preferences to relative prudence is

$$\mathcal{P}(x) = \gamma(x)[1 + s(x)\varepsilon(x)], \tag{3.3}$$

where $\gamma(x) = xa(x)$ is relative risk aversion as above. Therefore, *in the more general framework of Kreps-Porteus preferences, the strength of the precautionary saving motive is determined both by attitudes towards risk* and *attitudes towards intertemporal substitution.*

Three important special cases should be noted:

- For intertemporal expected utility maximization, $s(x) = 1/\gamma(x)$, so that $\mathcal{P}(x) = \gamma(x) + \varepsilon(x)$—which is the expression given in (1.3).
- For constant relative risk aversion, $\gamma(x) = \gamma$ and $\varepsilon(x) = 1$, so that $\mathcal{P}(x) = \gamma[1 + s(x)]$.
- For constant relative risk aversion $\gamma(x) = \gamma$, *and* constant (but in general distinct) elasticity of intertemporal substitution[8] $s(x) = s$, $\mathcal{P}(x) = \gamma[1 + s]$.

[8] This special case characterizes the framework used by Selden [20].

From (3.3), the local condition for *positive* precautionary saving is simply

$$\varepsilon(x) \geq -\frac{1}{s(x)}, \tag{3.4}$$

or, by calculating $\varepsilon(x)$ from (1.5) and moving one piece to the right-hand side of the equation:

$$\frac{-xv'''(x)}{v''(x)} \geq \gamma(x) - \rho(x), \tag{3.5}$$

where

$$\rho(x) = 1/s(x)$$

denotes the *resistance to intertemporal substitution*. The right-hand side of (3.5) is zero under intertemporal expected utility maximization, in which case (3.5) reduces to the familiar condition $v'''(x) \geq 0$.

Intriguingly, equation (3.5) shows that when one departs from intertemporal expected utility maximization, quadratic risk preferences do not in general lead to the absence of precautionary saving effects that goes under the name of "certainty equivalence."

Though the local measure $\mathcal{P}(x)$ cannot be used directly to establish global results, equation (3.1) does suggests several principles which are valid globally, as the next section shows. *First*, a change in risk preferences $v$ (holding the outer intertemporal utility function $U$ fixed) which raises both risk aversion and its rate of decline tends to strengthen the precautionary saving motive. *Second*, decreasing absolute risk aversion (which implies $\varepsilon(x) \geq 0$) guarantees that the precautionary saving motive is stronger than risk aversion. *Third*, raising intertemporal substitution $s(x)$, thus lowering the resistance to intertemporal substitution $\rho(x) = 1/s(x)$, while holding risk preferences constant widens the gap between the strength of the precautionary saving motive and risk aversion. In particular, when absolute risk aversion is decreasing, lowering the resistance to intertemporal substitution from the intertemporal utility maximizing level, so that $\rho(x)$ falls below $\gamma(x)$, makes the precautionary saving motive stronger than it is under intertemporal expected utility maximization. Conversely, a resistance to intertemporal substitution $\rho(x)$ greater than relative risk aversion $\gamma(x)$ makes the precautionary saving motive weaker than under intertemporal expected utility maximization.[9]

# 4 Large risks

## 4.1 Precautionary saving and decreasing absolute risk aversion

In the case of intertemporal expected utility maximization, Drèze and Modigliani [5] prove that decreasing absolute risk aversion leads to a precautionary saving motive stronger than risk aversion. Kimball [14] shows that there is a fundamental economic logic behind this result: decreasing absolute risk aversion means that greater saving makes it more desirable to take on a compensated risk. But the other side of such a complementarity between saving and a compensated risk is that a compensated risk makes saving more attractive.

This argument follows not from any model-specific features but rather from the logic of complementarity itself. To understand this claim, consider an arbitrary, concave indirect utility function of saving and the presence or absence of a

[9] There are several further minor but interesting propositions suggested by (3.1) which are valid globally. First, *constant absolute risk aversion* implies that $\epsilon = 0$, so that the precautionary saving motive and risk aversion are of exactly equal strength, regardless of intertemporal substitution. Second, two easy sets of sufficient conditions for a positive precautionary saving motive that are valid globally are (a) decreasing absolute risk aversion or (b) $v'''(\cdot) \geq 0$ and $\rho(x) \geq \gamma(x)$.

compensated risk: $J(x, \lambda)$, where $x$ is saving and $\lambda$ is 1 in the presence of the compensated risk $\tilde{y} + \pi^*$ and zero in its absence.[10] The definition of a compensated risk insures that

$$J(x, 1) - J(x, 0) = 0,$$

i.e., that the consumer is indifferent to the presence or absence of the risk $\tilde{y} + \pi^*$.

In the most fundamental sense (i.e., independently of any particular model), the property of decreasing absolute risk aversion means that a risk to which an agent is indifferent at one level of saved wealth will become desirable at a slightly higher level of saved wealth. Thus,

$$\frac{\partial}{\partial x}[J(x, 1) - J(x, 0)] = J_x(x, 1) - J_x(x, 0) \geq 0$$

if absolute risk aversion is decreasing. But this implies that if the first-order condition for optimal saving is satisfied in the absence of the compensated risk—i.e., if $J_x(x, 0) = 0$, then

$$J_x(x, 1) \geq 0,$$

implying that a compensated risk makes the agent want to save more. In other words, the risk premium needed to compensate the agent for the risk he or she is bearing does not eliminate the total effect of the (compensated) risk on saving. This is what we mean when we say that the precautionary saving motive is stronger than risk aversion.

We now examine more formally how this general logic manifests itself in the specific framework we are studying.

[10] In the Kreps-Porteus model, for instance, $J(x, \lambda) = u(w - x) + U[M(x + \lambda(\tilde{y} + \pi^*))]$. See equation (4.1) below for a precise definition of the compensating risk premium $\pi^*$.

### 4.1.1 Intertemporal expected utility maximization: a reminder

Under intertemporal expected utility maximization, decreasing absolute risk aversion implies that if $\pi^*$ is the compensating *risk* premium for $\tilde{y}$, which satisfies

$$\mathbf{E}\, v(x + \pi^* + \tilde{y}) = v(x), \tag{4.1}$$

then

$$\mathbf{E}\, v(x + \delta + \pi^* + \tilde{y}) \geq v(x + \delta) \tag{4.2}$$

for any $\delta \geq 0$.[11] Equation (4.1) and inequality (4.2) together imply that the derivative of the left-hand side of (4.2) at $\delta = 0$ must be greater than the derivative of the right-hand side of (4.2) at $\delta = 0$:

$$\mathbf{E}\, v'(x + \pi^* + \tilde{y}) \geq v'(x). \tag{4.3}$$

Equation (2.7) defining the compensating von Neumann-Morgenstern precautionary premium $\psi^*$ differs from (4.3) only by having $\psi^*$ in place of $\pi^*$ and by holding with equality. Therefore, since $v'$ is a decreasing function, $\psi^* \geq \pi^*$. In words, (4.3) says that a risk compensated by $\pi^*$, so that the agent is indifferent to the combination $\pi^* + \tilde{y}$, still raises expected marginal utility. Therefore, the von Neumann-Morgenstern precautionary premium $\psi^*$— which brings expected marginal utility back down to what it was under certainty—must be greater than $\pi^*$.

[11] In other words, the risk premium $\pi^*$ that compensates for the effect of the risk $\tilde{y}$ on utility when wealth is $x$ more than compensates for the effect of risk $\tilde{y}$ when wealth is $x + \delta$.

#### 4.1.2 Kreps-Porteus preferences

The same economic logic applies to the Kreps-Porteus case. Any property that ensures that extra saving makes compensated risks more attractive guarantees that extra compensated risk makes saving more desirable. Formally, decreasing absolute risk aversion guarantees that $M'(\cdot) \geq 1$, since differentiating the identity

$$M(x + \pi^*(x) + \tilde{y}) \equiv x \tag{4.4}$$

yields

$$(1 + \pi^{*\prime}(x))M'(x + \pi^*(x) + \tilde{y}) = 1. \tag{4.5}$$

Decreasing absolute risk aversion implies that $\pi^{*\prime}(x) \leq 0$. (Monotonicity of $v$ implies that $1 + \pi^{*\prime}(x) \geq 0$.) Thus,

$$U'(M(x + \pi^* + \tilde{y}))\, M'(x + \pi^* + \tilde{y}) = U'(x)\, M'(x + \pi^* + \tilde{y}) \geq U'(x), \tag{4.6}$$

where the equality on the left follows from (4.4),[12] and the inequality on the right follows from $M'(\cdot) \geq 1$. Combined with the assumption of a decreasing marginal utility of saving, (4.6) implies $\theta^* \geq \pi^*$. We have thus proved:

**Proposition 1** *Assuming a decreasing marginal utility of saving, if risk preferences $v$ exhibit decreasing absolute risk aversion, then the Kreps-Porteus precautionary premium is always greater than the risk premium ($\theta^* \geq \pi^*$).*

**Remark.** By reversing the direction of the appropriate inequalities above, one can see that globally *increasing* absolute risk aversion guarantees that $\theta^* \leq \pi^*$, just as

[12] Note that the risk premium $\pi^*$ is the same as it would be under intertemporal expected utility maximization.

globally *decreasing* absolute risk aversion guarantees that $\theta^* \geq \pi^*$. *If absolute risk aversion is constant, both inequalities must hold, implying that* $\theta^*(x) \equiv \pi^*(x) \equiv \text{constant}$,[13] *regardless of the form of the intertemporal utility function* $U$.

#### 4.1.3 Patent increases in risk

Proposition 1 can be extended to patent increases in risk, which, as defined by Kimball [13], are increases in risk for which the risk premium increases with risk aversion, at least when a utility function with decreasing absolute risk aversion is involved. The set of patent increases in risk includes increases in the scale of a risk coupled with any change of location, and the addition of any independent risk, but does not include all mean-preserving spreads.[14]

The compensating risk premium $\Pi^*(x)$ for the *difference* between two risks $\tilde{Y}$ and $\tilde{y}$ is defined by

$$\mathbf{E}\, v(x + \Pi^*(x) + \tilde{Y}) = \mathbf{E}\, v(x + \tilde{y}), \tag{4.7}$$

or equivalently, by

$$M(x + \Pi^*(x) + \tilde{Y}) = M(x + \tilde{y}). \tag{4.8}$$

If $\tilde{Y}$ is a patently greater risk than $\tilde{y}$, then decreasing absolute risk aversion guarantees that $\Pi^*(x)$ is a decreasing function of $x$. (Given decreasing absolute risk

[13] Weil [24] exploits the fact that $\theta^*(x) \equiv \pi^*(x)$ under constant absolute risk aversion to solve a multiperiod saving problem with Kreps-Porteus utility.

