Chapter 11

Wind and Brass Instruments

Where the string instruments of the previous chapter employ the vibration of strings to produce sound, wind and brass instruments use vibrations of air itself. Usually this involves setting up a standing wave, a stationary pattern of vibration, inside a pipe or cavity, then allowing a part of the vibration to escape in the form of sound through an opening in the pipe. Because they use air as their vibrating medium, wind and brass instruments generate sound more directly than string instruments. They do not require a soundboard, for instance, or any equivalent device. They directly produce vibrations of the air that we hear as sound.

We discuss both woodwind and brass instruments in this chapter and for convenience we will refer to them collectively as "wind" instruments. We divide them into three main classes depending on the method they use to set the air in the instrument vibrating. Reed instruments make use of a thin, flexible reed to inject bursts of air into a pipe. Brass instruments use a similar strategy but replace the reed with the player’s lips, while flutes use a different technique in which a jet of air oscillates in and out of the pipe. Wind instruments are usually powered directly by the breath of the player, although in a few cases, such as pipe organ, pedal harmonium, and accordion, they are driven mechanically by an air pump.

In this chapter we look first at the phenomenon of standing waves in pipes and tubes and then at how those standing waves are put into use in a range of wind instruments, from simple recorders to complex instruments like the clarinet, flute, and trumpet.

11.1 Vibration of Air in a Pipe

As discussed in Chapter 9, all musical instruments make use of some sort of vibrating element in order to produce sound. In wind instruments that vibrating element is a
Figure 11.1: Vibration of air in a pipe. If we take a short section of pipe, closed at both ends, and push the air inside it to one end (top) the build-up of pressure at that end provides the restoring force needed to push it back towards the other end, where the pressure builds up again (middle). The air will continue to rock back and forth until the vibration dies away. The bottom panel shows the pressure profile within the pipe over time.

Consider a short pipe—a straight, cylindrical tube such as copper water pipe or garden hose, perhaps a meter long or less—and suppose we close off both ends of the pipe so that the air inside cannot escape. If we nudge it in just the right way, the enclosed air can be made vibrate as shown Fig. 11.1. (For the moment we ignore the important question of how we set it vibrating, but we will discuss that soon.) Suppose we somehow push the air over to the left in the picture, compressing it against the left-hand cap of the pipe. This drives up the pressure at the left end and this increased pressure pushes the air back the other way until it runs up against the right-hand end of the pipe, where the pressure again builds up and pushes back to the left again, and so the motion repeats with the air sloshing back and forth along the pipe. This is a standing wave. It is an oscillation of the air, similar in some ways to the sound waves we are already familiar with, but unlike normal sound waves it is fixed in place, trapped by the ends of the pipe, rather than traveling along at the speed of sound.

More formally, as discussed in Section 9.1, a vibrating system must have three crucial features: an equilibrium position, a restoring force, and inertia. The air in our

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1 An exception is the “hard reed” instruments, such as harmonica, which do not have a pipe and which we discuss separately in Section 11.5.
Figure 11.2: Wavelength of a standing wave. We can imagine the standing pressure wave of Fig. 11.1 as a portion of a (fictitious) larger wave as indicated by the dashed line. The length of the portion inside the pipe is one half of the wavelength of the full wave.

pipe has all three. Equilibrium occurs when the air is uniformly distributed and at rest in the pipe. When the air is displaced to one side or the other there is a restoring force pushing it back towards the middle that comes from the build-up of pressure against the closed-off end of the pipe. And the moving air has inertia: all objects that have weight or mass have inertia, and air has mass. The mass of air is not very great—a cubic meter of air weighs only about one kilogram—but it is enough to provide the inertia we need to drive the vibration.

Another way of representing the movement of the air in the pipe is shown in the last panel of Fig. 11.1, where we plot the pressure along the pipe at various times during the oscillation. Initially, for instance, the pressure might be high at the left end of the pipe and low on the right (the bold curve in the figure). Then over time it equalizes and then builds up on the right side before moving back to the left again. The result is a variety of pressure profiles over time as shown.

Now take a look at Fig. 11.2 and consider the bold curve again. This curve shows the same pressure profile as in Fig. 11.1 but we now think of it as a portion of a larger pressure wave that extends outside the pipe and far to the left and right. This larger wave does not actually exist—this is just a thought experiment—but it is useful nonetheless. Note how the portion of the wave in the pipe (the portion that actually exists) amounts to exactly one half of a wavelength of the larger wave. In Section 2.1.1 we saw that for pressure waves in air there is a simple relationship between the wavelength $\lambda$ and the frequency $f$:

$$f = \frac{c}{\lambda}. \quad (11.1)$$

We derived this result for a traveling sound wave, whereas the wave inside our pipe is a standing wave, but it turns out that Eq. (11.1) applies to standing waves also. We will prove this in Section 11.1.5, but for now let us see what it can tell us.

Suppose the length of our pipe, from end to end, is $L$. Since, as we have said, there is one half wavelength of our standing wave in the pipe, we have $\frac{1}{2} \lambda = L$ and
hence \( \lambda = 2L \). Feeding this value into Eq. (11.1), we get

\[
f = \frac{c}{2L}.
\]

(11.2)

This equation is the equivalent for air in a pipe of Mersenne’s law for the vibration of a string. It tells us the frequency with which the air will vibrate. As