[14] As noted in Kimball [13], "... the results of Pratt [18] make it clear that $\tilde{X}$ is patently more risky than $\tilde{x}$ if $\tilde{X}$ can be obtained from $\tilde{x}$ by adding to $\tilde{x}$ a random variable $\nu$ that is positively related to $\tilde{x}$ in the sense of having a distribution conditional on $\tilde{x}$ which improves according to third-order stochastic dominance for higher realizations of $\tilde{x}$. This sufficient condition for patently greater risk includes as polar special cases $\nu$ perfectly correlated with $\tilde{x}$—which makes the movement from $\tilde{x}$ to $\tilde{X}$ a simple change of location and increase in scale—and $\nu$ statistically independent of $\tilde{x}$."

aversion, reducing $x$ increases risk aversion and therefore increases $\Pi^*$ by the definition of a patently greater risk.) Differentiating equation (4.8) with respect to $x$, one finds that

$$\begin{aligned} M'(x+\tilde{y}) &= [1+\Pi^{*\prime}(x)]\, M'(x+\Pi^*(x)+\tilde{Y}) \\ &\leq M'(x+\Pi^*(x)+\tilde{Y}), \end{aligned} \tag{4.9}$$

where the inequality on the second line follows from $\Pi^{*\prime}(x) \leq 0$. In combination, equations (4.8) and (4.9) imply that

$$U'(M(x+\Pi^*(x)+\tilde{Y}))\, M'(x+\Pi^*(x)+\tilde{Y}) \geq U'(M(x+\tilde{y}))\, M'(x+\tilde{y}). \tag{4.10}$$

Defining the compensating Kreps-Porteus precautionary premium $\Theta^*(x)$ for the difference between the two risks $\tilde{Y}$ and $\tilde{y}$ by

$$U'(M(x+\Theta^*(x)+\tilde{Y}))\, M'(x+\Theta^*(x)+\tilde{Y}) = U'(M(x+\tilde{y}))\, M'(x+\tilde{y}), \tag{4.11}$$

a decreasing marginal utility of saving implies that $\Theta^*(x) \geq \Pi^*(x)$.

**Proposition 2 (extension of Proposition 1)** *Assuming a decreasing marginal utility of saving, if the inner interpossibility function $v$ exhibits decreasing absolute risk aversion and $\tilde{Y}$ is patently riskier than $\tilde{y}$, then the Kreps-Porteus precautionary premium for the difference between $\tilde{Y}$ and $\tilde{y}$ is always greater than the risk premium for the difference between $\tilde{Y}$ and $\tilde{y}$—i.e., $\Theta^* \geq \Pi^*$.*

*Proof:* See above. □

## 4.2 Comparative statics

We now examine how the strength of the precautionary saving motive is affected by changes in intertemporal substitution, risk aversion, and wealth.

### 4.2.1 Intertemporal substitution

Before analyzing the effect of intertemporal substitution on the size of the precautionary premium in general, it is instructive to examine how departures from intertemporal expected utility maximization affect the precautionary premium:

**Proposition 3** *Assuming a decreasing marginal utility of saving for the Kreps-Porteus utility function $u(c_1) + \phi(\mathbf{E}\, v(\tilde{c}_2))$, if the von Neumann-Morgenstern precautionary premium $\psi^*$ is greater than the risk premium $\pi^*$ (as it will be if absolute risk aversion is decreasing), then the Kreps-Porteus precautionary premium $\theta^* \leq \psi^*$ when the function $\phi(\cdot)$ is concave, but $\theta^* \geq \psi^*$ when $\phi(\cdot)$ is convex. If $\psi^* \leq \pi^*$ (as it will be if absolute risk aversion is increasing), then $\theta^* \geq \psi^*$ when $\phi(\cdot)$ is concave, but $\theta^* \leq \psi^*$ when $\phi(\cdot)$ is convex.*

*Proof:* Assume first that $\psi^* \geq \pi^*$ and $\phi(\cdot)$ is concave. Then

$$\begin{aligned} \phi'(\mathbf{E}\, v(x+\psi^*+\tilde{y}))\mathbf{E}\, v'(x+\psi^*+\tilde{y}) &= \phi'(\mathbf{E}\, v(x+\psi^*+\tilde{y}))v'(x) \\ &\leq \phi'(v(x))v'(x). \end{aligned} \tag{4.12}$$

The equality on the first line follows from the definition of $\psi^*$, and the inequality on the second line follows from $\psi^* \geq \pi^*$, the monotonicity of $v$ and the concavity of $\phi(\cdot)$, which makes $\phi'(\cdot)$ a decreasing function. Together with a decreasing marginal utility of saving, (4.12) implies that

$$\theta^* \leq \psi^*. \tag{4.13}$$

Either the convexity of $\phi(\cdot)$ or the inequality $\psi^* \leq \pi^*$ would reverse the direction of the inequalities in both (4.12) and (4.13), while convexity of $\phi(\cdot)$ and $\psi^* \leq \pi^*$ together leave the direction of the inequalities unchanged. $\square$

The results of proposition 3 are summarized in Table 1.

| | $\phi$ concave ($\rho > \gamma$) | $\phi$ convex ($\rho < \gamma$) |
|---|---|---|
| $v$ DARA | $\pi^* \leq \theta^* \leq \psi^*$ | $\pi^* \leq \psi^* \leq \theta^*$ |
| $v$ IARA | $\psi^* \leq \theta^* \leq \pi^*$ | $\theta^* \leq \psi^* \leq \pi^*$ |

Table 1: Summary of Proposition 3

Concavity of $\phi$ brings the Kreps-Porteus precautionary premium $\theta^*$ closer than the von Neumann-Morgenstern precautionary premium $\psi^*$ to the risk premium $\pi^*$. Convexity of $\phi$ pushes $\theta^*$ further than $\psi^*$ from $\pi^*$. For example, for quadratic utility—which exhibits IARA and for which $\psi^* = 0$— the precautionary premium $\theta^*$ is positive if $\phi$ is concave and negative if $\phi$ is convex.

Since concavity of $\phi$ is equivalent to the elasticity of intertemporal substitution $s(x)$ below relative risk tolerance $1/\gamma(x)$, and convexity of $\phi$ is equivalent to the elasticity of intertemporal substitution $s(x)$ above relative risk tolerance $1/\gamma(x)$, one can see that the pattern is one of greater intertemporal substitution increasing the distance between the risk premium $\pi^*$ and the Kreps-Porteus precautionary premium $\theta^*$.

Proposition 4 indicates that, holding risk preferences fixed, higher intertemporal substitution (or equivalently, a lower resistance to intertemporal substitution) leads quite generally to a Kreps-Porteus precautionary premium that is further from the risk premium:

**Proposition 4** *If, for two Kreps-Porteus utility functions* $u_i(c_1) + U_i(M(\tilde{c}_2))$, $i = 1, 2$, *with the same risk preferences,*

**(a)** *the optimal amount of saving under certainty is the same,*

**(b)** *the marginal utility of saving is decreasing for both utility functions, and*

**(c)** $U_2$ *has greater resistance to intertemporal substitution than* $U_1$*—that is,* $\frac{-U_2''(x)}{U_2'(x)} \geq \frac{-U_1''(x)}{U_1'(x)}$ *for all* $x$,

*then* $|\theta_2^* - \pi^*| \leq |\theta_1^* - \pi^*|$, *where* $\theta_i$ *is the Kreps-Porteus precautionary premium for utility function* $i$ *and* $\pi^*$ *is the risk premium for both utility functions. Also,* $\theta_2^* - \pi^*$ *and* $\theta_1^* - \pi^*$ *have the same sign.*

*Proof:* See Appendix C. □

**Remark.** Using the machinery for the proof of Proposition 4, Appendix C also provides graphical intuition for why intertemporal substitution, risk aversion and the rate of decline of risk aversion matter in the way they do for the strength of the precautionary saving motive.

#### 4.2.2 Risk aversion

Next, we examine how the strength of the precautionary saving motive is affected by changes in risk aversion.

Given two inner interpossibility functions $v_1$ and $v_2$, define

$$M_i(x + \tilde{y}) = v_i^{-1}(\mathbf{E}\, v_i(x + \tilde{y})) \tag{4.14}$$

for $i = 1, 2$, and define $\theta_1^*$ and $\theta_2^*$ by appropriately subscripted versions of (2.6). Then one can state the following lemma:

**Lemma 1** *Assuming a decreasing marginal utility of saving and a concave intertemporal utility function $U$ that is held fixed when risk preferences are altered, if*

$$U'(M_2(\xi + \tilde{y}))\, M_2'(\xi + \tilde{y}) \geq U'(M_1(\xi + \tilde{y}))\, M_1'(\xi + \tilde{y})$$

*for all $\xi$, then $\theta_2^*(x) \geq \theta_1^*(x)$.*

*Proof:* Letting $\xi = x + \theta_1^*(x)$,

$$\begin{aligned} U'(M_2(x+\theta_1^*(x) + \tilde{y}))\, M_2'(x + \theta_1^*(x) + \tilde{y}) \\ \geq U'(M_1(x + \theta_1^*(x) + \tilde{y}))\, M_1'(x + \theta_1^*(x) + \tilde{y}) \\ = U'(x). \end{aligned} \tag{4.15}$$

Thus, the Kreps-Porteus precautionary premium $\theta_1^*$ for the first set of risk preferences is insufficient under the second set of risk preferences to bring the marginal utility of saving back down to $U'(x)$. By the assumption of a decreasing marginal utility of saving, this means that $\theta_2^*$ must be greater than $\theta_1^*$ to bring the marginal utility of saving all the way back down to $U'(x)$. □

This lemma has an obvious corollary:

**Corollary 1** *Assuming a decreasing marginal utility of saving and a concave outer intertemporal utility function $U$ that is held fixed when risk preferences are altered, if $M_2(\xi + \tilde{y}) \leq M_1(\xi + \tilde{y})$ and $M_2'(\xi + \tilde{y}) \geq M_1'(\xi + \tilde{y})$ at $\xi = x + \theta_1^*(x)$, then $\theta_2^*(x) \geq \theta_1^*(x)$.*

The corollary is less helpful than one might hope because it is not easy to find suitable conditions to guarantee that $M_2' \geq M_1'$. To see why it is not easy, consider the case of a two point risk and two interpossibility functions that both go from infinite positive risk aversion at zero, to zero risk aversion at infinity. Then, at zero, both certainty equivalents will converge to the lower outcome of the two-point risk, while at infinity, both certainty equivalents will converge to the mean value of the two-point risk. Since the certainty equivalents start and end at the same values, one cannot always have a higher slope than the other. There is, though, at least one case in which it is easy to establish that $M_2' \geq M_1'$: when $v_1$ has increasing (or constant) absolute risk aversion, so that $M_1' \leq 1$, while $v_2$ has decreasing (or constant) absolute risk aversion, so that $M_2' \geq 1$. This result is equivalent to Proposition 3 and Table 1.

In assessing the results that follow, we are especially interested in their application to utility functions of the constant elasticity of intertemporal substitution, constant relative risk aversion (CEIS-CRRA) class.

The next proposition focuses on the effect of risk aversion on the strength of the precautionary saving motive. Within the CEIS-CRRA class, it takes care of the case where the resistance to intertemporal substitution (the reciprocal of the elasticity of intertemporal substitution) is greater than risk aversion.

**Proposition 5** *Suppose two Kreps-Porteus utility functions share the same intertemporal function $U$ but have different inner interpossibility functions $v_1$ and $v_2$. If*

**(a)** *the resistance to intertemporal substitution is greater than the risk aversion of $v_1$ (that is, $U$ is an increasing, concave transformation of $v_1$),*

**(b)** *$v_2$ is globally more risk averse than $v_1$,*

**(c)** *$v_2$ is globally more prudent than $v_1$,*

**(d)** *$v_2$ has decreasing absolute risk aversion, and*

**(e)** *both Kreps-Porteus utility functions have a decreasing marginal utility of saving,*

*then the Kreps-Porteus precautionary premium given $v_2$ is greater than the Kreps-Porteus precautionary premium given $v_1$.*

*Proof:* See Appendix D. □

Within the CEIS-CRRA class, all the conditions of Proposition 5 are satisfied when $v_2$ is more risk averse than $v_1$ and $U$ also has greater curvature than $v_1$.

A second proposition about the effect of greater risk aversion on the strength of the precautionary saving motive takes care of the case within the CEIS-CRRA class where the elasticity of intertemporal substitution is less than one:

**Proposition 6** *Suppose two Kreps-Porteus utility functions share the same outer intertemporal function $U$ but have different inner interpossibility functions $v_1$ and $v_2$. If*

**(a)** *the resistance to intertemporal substitution is greater than 1 (that is, $U(x)$ is an increasing, concave transformation of $\ln(x)$),*

**(b)** *$v_2$ is globally more risk averse than $v_1$,*

**(c)** *$v_1$ has constant or increasing relative risk aversion*

**(d)** *$v_2$ has constant or decreasing relative risk aversion, and*

**(e)** *both Kreps-Porteus utility functions have a decreasing marginal utility of saving,*

*then the Kreps-Porteus precautionary premium given $v_2$ is greater than the Kreps-Porteus precautionary premium given $v_1$.*

*Proof:* See Appendix E. □

Within the CEIS-CRRA class, all the conditions of Proposition 6 are satisfied when $v_2$ is more risk averse than $v_1$ and the elasticity of intertemporal substitution is less than or equal to one.

Empirically, there is strong evidence that the elasticity of intertemporal substitution is no larger than one, so this proposition takes care of all the empirically relevant cases within the CEIS-CRRA class.[15]

In Proposition 6, as in Proposition 5, an assumption is needed that insures that the absolute risk aversion of $v_2$ decreases fast enough relative to the absolute risk aversion of $v_1$. In Proposition 5, the assumption of greater prudence for $v_2$ plays that role. In Proposition 6, the assumption that $v_2$ has constant or decreasing relative risk aversion, while $v_1$ has constant or increasing relative risk aversion plays that role.

Together, Propositions 5 and 6 guarantee that if a CEIS-CRRA utility function has values of $\gamma$ and $\rho$ in the shaded region of Figure 2, while the second function has parameter values directly to the right (whether or not in the shaded region), the precautionary premium will be greater for the second utility function.

#### 4.2.3 Limitations on what can be proven about the effect of risk aversion on the strength of the precautionary saving motive

It is reasonable to wonder if stronger results are possible. In particular, what about those utility functions within the CEIS-CRRA class not addressed by Propositions 5 and 6? Proposition 7 says that there are no corresponding results when raising

[15] See for example Hall [8] and Barsky, Juster, Kimball, and Shapiro [2].

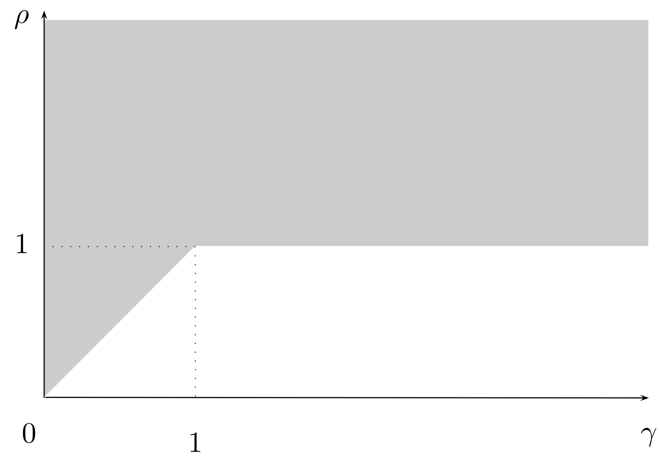


Figure 2: Risk aversion and precautionary saving: CEIS-CRRA case

risk aversion starting from any CEIS-CRRA utility function in the interior of the unshaded region of Figure 2.

**Proposition 7** *For every Kreps-Porteus utility function* $u(c_1) + \frac{[\mathbf{E}\,(\tilde{c}_2)^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}}}{1-\rho}$ *with* $\gamma \neq 1$ *and* $\rho \in (0, \min\{\gamma, 1\})$, *and (when* $\gamma = 1$*) for every Kreps-Porteus utility function* $u(c_1) + \frac{e^{(1-\rho)[\mathbf{E}\,\ln(\tilde{c}_2)]}}{1-\rho}$ *with* $\rho \in (0, 1)$, *there are two-point risks for which local increases in* $\gamma$ reduce *the size of the precautionary premium* $\theta^*$.

*Proof:* See Appendix F. □

#### 4.2.4 Wealth

Kimball [11, 12] shows that a precautionary premium that declines with wealth implies that labor income risk raises the marginal propensity to consume out of wealth at a given level of first-period consumption. As can be seen from studying Figure 1, this connection between a decreasing precautionary premium and the marginal propensity to consume holds for Kreps-Porteus preferences as well. In the two-period case with intertemporal expected utility maximization, decreasing absolute prudence is necessary and sufficient for a decreasing precautionary premium. Section 3 has identified a necessary condition based on small risks for a decreasing precautionary premium. This subsection identifies sufficient conditions for a decreasing precautionary premium with Kreps-Porteus preferences and large risks.

Given a decreasing marginal utility of saving, define $Q(\tilde{w})$ by the identity

$$U'(Q(\tilde{w})) \equiv U'(M(\tilde{w}))M'(\tilde{w}). \tag{4.16}$$

Then

$$Q(x + \theta^*(x) + \tilde{y}) \equiv x,$$

and

$$[1 + \theta^{*'}(x)]Q'(x + \theta^*(x) + \tilde{y}) = 1.$$

As a consequence, the precautionary premium is decreasing with $x$ if and only if $Q'(\tilde{w}) \geq 1$ with $\tilde{w} = x + \theta^*(x, \tilde{y}) + \tilde{y}$.

Differentiation of (4.16) yields

$$U''(Q(\tilde{w}))\, Q'(\tilde{w}) = U''(M(\tilde{w}))[M'(\tilde{w})]^2 + U'(M(\tilde{w}))M''(\tilde{w}).$$

Changing signs and dividing by the square of (4.16), we find

$$\left(-\frac{U''(Q)}{[U'(Q)]^2}\right) Q' = \left(-\frac{U''(M)}{[U'(M)]^2}\right) + \left(\frac{1}{U'(M)}\right)\left(-\frac{M''}{[M']^2}\right), \tag{4.17}$$

with the argument $\tilde{w}$ suppressed for clarity.

Appendix A shows, following Hardy, Littlewood, and Polya [10, pp. 85-88], that concave absolute risk tolerance of $v$ is sufficient to guarantee that the certainty equivalent function $M$ is concave. Since CRRA, exponential and quadratic utility, with or without a shifted origin yield linear absolute risk tolerance, weakly concave absolute risk tolerance holds for an important set of inner interpossibility functions $v$. This result makes the following proposition very useful:

**Proposition 8** *Given the Kreps-Porteus utility function $u(c_1) + U(v^{-1}(\mathbf{E}\, v(\tilde{c}_2)))$ with $U$ and $v$ increasing and concave, if*

**(a)** *absolute risk aversion is decreasing,*

**(b)** *the certainty-equivalent function $M$ for the inner interpossibility function $v$ is concave, and*

**(c)** $-\frac{U''(x)}{[U'(x)]^2}$ *is (weakly) increasing,*

*then the precautionary premium is decreasing in wealth.*

*Proof:* First, as shown in Appendix A, if both $M(\tilde{w})$ and $U(x)$ are increasing and concave, then the marginal utility of saving is decreasing. Second, decreasing absolute risk aversion guarantees not only that $M' \geq 1$, but also, by (4.16), that $Q(\tilde{w}) \leq M(\tilde{w})$. Since $-\frac{U''(x)}{[U'(x)]^2}$ is increasing,

$$-\frac{U''(Q)}{[U'(Q)]^2} \leq -\frac{U''(M)}{[U'(M)]^2}.$$

Therefore,

$$\left(-\frac{U''(Q)}{[U'(Q)]^2}\right) \leq \left(-\frac{U''(M)}{[U'(M)]^2}\right) + \left(\frac{1}{U'(M)}\right)\left(-\frac{M''}{[M']^2}\right),$$

implying by (4.17) that $Q' \geq 1$. □

**Remark.** If the elasticity of intertemporal substitution is constant, condition (c), which requires $-\frac{U''(x)}{[U'(x)]^2}$ to be increasing, is equivalent to $\rho \geq 1$, or to an elasticity of intertemporal substitution $s$ less than or equal to 1. More generally, this condition is equivalent to $s'(x) \leq \frac{1-s(x)}{x}$, where $s(x) = -\frac{U'(x)}{xU''(x)}$.

For the effect of greater risk aversion on the precautionary premium, Section 4.2.2 has provided results that cover the CEIS-CRRA case not only for $\rho \geq 1$, but also for $\rho \geq \gamma$. We can do the same for the effect of wealth on the precautionary premium. Proposition 9 shows that if the precautionary premium is decreasing for a utility function at any point in Figure 2, the precautionary premium will also be decreasing at any utility function directly above it on Figure 2, up to the point where $\rho = 1$. Proposition 10 combines this result with the fact that the precautionary premium is decreasing for the CRRA intertemporal expected utility case $\rho = \gamma$ to demonstrate that within the CEIS-CRRA class, $\rho \geq \gamma$ is enough to guarantee a decreasing precautionary premium, even if $\rho < 1$.

**Proposition 9** *Suppose two Kreps-Porteus utility functions share the same inner interpossibility function $v$ but have different increasing, concave intertemporal functions $U_1$ and $U_2$ and associated precautionary certainty equivalence functions $Q_1(\tilde{w})$ and $Q_2(\tilde{w})$. If*

**(a)** *$v$ has decreasing absolute risk aversion,*

**(b)** *$Q_1'(\tilde{w}) \geq 1$ so that the precautionary premium for the first utility function is decreasing,*

**(c)** $-\frac{U_2''(x)}{U_2'(x)} \geq -\frac{U_1''(x)}{U_1'(x)}$, *or equivalently* $U_2$ *has a greater resistance to intertemporal substitution than* $U_1$,

**(d)** $2\frac{U_2''(x)}{U_2'(x)} - \frac{U_2'''(x)}{U_2''(x)} \leq 2\frac{U_1''(x)}{U_1'(x)} - \frac{U_1'''(x)}{U_1''(x)}$, *and*

**(e)** *either* $2\frac{U_1''(x)}{U_1'(x)} - \frac{U_1'''(x)}{U_1''(x)}$ *or* $2\frac{U_2''(x)}{U_2'(x)} - \frac{U_2'''(x)}{U_2''(x)}$ *is a decreasing function,*

*then* $Q_2'(\tilde{w}) \geq 1$, *so that the precautionary premium for the second utility function is decreasing at the corresponding point.*

*Proof:* See Appendix G. □

**Remark.** When the elasticity of intertemporal substitution $s$ and its reciprocal $\rho$ are constant, condition (c) boils down to $\frac{\rho_2}{x} \geq \frac{\rho_1}{x}$, condition (d) boils down to $\frac{1-\rho_2}{x} \leq \frac{1-\rho_1}{x}$, and condition (e) boils down to $\frac{1-\rho_i}{x}$ being a decreasing function for one of the functions, which is guaranteed if either $\rho_i \leq 1$. Thus, for the CEIS-CRRA utility functions, Proposition 9 applies in exactly the region where Proposition 8 cannot be used.

The foregoing proposition has an obvious corollary. Set $U_1(x) = v(x)$ and denote $U_2(x) = U(x)$:

**Proposition 10** *(***corollary to Proposition 9***) Suppose a Kreps-Porteus utility function satisfies the conditions*

**(a)** *the inner interpossibility function* $v$ *is increasing, concave, has decreasing absolute risk aversion and decreasing absolute prudence,*

**(b)** $-\frac{U''(x)}{U'(x)} \geq -\frac{v''(x)}{v'(x)}$, *or equivalently* $\rho(x) \geq \gamma(x)$,

**(c)** $2\frac{U''(x)}{U'(x)} - \frac{U'''(x)}{U''(x)} \leq 2\frac{v''(x)}{v'(x)} - \frac{v'''(x)}{v''(x)}$, *and*

**(d)** *either* $2\frac{v''(x)}{v'(x)} - \frac{v'''(x)}{v''(x)}$ *or* $2\frac{U''(x)}{U'(x)} - \frac{U'''(x)}{U''(x)}$ *is a decreasing function,*

*then the precautionary premium* $\theta^*(x)$ *is decreasing.*

The condition that $v$ has decreasing absolute prudence is needed to guarantee that $Q_1' \geq 1$ with $U_1 = v$.

Within the CEIS-CRRA class, the conditions of Proposition 10 are all guaranteed when $1 > \rho > \gamma$. Propositions 8 and 10 taken together thus establish that the precautionary premium is decreasing for large risks for any set of parameters in the shaded region of Figure 2. This is the same region in which greater risk aversion increases the size of the precautionary premium. Conversely, Proposition 11 demonstrates that one cannot guarantee a decreasing precautionary premium for any of the utility functions in the interior of the unshaded region in Figure 2. Thus, the utility functions in the unshaded region—those with a resistance to intertemporal substitution below one *and* below relative risk aversion—are badly behaved in relation to both the effect of risk aversion and the effect of wealth on the strength of the precautionary saving motive:

**Proposition 11** *For every Kreps-Porteus utility function* $u(c_1) + \frac{[\mathbf{E}\,(\tilde{c}_2)^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}}}{1-\rho}$ *with* $\gamma \neq 1$ *and* $\rho \in (0, \min\{\gamma, 1\})$*, and (when* $\gamma = 1$*) for every Kreps-Porteus utility function* $u(c_1) + \frac{e^{(1-\rho)[\mathbf{E}\ \ln(\tilde{c}_2)]}}{1-\rho}$ *with* $\rho \in (0, 1)$*, there are two-point risks for which increases in wealth* raise *the size of the precautionary premium* $\theta^*$*.*

*Proof:* See Appendix H. □

## 5 Conclusion

We have made considerable progress in understanding the determinants of the strength of the precautionary saving motive under Kreps-Porteus preferences in the two-period case. Perhaps one of the more surprising results is that greater risk aversion tends to increase the strength of the precautionary saving motive, while greater resistance to intertemporal substitution *reduces* the strength of the precautionary saving motive in the typical case of decreasing absolute risk aversion. Thus, distinguishing between risk aversion and the resistance to intertemporal substitution is very important in discussing the determinants of the strength of the precautionary saving motive, since these two parameters—which are forced to be equal under intertemporal expected utility maximization—have opposite effects when allowed to vary separately.

Our results also bear on judgments about the likely empirical magnitude of the precautionary saving motive. Remember that we show in Section 3 that, for preferences exhibiting a constant elasticity of intertemporal substitution and constant relative risk aversion, the local measure (in relative terms) of the strength of the precautionary saving motive is

$$\mathcal{P} = \gamma(1 + s) = \gamma(1 + \rho^{-1}),$$

where $\gamma$ is the coefficient of relative risk aversion, $s$ is the elasticity of intertemporal substitution, and $\rho = 1/s$ is the resistance to intertemporal substitution.

To give some idea of what this formula means in practice, Table 2 shows the implied value of $\mathcal{P}$ for $\gamma$ and $\rho$ both ranging over the three representative "low," "medium," and "high" values 2, 6 and 18. The words "low," "medium," and "high,"

| | $\gamma = 2$ | $\gamma = 6$ | $\gamma = 18$ |
|---|---|---|---|
| $\rho = 2$ | 3 | 9 | 27 |
| $\rho = 6$ | 2.33 | 7 | 21 |
| $\rho = 18$ | 2.11 | 6.33 | 19 |

Table 2: The local coefficient of relative prudence $\mathcal{P} = \gamma(1 + \rho^{-1})$ for isoelastic Kreps-Porteus preferences

are in quotation marks because the values of both $\gamma$ and $\rho$ and what values would count as "low," "medium" and "high," in relation to the data are hugely controversial. It is not hard to imagine the following situation. Economist A finds $\gamma = 2$ intuitively reasonable, while being convinced by consumption Euler equation evidence like that of Hall [8] that the elasticity of intertemporal substitution $s$ is very small, with $s$ about .055 and thus $\rho = 18$. Economist B finds the value $\rho = 2$, and the higher implied $s = .5$, intuitively more reasonable in relation to what is needed to explain business cycles, but is convinced by the evidence behind the equity premium puzzle that relative risk aversion should be at least $\gamma = 18$.

If, however, they do not know how to relax the von Neumann-Morgenstern restriction $\rho = \gamma$, these economists with quite divergent views about risk aversion and intertemporal substitution might each compromise on the geometric mean of 2 and 18 for both $\gamma$ and $\rho$, thus agreeing on $\rho = \gamma = 6$ and a precautionary saving motive strength given by the relative prudence of 7. This agreement is artificial as, freed by Kreps-Porteus preferences from the restriction $\rho = \gamma$, economists A and B would differ on the equilibrium equity premium and real interest rate implied by their model. In addition, they would strongly disagree about the strength of the precautionary saving motive. Economist A would expect a very strong precautionary saving motive with $\mathcal{P} = 27$, while Economist B would expect a much weaker precautionary saving motive with $\mathcal{P} = 2.11$.

The dramatic quantitative difference in this example underlines the importance of refining the empirical evidence for the magnitude of risk aversion, and of obtaining distinct estimates of the resistance to intertemporal substitution.[16] Kreps-Porteus preferences provide the conceptual foundation of this further empirical research program. In turn, understanding the precautionary saving effects of Kreps-Porteus preferences is central to a sound theoretical and empirical understanding of consumption and saving behavior.

[16]One of the key practical implications of distinguishing between risk aversion and the resistance to intertemporal substitution is that, without the restriction that they be equal, there is only half as much empirical evidence on the value of each as one might otherwise think.

# Appendix A Conditions guaranteeing a decreasing marginal utility of saving

Almost every proof in the preceding sections relies in some way on the assumption of a decreasing marginal utility of saving. If that assumption fails, the marginal utility of saving $U'(M(x+\tilde{y}))M'(x+\tilde{y})$ can intersect first-period marginal utility $u'(w-x)$ more than once when the first-period utility function $u$ is close to being linear. Regardless of the shape of the first-period utility function, an increasing marginal utility of saving results in a negative first-period marginal propensity to consume out of wealth, as illustrated in Figure 3. In Figure 3, the first-period marginal utility of consumption $u'$ rises as wealth

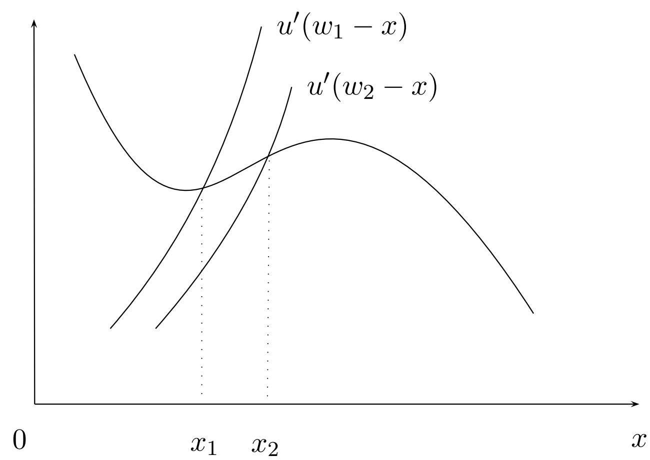


Figure 3: A marginal utility of saving increasing on one interval

rises from $w_1$ to $w_2$, indicating that first-period consumption *falls*.

In order to assess how restrictive this assumption is, we now derive conditions sufficient to guarantee that the marginal utility of saving will be decreasing.

The simplest sufficient condition for a decreasing marginal utility of saving is concavity of $\phi$ and $v$—or equivalently, $\rho \geq \gamma > 0$.

**Proposition 12** *If both $v$ and $\phi$ are monotonically increasing and concave—that is, if the resistance to intertemporal substitution is greater than risk aversion, which in turn is positive—then the marginal utility of saving is decreasing.*

*Proof:* Concavity of both $v$ and $\phi$ guarantees that both factors of the Kreps-Porteus representation of the marginal utility of saving, $\phi'(\mathbf{E}\, v(x+\tilde{y}))\mathbf{E}\, v'(x+\tilde{y})$, decrease as $x$ increases. $\square$

Alternatively, concavity of both the outer intertemporal function $U$ and of the certainty equivalent function $M$, is enough to guarantee a decreasing marginal utility of saving:

**Proposition 13** *If both the outer intertemporal utility function $U$ and the certainty equivalent function $M(x+\tilde{y})$ are increasing and concave in $x$, then the marginal utility of saving is decreasing.*

*Proof:* Concavity of both $U$ and $M$ guarantees that both factors of the Selden representation of the marginal utility of saving, $U'(M(x+\tilde{y}))M'(x+\tilde{y})$, decrease as $x$ increases. $\square$

This result becomes very useful in conjunction with Proposition 14 below (adapted from Hardy, Littlewood, and Polya's classic book *Inequalities*) that the certainty equivalent function $M$ is always concave for any atemporal-von Neumann Morgenstern utility function $v$ with concave absolute risk tolerance, which includes any $v$ in the hyperbolic absolute risk aversion class, including quadratic, exponential, linear, and constant relative risk aversion utility functions, which all have linear absolute risk tolerance.[17]

**Proposition 14** *If $v'(x) > 0$, $v''(x) < 0$, and $-\frac{v'(x)}{v''(x)}$ is a concave function of $x$, then $M(\xi+\tilde{y}) = v^{-1}(\mathbf{E}\, v(\xi+\tilde{y}))$ is a concave function of $\xi$.*

[17] Utility functions in the hyperbolic absolute risk aversion class are those that can be expressed in the form $v(x) = \frac{\gamma}{1-\gamma}\left\{\left(\frac{x-b}{\gamma}\right)^{1-\gamma} - 1\right\}$ together with the logarithmic limit as $\gamma \to 1$ and the exponential limit as $\gamma \to \infty$ with $\frac{b}{\gamma} \to -a$.

*Proof:* As noted above, this proof is adapted from Hardy et al. [10]. Write $\xi + \tilde{y} = \tilde{w}$. Differentiating the identity

$$v(M(\tilde{w})) \equiv \mathbf{E}\, v(\tilde{w})$$

yields

$$v'(M(\tilde{w}))M'(\tilde{w}) = \mathbf{E}\, v'(\tilde{w}) \tag{A.1}$$

and

$$v'(M(\tilde{w}))M''(\tilde{w}) + v''(M(\tilde{w}))[M'(\tilde{w})]^2 = \mathbf{E}\, v''(\tilde{w}). \tag{A.2}$$

Dividing (A.2) by the square of (A.1) and rearranging,

$$\frac{M''(\tilde{w})}{v'(M(\tilde{w}))[M'(\tilde{w})]^2} = \left(\frac{-v''(M(\tilde{w}))}{[v'(M(\tilde{w}))]^2}\right) - \left(\frac{-\mathbf{E}\, v''(\tilde{w})}{[\mathbf{E}\, v'(\tilde{w})]^2}\right) \tag{A.3}$$

The sign of $M''$ is the same as the sign of the right-hand-side of (A.3).

For any particular realization of $\tilde{w}$, the matrix

$$\begin{bmatrix} -\frac{[v'(w)]^2}{v''(w)} & v'(w) \\ v'(w) & -v''(w) \end{bmatrix}$$

is positive semi-definite. Since an expectation over positive semi-definite matrices is positive semi-definite,

$$\begin{bmatrix} -\mathbf{E}\, \frac{[v'(\tilde{w})]^2}{v''(\tilde{w})} & \mathbf{E}\, v'(\tilde{w}) \\ \mathbf{E}\, v'(\tilde{w}) & -\mathbf{E}\, v''(\tilde{w}) \end{bmatrix}$$

is also positive semi-definite. Therefore,

$$\left[\mathbf{E}\, \frac{[v'(\tilde{w})]^2}{v''(\tilde{w})}\right] \mathbf{E}\, v''(\tilde{w}) \geq [\mathbf{E}\, v'(\tilde{w})]^2.$$

This implies the first inequality in (A.4):

$$\begin{aligned} \frac{[\mathbf{E}\, v'(\tilde{w})]^2}{[-\mathbf{E}\, v''(\tilde{w})]} &\leq \mathbf{E}\left(\frac{[v'(\tilde{w})]^2}{[-v''(\tilde{w})]}\right) \\ &\leq -\frac{[v'(M(\tilde{w}))]^2}{v''(M(\tilde{w}))}. \end{aligned} \tag{A.4}$$

The second inequality in (A.4) follows from the concavity of absolute risk tolerance $-\frac{v'(x)}{v''(x)}$. Consider the function $\ell(z)$ defined by

$$\ell(z) \equiv -\frac{[v'(v^{-1}(z))]^2}{v''(v^{-1}(z))}.$$

Its first derivative is

$$\ell'(z) \equiv \frac{v'''(v^{-1}(z))v'(v^{-1}(z))}{[v''(v^{-1}(z))]^2} - 2.$$

For comparison,

$$\frac{\partial}{\partial x}\left(\frac{-v'(x)}{v''(x)}\right) \equiv \frac{v'''(x)v'(x)}{[v''(x)]^2} - 1.$$

Let $z = v(x)$. Since $\frac{dz}{dx} > 0$, $\ell'(z)$ is decreasing if and only if $\frac{\partial}{\partial x}\left(\frac{-v'(x)}{v''(x)}\right)$ is decreasing. Therefore, $\ell(z)$ is concave as a function of $z$ if and only if absolute risk tolerance $\frac{-v'(x)}{v''(x)}$ is concave as a function of $x$. In turn, concavity of $\ell$ implies

$$\mathbf{E}\left(\frac{[v'(\tilde{w})]^2}{[-v''(\tilde{w})]}\right) = \mathbf{E}\,\ell(v(\tilde{w})) \leq \ell(\mathbf{E}\, v(\tilde{w})) = -\frac{[v'(M(\tilde{w}))]^2}{v''(M(\tilde{w}))}.$$

Inequality (A.5) and equation (A.3) prove that $M$ is concave. □

As for the converse of Proposition 13, a necessary condition on $v$ for $U'(M(x + \tilde{y}))M'(x + \tilde{y})$ to be decreasing for *any* concave $U$ (that is, regardless of the size of the resistance to intertemporal substitution) is for $M$ to be concave.

Only a partial converse is available for Proposition 14. For small risks, the necessary and sufficient condition for $M$ to be concave is for absolute risk *aversion* $\frac{-v''(x)}{v'(x)}$ to be *convex*. (See equation (B.2).) This is a weaker condition than the *concavity* of absolute risk *tolerance* that guarantees concavity of $M$ for large risks, so this does not provide a full converse to Proposition 14. Still, it does indicate how to find cases where $M$ is convex, which have the potential to violate the assumption of a decreasing marginal utility of saving for a low enough resistance to intertemporal substitution. Hardy, Littlewood, and Polya [10, pp. 85-88] look at a more general result than Proposition 14 (in our notation, they are interested in $M(\tilde{y} + x\tilde{z})$ being concave in $x$ for any random variable $\tilde{z}$), so their converse does not apply.

## Appendix B Local approximation of section 3

Define the *equivalent Kreps-Porteus precautionary premium* $\theta$, expressed as a function of $x$, as the solution to the equation

$$U'(M(x))M'(x) = U'(x - \theta(x)). \tag{B.1}$$

It is easiest to find the small-risk approximation for the equivalent Kreps-Porteus precautionary premium $\theta$ first, and then the small risk approximation for the compensating Kreps-Porteus precautionary premium $\theta^*$. For a small risk $\tilde{y}$ with mean zero and variance $\sigma^2$, Pratt [17] shows that

$$M(x + \tilde{y}) = x - a(x)\frac{\sigma^2}{2} + o(\sigma^2). \tag{B.2}$$

As long as $M''(\cdot)$ is bounded in the neighborhood of $x$, (B.2) can be differentiated to obtain

$$M'(x + \tilde{y}) = 1 - a'(x)\frac{\sigma^2}{2} + o(\sigma^2). \tag{B.3}$$

Finally, substituting from (B.2) and (B.3) into (B.1), and doing a Taylor expansion of $U(x-\theta)$ around $x$,

$$\begin{aligned} U'(M(x))M'(x) &= [U'(x) - U''(x)a(x)\frac{\sigma^2}{2} + o(\sigma^2)]\,[1 - a'(x)\frac{\sigma^2}{2} + o(\sigma^2)] \\ &= U'(x) - [U''(x)a(x) + U'(x)a'(x)]\frac{\sigma^2}{2} + o(\sigma^2) \\ &= U'(x) - U''(x)\theta(x) + o(\theta(x)). \end{aligned} \tag{B.4}$$

Therefore,

$$\theta(x) = \left[a(x) + \frac{U'(x)}{U''(x)}a'(x)\right]\frac{\sigma^2}{2} + o(\sigma^2). \tag{B.5}$$

Inspecting (2.6) and (B.1) and then using (B.5) makes it clear that

$$\begin{aligned} \theta^*(x) &= \theta(x + \theta^*(x)) \\ &= \left[a(x+\theta^*(x)) + \frac{U'(x+\theta^*(x))}{U''(x+\theta^*(x))}a'(x+\theta^*(x))\right]\frac{\sigma^2}{2} + o(\sigma^2) \\ &= \left[a(x) + \frac{U'(x)}{U''(x)}a'(x)\right]\frac{\sigma^2}{2} + o(\sigma^2), \end{aligned} \tag{B.6}$$

as long as $a'$ and $\frac{U'}{U''}$ are continuous at $x$. Using (3.2), (B.6) can be rewritten as

$$\theta^*(x) = a(x)[1 + s(x)\varepsilon(x)]\frac{\sigma^2}{2} + o(\sigma^2), \tag{B.7}$$

which establishes (3.1) in the text.

# Appendix C Proof of Proposition 4

The optimal amount of saving for the two utility functions is the same under certainty only if there is an $x_0$ for which

$$u_i'(w - x_0) = U_i'(x_0) \quad i = 1, 2. \tag{C.1}$$

Define the normalized marginal utility of saving functions $f_1$ and $f_2$ by

$$f_i(x) = \frac{U_i'(M(x+\tilde{y}))M'(x+\tilde{y})}{U_i'(x_0)}. \tag{C.2}$$

For either utility function, the normalized marginal utility of saving $f_i$ equals 1 when $x = x_0 + \theta_i^*(x_0)$. Therefore, $f_1$ and $f_2$ can be used to establish comparative statics for the Kreps-Porteus precautionary premium $\theta^*$.

Equation (C.1) says that $f_1$ and $f_2$ meet at $x = x_0 + \pi^*(x_0)$, since

$$\begin{aligned} \frac{U_i'(M(x_0+\pi^*(x_0)+\tilde{y}))\, M'(x_0+\pi^*(x_0)+\tilde{y})}{U_i'(x_0)} &= \frac{U_i'(x_0)\, M'(x_0+\pi^*(x_0)+\tilde{y})}{U_i'(x_0)} \\ &= M'(x_0+\pi^*(x_0)+\tilde{y}) \end{aligned} \tag{C.3}$$

for $i = 1, 2$. Since the identity $M(x+\pi^*(x)+\tilde{y}) \equiv x$ can be differentiated to obtain

$$[1+\pi^{*\prime}(x_0)]M'(x_0+\pi^*(x_0)+\tilde{y}) = 1, \tag{C.4}$$

both $f_1(x_0+\pi^*(x_0))$ and $f_2(x_0+\pi^*(x_0))$ are greater than 1 if $\pi'(x_0) \le 0$ and both are less than 1 if $\pi^{*\prime}(x_0) \ge 0$.

The ratio between $f_2$ and $f_1$ simplifies as follows:

$$\frac{f_2(x)}{f_1(x)} = \frac{U_2'(M(x+\tilde{y}))}{U_1'(M(x+\tilde{y}))}\frac{U_1'(x_0)}{U_2'(x_0)}. \tag{C.5}$$

The condition $\frac{-U_2''(x)}{U_2'(x)} \ge \frac{-U_1''(x)}{U_1'(x)}$ implies that the ratio $\frac{U_2'(x)}{U_1'(x)}$ is a decreasing function of $x$ since

$$\frac{d}{dx}\ln\left(\frac{U_2'(x)}{U_1'(x)}\right) = \frac{U_2''(x)}{U_2'(x)} - \frac{U_1''(x)}{U_1'(x)} \le 0. \tag{C.6}$$

Since $M(x+\tilde{y})$ is an increasing function of $x$, this means that $\frac{f_2(x)}{f_1(x)}$ is a decreasing function of $x$. Therefore, $f_2(x) \ge f_1(x)$ for $x \le x_0 + \pi^*(x_0)$ and $f_2(x) \le f_1(x)$ for $x \ge x_0 + \pi^*(x_0)$.

We are now in a position to draw an instructive graph of $f_1$ and $f_2$. Figures 4 and 5 depict the two main cases. Decreasing marginal utility of saving means that $f_i$ is decreasing for both utility functions. If $\pi^{*\prime}(x_0) \leq 0$, then $f_2$ and $f_1$ are both above 1 at $x = x_0 + \pi^*(x_0)$ and $f_2$ must hit 1 first as $x$ moves to the right from $x_0 + \pi^*(x_0)$ since $f_2$ is below $f_1$ to the right of $x_0 + \pi^*(x_0)$. Therefore $x_0 + \theta_2^*(x_0) \leq x_0 + \theta_1^*(x_0)$ and $\theta_2^*(x_0) \leq \theta_1^*(x_0)$.

If $\pi^{*\prime}(x_0) \geq 0$, then $f_2$ and $f_1$ are both below 1 at $x = x_0 + \pi^*(x_0)$ and $f_2$ must hit 1 first as $x$ moves to the left from $x_0 + \pi^*(x_0)$ since $f_2$ is above $f_1$ to the left of $x_0 + \pi^*(x_0)$. Therefore $x_0 + \theta_2^*(x_0) \geq x_0 + \theta_1^*(x_0)$ and $\theta_2^*(x_0) \geq \theta_1^*(x_0)$.

If one or the other of $f_i(x)$ never hits 1, these inequalities remain valid if one writes $\theta_i^*(x_0) = +\infty$ when $f_i(x) \geq 1$ for all $x$, and $\theta_i^*(x_0) = -\infty$ when $f_i(x) \leq 1$ for all $x$. In all of these cases, $\theta_1(x_0)$ and $\theta_2(x_0)$ are on the same side of $\pi^*(x_0)$ and $|\theta_2^*(x_0) - \pi^*(x_0)| \leq |\theta_1^*(x_0) - \pi^*(x_0)|$.

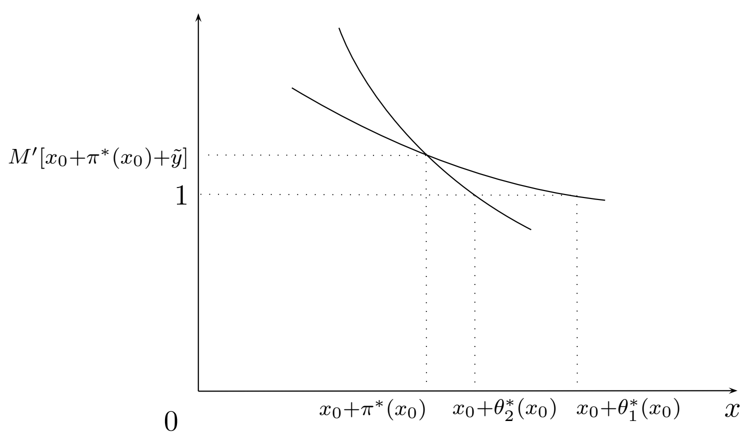


Figure 4: $\pi^{*\prime}(x_0) < 0$

Figures 4 and 5 can be used to illustrate not only the effects of changing the elasticity of intertemporal substitution but also the effects of changing the level and rate of decline of risk aversion. Let us begin by presenting what would be a satisfying intuitive story except that it neglects one effect. Focusing on the point $(x_0 + \pi^*(x_0), M'(x_0 + \pi^*(x_0) + \tilde{y}))$

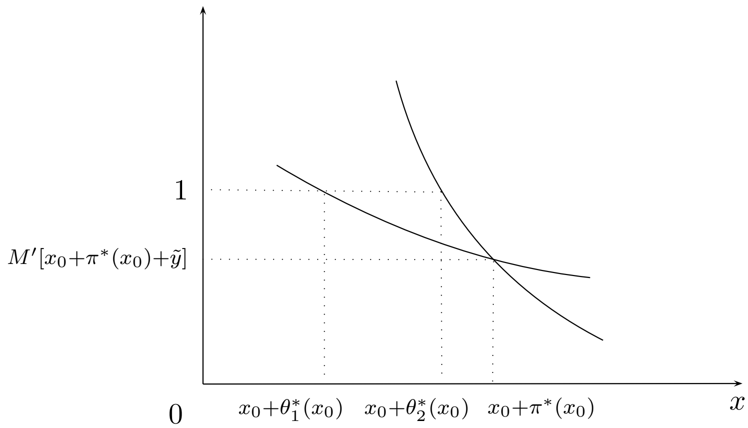


Figure 5: $\pi^{*\prime}(x_0) > 0$

at which the curves $f_1$ and $f_2$ intersect, increases in risk aversion move this point to the right by increasing $\pi^*$, more quickly declining risk aversion tends to move this point upward by increasing $M'$, while increases in the elasticity of intertemporal substitution swivel the curve describing $f$ counterclockwise around this point (increases in the *resistance* to intertemporal substitution swivel the curve clockwise around this point.) The effects of each change on the strength of the precautionary saving motive can be seen in the horizontal movement of the intersection with the dashed line $f(x) = 1$. The message of equation (3.3) about the determinants of the strength of the precautionary saving motive under Kreps-Porteus preferences is clearly confirmed by such an exercise. The fly in the ointment is the one neglected effect: like the rate of change of risk aversion, changes in the level of risk aversion can affect $M'$ and so move the $(x_0 + \pi^*(x_0), M'(x_0 + \pi^*(x_0) + \tilde{y}))$ point vertically as well as horizontally. This vertical movement can be either up or down in response to an increase in the level of risk aversion. The subtleties of section 4.2.2 stem from this complication.

# Appendix D Proof of Proposition 5

By Lemma 1, together with the assumption of a decreasing marginal utility of saving, it is enough to show that $U'(M_2)M_2' > U'(M_1)M_1'$. Write $\tilde{w} = x + \theta_1 + \tilde{y}$. Take the derivative of both sides of the identity

$$v(M_i(\tilde{w})) \equiv \mathbf{E}\, v(\tilde{w})$$

with respect to the addition of nonstochastic constants to get

$$v'(M_i(\tilde{w}))M_i'(\tilde{w}) = \mathbf{E}\, v'(\tilde{w}).$$

Also define $N_i(\tilde{w})$ by

$$v'(N_i(\tilde{w})) = \mathbf{E}\, v'(\tilde{w}),$$

so that

$$M_i'(\tilde{w}) = \frac{v_i'(N_i(\tilde{w}))}{v_i'(M_i(\tilde{w}))}.$$

Suppressing the arguments for $M_i$, $M_i'$ and $N_i$, we find that

$$\begin{aligned}
\frac{U'(M_2)M_2'}{U'(M_1)M_1'} &= \frac{U'(M_2)\left(\frac{v_2'(N_2)}{v_2'(M_2)}\right)}{U'(M_1)\left(\frac{v_1'(N_1)}{v_1'(M_1)}\right)} \\
&= \left\{\frac{\left(\frac{U'(M_2)}{v_1'(M_2)}\right)}{\left(\frac{U'(M_1)}{v_1'(M_1)}\right)}\right\}\left\{\frac{\left(\frac{v_2'(N_2)}{v_1'(N_2)}\right)}{\left(\frac{v_2'(M_2)}{v_1'(M_2)}\right)}\right\}\left\{\frac{v_1'(N_2)}{v_1'(N_1)}\right\} \\
&\geq 1.
\end{aligned}$$

First,

$$\frac{\left(\frac{U'(M_2)}{v_1'(M_2)}\right)}{\left(\frac{U'(M_1)}{v_1'(M_1)}\right)} \geq 1$$

because resistance to intertemporal substitution greater than risk aversion means that $\ln(U')$ declines more quickly than $\ln(v_1')$, making $\frac{U'}{v_1'}$ a decreasing function, and $v_2$ being more risk averse than $v_1$ guarantees that $M_2(\tilde{w}) \leq M_1(\tilde{w})$. Second,

$$\frac{\left(\frac{v_2'(N_2)}{v_1'(N_2)}\right)}{\left(\frac{v_2'(M_2)}{v_1'(M_2)}\right)} \geq 1$$

because $v_2$ being more risk averse than $v_1$ makes $\frac{v_2'}{v_1'}$ a decreasing function, and the decreasing absolute risk aversion of $v_2$ guarantees that $N_2(\tilde{w}) \leq M_2(\tilde{w})$. Third,

$$\frac{v_1'(N_2)}{v_1'(N_1)} \geq 1$$

because $v_2$ being more prudent than $v_1$ guarantees that $N_2(\tilde{w}) \leq N_1(\tilde{w})$.

## Appendix E Proof of Proposition 6

Using a slightly different rearrangement than in Appendix D,

$$\begin{aligned}
\frac{U'(M_2)M_2'}{U'(M_1)M_1'} &= \frac{U'(M_2)\left(\frac{\mathbf{E}\, v_2'(\tilde{w})}{v_2'(M_2)}\right)}{U'(M_1)\left(\frac{\mathbf{E}\, v_1'(\tilde{w})}{v_1'(M_1)}\right)} \\
&= \left\{\frac{M_2 U'(M_2)}{M_1 U'(M_1)}\right\} \left\{\frac{\left(\frac{\mathbf{E}\, \tilde{w} v_1'(\tilde{w})}{\mathbf{E}\, v_1'(\tilde{w})}\right)}{\left(\frac{\mathbf{E}\, \tilde{w} v_2'(\tilde{w})}{\mathbf{E}\, v_2'(\tilde{w})}\right)}\right\} \left\{\frac{\left(\frac{\mathbf{E}\, \tilde{w} v_2'(\tilde{w})}{M_2 v_2'(M_2)}\right)}{\left(\frac{\mathbf{E}\, \tilde{w} v_1'(\tilde{w})}{M_1 v_1'(M_1)}\right)}\right\} \\
&\geq 1.
\end{aligned}$$

Having the resistance to intertemporal substitution greater than one (or equivalently, the elasticity of intertemporal substitution less than one) implies that $xU'(x)$ is a decreasing function. Combined with $v_2$ being more risk averse than $v_1$, so that $M_2 \leq M_1$, this guarantees that

$$\frac{M_2 U'(M_2)}{M_1 U'(M_1)} \geq 1.$$

Next, $v_2$ being more risk averse than $v_1$ implies that the function

$$j(w) = \frac{v_1'(w)}{E_{\mathcal{W}} v_1'(\tilde{\mathcal{W}})} - \frac{v_2'(w)}{E_{\mathcal{W}} v_2'(\tilde{\mathcal{W}})}$$

is single-crossing upwards. Let $w_0$ be the crossing point. Then, when $\tilde{\mathcal{W}}$ has a distribution that is identical to, but independent from $\tilde{w}$, $\mathbf{E}\, j(\tilde{w}) = 0$ and so

$$E_w \tilde{w} \left[ \frac{v_1'(\tilde{w})}{E_{\mathcal{W}} v_1'(\tilde{\mathcal{W}})} - \frac{v_2'(\tilde{w})}{E_{\mathcal{W}} v_2'(\tilde{\mathcal{W}})} \right] = E_w (\tilde{w} - w_0) \left[ \frac{v_1'(\tilde{w})}{E_{\mathcal{W}} v_1'(\tilde{\mathcal{W}})} - \frac{v_2'(\tilde{w})}{E_{\mathcal{W}} v_2'(\tilde{\mathcal{W}})} \right] \geq 0.$$

This implies that

$$\frac{\left( \frac{\mathbf{E}\, \tilde{w} v_1'(\tilde{w})}{\mathbf{E}\, v_1'(\tilde{w})} \right)}{\left( \frac{\mathbf{E}\, \tilde{w} v_2'(\tilde{w})}{\mathbf{E}\, v_2'(\tilde{w})} \right)} \geq 1.$$

Finally, defining the functions $g_i(\cdot)$ by

$$x v_i'(x) \equiv g_i(v_i(x)), \tag{E.1}$$

the constant or decreasing relative risk aversion of $v_2$ implies that $g_2$ is convex, while the constant or increasing relative risk aversion of $v_1$ implies that $g_1$ is concave. To show this, differentiate (E.1) and divide by $v_i'$ to get

$$1 - \left( -\frac{x v_i''(x)}{v_i'(x)} \right) = g_i'(v_i(x)). \tag{E.2}$$

Thus, $g_1'$ is decreasing (making $g_1$ concave) because relative risk aversion $\frac{-x v_1''(x)}{v_1'(x)}$ is *in-*

*creasing* and $g_2'$ is increasing (making $g_2$ convex) because $\frac{-xv_2''(x)}{v_2'(x)}$ is *decreasing*.

The convexity of $g_2$ implies

$$\mathbf{E}\,\tilde{w}v_2'(\tilde{w}) = \mathbf{E}\,g_2(v_2(\tilde{w})) \geq g_2(\mathbf{E}\,v_2(\tilde{w})) = g_2(v_2(M_2(\tilde{w}))) = M_2(\tilde{w})v'(M_2(\tilde{w})).$$

The concavity of $g_1$ implies

$$\mathbf{E}\,\tilde{w}v_1'(\tilde{w}) = \mathbf{E}\,g_1(v_1(\tilde{w})) \leq g_1(\mathbf{E}\,v_1(\tilde{w})) = g_1(v_1(M_1(\tilde{w}))) = M_1(\tilde{w})v'(M_1(\tilde{w})).$$

As a result,

$$\frac{\left(\frac{\mathbf{E}\,\tilde{w}v_2'(\tilde{w})}{M_2 v_2'(M_2)}\right)}{\left(\frac{\mathbf{E}\,\tilde{w}v_1'(\tilde{w})}{M_1 v_1'(M_1)}\right)} \geq 1,$$

completing the proof.

## Appendix F Proof of Proposition 7

Substituting in for $M'$ as in the previous appendix, and for the particular functional form used in Proposition , we have

$$\begin{aligned} U'(M(\tilde{w}))M'(\tilde{w}) &= U'(M(\tilde{w}))\frac{\mathbf{E}\,v'(\tilde{w})}{v'(M(\tilde{w}))} \\ &= [M(\tilde{w};\gamma)]^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}. \end{aligned}$$

We need to show that the derivative of this expression with respect to $\gamma$ can be negative. Fortunately $M(\tilde{w};\gamma)$ is continuously differentiable with respect to $\gamma$, even across the $\gamma = 1$ boundary. Consider the two-point risk equal to $\epsilon < 1$ with probability $p$ and $\zeta > 1$ with probability $1 - p$. When $\gamma \neq 1$, calculate

$$
\begin{aligned}
\frac{\partial}{\partial\gamma}\ln[M(\tilde{w};\gamma)^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}] &= \frac{\partial}{\partial\gamma}\Big[\frac{\gamma-\rho}{1-\gamma}\ln(p\epsilon^{1-\gamma}+(1-p)\zeta^{1-\gamma}) \\
&\qquad +\ln(p\epsilon^{-\gamma}+(1-p)\zeta^{-\gamma})\Big] \\
&= \frac{1-\rho}{(1-\gamma)^2}\ln(p\epsilon^{1-\gamma}+(1-p)\zeta^{1-\gamma}) \\
&\qquad +\left(\frac{\gamma-\rho}{1-\gamma}\right)\left(\frac{-\ln(\epsilon)p\epsilon^{1-\gamma}-\ln(\zeta)(1-p)\zeta^{1-\gamma}}{p\epsilon^{1-\gamma}+(1-p)\zeta^{1-\gamma}}\right) \\
&\qquad +\frac{[-\ln(\epsilon)p\epsilon^{-\gamma}-\ln(\zeta)(1-p)\zeta^{-\gamma}]}{p\epsilon^{-\gamma}+(1-p)\zeta^{-\gamma}}.
\end{aligned} \tag{F.1}
$$

For given $\epsilon$, choose $p$ and $\zeta$ so that

$$p\epsilon^{1-\gamma}=q$$

and

$$(1-p)\zeta^{1-\gamma}=1-q.$$

That is,

$$p=q\epsilon^{\gamma-1} \tag{F.2}$$

and

$$\zeta=\left(\frac{1-q}{1-q\epsilon^{\gamma-1}}\right)^{\frac{1}{1-\gamma}}. \tag{F.3}$$

When $\gamma<1$, Equation (F.2) requires $\epsilon$ to be in $(q^{1/(1-\gamma)},1)$ to keep $p$ in (0,1). When $\gamma>1$, Equation (F.2) allows $\epsilon$ to be anywhere in $(0,1)$. As $\gamma\to 1$, (F.2) and (F.3) limit into $p=q$ and $\zeta=\epsilon^{-q/(1-q)}$.

For all values of $\gamma$, the choice in (F.2) and (F.3) makes $M(\tilde{w};\gamma)=1$. When $\gamma\neq 1$, it allows (F.1) to be written in terms of $\epsilon$ as follows:

$$\frac{\partial \ln[M(\tilde{w};\gamma)^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}]}{\partial \ln \gamma} = \left(\frac{\gamma-\rho}{1-\gamma}\right)[-q\ln(\epsilon)-(1-q)\ln(\zeta)] \\ + \frac{[-q\ln(\epsilon)\epsilon^{-1}-(1-q)\ln(\zeta)\zeta^{-1}]}{q\epsilon^{-1}+(1-q)\zeta^{-1}} \tag{F.4}$$

**Case 1:** $\gamma > 1$**.** Choose $q \in \left(\frac{\gamma-1}{\gamma-\rho}, 1\right)$. This is possible because $\rho < 1 < \gamma$ in this case. Consider the limit as $\epsilon \to 0$, with $p$ given by (F.2) and $\zeta$ given by (F.3). With $\gamma > 1$,

$$\lim_{\epsilon\to 0}\zeta = (1-q)^{\frac{1}{1-\gamma}},$$

and

$$\lim_{\epsilon\to 0}\frac{\partial \ln[M(\tilde{w};\gamma)^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}]}{\partial \ln \gamma} = \left[q\frac{\gamma-\rho}{\gamma-1}-1\right]\ln(\epsilon)-(1-q)\ln(1-q)\frac{\gamma-\rho}{(\gamma-1)^2} = -\infty.$$

Therefore, there are values of $\epsilon$ small enough that the derivative of the marginal utility of saving with respect to $\gamma$ is negative, which in turn implies a precautionary premium that decreases with $\gamma$ over some range.

**Case 2:** $\gamma < 1$**.** In this case, choose $q = .5$ and consider the limit as $\epsilon \to 2^{-1/(1-\gamma)}$ from above. Then $\zeta \to \infty$ and

$$\lim_{\epsilon\to\left(2^{-1/(1-\gamma)}\right)^+}\frac{\partial \ln[M(\tilde{w};\gamma)^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}]}{\partial \ln \gamma} = \frac{(2-\gamma-\rho)\ln(2)}{2(1-\gamma)^2} - \frac{(\gamma-\rho)\ln(\zeta)}{2(1-\gamma)} = -\infty.$$

Again, this implies that for values of $\epsilon$ close enough above $2^{-1/(1-\gamma)}$, the precautionary premium is decreasing in $\gamma$.

**Case 3:** $\gamma = 1$**.** To deal with this case, let the two-point risk be $\epsilon$ with probability .5 and $\zeta = \epsilon^{-1}$ with probability .5. Then the logarithmic derivative of the marginal utility of saving is

$$\frac{\partial}{\partial \ln \gamma} \ln[M(\tilde{w};\gamma)^{\gamma-\rho}\mathbf{E}\,\tilde{w}^{-\gamma}] = \ln(M(\tilde{w},\gamma)) + (\gamma-\rho)\left\{\frac{\ln(.5\epsilon^{1-\gamma}+.5\epsilon^{\gamma-1})}{(1-\gamma)^2} + \frac{\ln(\epsilon)[\epsilon^{\gamma-1}-\epsilon^{1-\gamma}]}{(1-\gamma)[\epsilon^{\gamma-1}+\epsilon^{1-\gamma}]}\right\} + \ln(\epsilon)\frac{\epsilon^{\gamma}-\epsilon^{-\gamma}}{\epsilon^{\gamma}+\epsilon^{-\gamma}} \tag{F.5}$$

The logarithmic derivative of the marginal utility of saving with respect to $\gamma$ has a well-defined limit as $\gamma \to 1$ from either side of 1. Using the fact that $\lim_{\gamma\to 1} M(\tilde{w},\gamma) = 1$ and a messy application of L'Hopital's rule, the limit as $\gamma \to 1$ of the expression on the right-hand-side of (F.5) is

$$-\ln(\epsilon)\left(\frac{1-\epsilon^2}{1+\epsilon^2}\right) - .5(1-\rho)[\ln(\epsilon)]^2$$

Clearly, for small enough $\epsilon$, the marginal utility of saving—and therefore the precautionary premium—is decreasing in $\gamma$ on both sides of $\gamma = 1$.

# Appendix G Proof of Proposition 9

For each utility function $U_i$, define

$$h_i(\lambda) = -\left(\frac{U_i''(U_i'^{-1}(\lambda U_i'(M(\tilde{w}))))}{\lambda^2 U_i'(M(\tilde{w}))}\right).$$

Using the fact that $U_i'(Q_i) = U_i'(M)M'$, observe that the counterpart of equation (4.17) for each utility function can be rewritten, after multiplying through by $U_i'(M(\tilde{w}))$, as

$$h_i(M')Q_i' = h_i(1) - \frac{M''}{[M']^2},$$

or

$$h_i(M')(Q_i' - 1) = -\frac{M''}{[M']^2} - [h_i(M') - h_i(1)]. \tag{G.1}$$

The fact that $U_i$ is increasing and concave guarantees that $h_i$ is positive. Therefore, condition **(a)** in Proposition 9, $Q_1' \geq 1$, is equivalent to the right-hand-side of (G.1) being positive, or

$$h_1(M') - h_1(1) \leq -\frac{M''}{[M']^2}.$$

We need to find conditions under which this inequality implies that

$$h_2(M') - h_2(1) \leq -\frac{M''}{[M']^2} \tag{G.2}$$

as well.

One approach is to try to establish that $h_2'(\lambda) \leq 0$ and that $M'' \leq 0$, so that

$$h_2(M') - h_2(1) \leq 0 \leq -\frac{M''}{[M']^2}.$$

This is exactly what Proposition 8 does. The other approach—the essence of Proposition 9—is to establish that $h_2'(\lambda) \leq h_1'(\lambda)$ for $\lambda \geq 1$, so that, given $M' \geq 1$ (a consequence of decreasing absolute risk aversion),

$$h_2(M') - h_2(1) \leq h_1(M') - h_1(1) \leq -\frac{M''}{[M']^2}. \tag{G.3}$$

Straightforward differentiation yields

$$h_i'(\lambda) = \frac{1}{\lambda^2}\left\{2\frac{U''(U_i'^{-1}(\lambda U_i'(M)))}{\lambda U'(M)} - \frac{U'''(U_i'^{-1}(\lambda U_i'(M)))}{U''(U_i'^{-1}(\lambda U_i'(M(\tilde{w}))))}\right\}.$$

Defining

$$q_i(\lambda) = U_i'^{-1}(\lambda U_i'(M)),$$

one can write

$$Q_i = q_i(M')$$

and

$$h_i'(\lambda) = \frac{1}{\lambda^2}\left\{2\frac{U_i''(q_i(\lambda))}{U_i'(q_i(\lambda))} - \frac{U_i'''(q_i(\lambda))}{U_i''(q_i(\lambda))}\right\}. \tag{G.4}$$

(Note that if $-\frac{U''}{[U']^2}$ is increasing, as assumed in Proposition 8, then $h'(\lambda) \leq 0$.)

The assumption that $-\frac{U_2''(x)}{U_2'(x)} \geq -\frac{U_1''(x)}{U_1'(x)}$ (condition **(c)** in Proposition 9) implies that

$$\frac{U_2'(x)}{U_2'(M)} \geq \frac{U_1'(x)}{U_1'(M)}.$$

for any $x \leq M$. Therefore, for any $\lambda \geq 1$,

$$q_2(\lambda) = U_2'^{-1}(\lambda U_2'(M)) \geq U_1'^{-1}(\lambda U_1'(M)) = q_1(\lambda).$$

If $2\frac{U_2''(x)}{U_2'(x)} - \frac{U_2'''(x)}{U_2''(x)}$ is decreasing (condition **(e)** in Proposition 9) then

$$\begin{aligned}
h_2'(\lambda) &= \frac{1}{\lambda^2}\left\{2\frac{U_2''(q_2(\lambda))}{U_2'(q_2(\lambda))} - \frac{U_2'''(q_2(\lambda))}{U_2''(q_2(\lambda))}\right\} \\
&\leq \frac{1}{\lambda^2}\left\{2\frac{U_2''(q_1(\lambda))}{U_2'(q_1(\lambda)} - \frac{U_2'''(q_1(\lambda))}{U_2''(q_1(\lambda))}\right\} \\
&\leq \frac{1}{\lambda^2}\left\{2\frac{U_1''(q_1(\lambda))}{U_1'(q_1(\lambda)} - \frac{U_1'''(q_1(\lambda))}{U_1''(q_1(\lambda))}\right\} \\
&= h_1'(\lambda)
\end{aligned}$$

Alternatively, if $2\frac{U_1''(x)}{U_1'(x)} - \frac{U_1'''(x)}{U_1''(x)}$ is decreasing (condition **(e)** in Proposition 9) then

$$\begin{aligned} h_2'(\lambda) &= \frac{1}{\lambda^2}\left\{2\frac{U_2''(q_2(\lambda))}{U_2'(q_2(\lambda)} - \frac{U_2'''(q_2(\lambda))}{U_2''(q_2(\lambda))}\right\} \\ &\leq \frac{1}{\lambda^2}\left\{2\frac{U_1''(q_2(\lambda))}{U_1'(q_2(\lambda)} - \frac{U_1'''(q_2(\lambda))}{U_1''(q_2(\lambda))}\right\} \\ &\leq \frac{1}{\lambda^2}\left\{2\frac{U_1''(q_1(\lambda))}{U_1'(q_1(\lambda)} - \frac{U_1'''(q_1(\lambda))}{U_1''(q_1(\lambda))}\right\} \\ &= h_1'(\lambda). \end{aligned}$$

Both cases use condition **(d)** in Proposition 9, namely the assumption that $2\frac{U_2''(x)}{U_2'(x)} - \frac{U_2'''(x)}{U_2''(x)} \leq 2\frac{U_1''(x)}{U_1'(x)} - \frac{U_1'''(x)}{U_1''(x)}$.

# Appendix H Proof of Proposition 11

Substituting $U(x) = \frac{x^{1-\rho}}{1-\rho}$ into the identity $U'(Q) = U'(M)M'$ and solving for $Q$,

$$Q = M[M']^{-1/\rho}.$$

Differentiating with respect to addition of a nonstochastic constant,

$$Q'(\tilde{w}) = \left(1 - \frac{M(\tilde{w})M''(\tilde{w})}{\rho[M'(\tilde{w})]^2}\right)[M'(\tilde{w})]^{1-(1/\rho)}. \tag{H.1}$$

Using equations (A.1) and (A.2), we have

$$M' = \frac{\mathbf{E}\, v'(\tilde{w})}{v'(M)} = M^\gamma \mathbf{E}\, \tilde{w}^{-\gamma}, \tag{H.2}$$

and

$$\begin{aligned} \frac{MM''}{[M']^2} &= Mv'(M)\left[\frac{\mathbf{E}\, v''(\tilde{w})}{[\mathbf{E}\, v'(\tilde{w})]^2} - \frac{v''(M)}{[v'(M)]^2}\right] \\ &= \gamma - \gamma M^{1-\gamma}\frac{\mathbf{E}\, \tilde{w}^{-\gamma-1}}{[\mathbf{E}\, \tilde{w}^{-\gamma}]^2}. \end{aligned} \tag{H.3}$$

Substituting into (H.1), we find that

$$Q'(\tilde{w}) = M^{\gamma-\frac{\gamma}{\rho}}[\mathbf{E}\,\tilde{w}^{-\gamma}]^{1-\rho^{-1}}\left(1-\frac{\gamma}{\rho}\right)+\frac{\gamma}{\rho}M^{1-\frac{\gamma}{\rho}}[\mathbf{E}\,\tilde{w}^{-\gamma-1}][\mathbf{E}\,\tilde{w}^{-\gamma}]^{-1-\rho^{-1}}. \quad \text{(H.4)}$$

As in the proof of Proposition 7 in Appendix F, we focus on two-point risks equal to $\epsilon < 1$ with probability $p$ and $\zeta > 1$ with probability $1-p$ that satisfy $M(\tilde{w}) = 1$. Combining this choice with (H.4) yields

$$\begin{aligned} Q' &= \frac{\gamma}{\rho}[p\epsilon^{-\gamma-1}+(1-p)\zeta^{-\gamma-1}][p\epsilon^{-\gamma}+(1-p)\zeta^{-\gamma}]^{-1-\rho^{-1}} \\ &\quad -\frac{(\gamma-\rho)}{\rho}[p\epsilon^{-\gamma}+(1-p)\zeta^{-\gamma}]^{1-\rho^{-1}}. \end{aligned} \quad \text{(H.5)}$$

In this case, however, let

$$p = \epsilon^{\gamma+\alpha-1}$$

where $\alpha \in (\max\{0, 1-\gamma\}, 1-\rho)$. Note that the condition $\rho < \max\{1, \gamma\}$ is exactly what is needed to guarantee that this set is nonempty. To redeem the pledge that $M(\tilde{w}) = 1,\quad \zeta$ must be given by

$$\zeta = \left(\frac{1-\epsilon^{\alpha}}{1-\epsilon^{\gamma+\alpha-1}}\right)^{\frac{1}{1-\gamma}},$$

which limits into

$$\zeta = \epsilon^{-\frac{\epsilon^{\alpha}}{1-\epsilon^{\alpha}}}$$

when $\gamma \to 1$.

Having $\alpha \in (\max\{0, 1-\gamma\}, 1-\rho)$guarantees that, as $\epsilon \to 0^+$,

$$p = \epsilon^{\gamma+\alpha-1} \to 0,$$

$$p\epsilon^{1-\gamma} = \epsilon^{\alpha} \to 0,$$

$$\zeta \to 1,$$

and, consequently, that

$$Q' \to 0.$$

Since $[1 + \theta^{*'}(x)]Q' = 1$, this implies not only that $\theta^*$ can be increasing, but also the stronger result that $\theta^{*'}(x)$ can be made as large a positive number as desired. Intuitively, given the relatively high elasticity of intertemporal substitution, which yields a strong Drèze-Modigliani substitution effect, the consumer can "afford" more of the extra precautionary saving beyond risk aversion (and for these parameter values, even beyond prudence) as he or she moves up from a desperately impoverished situation.

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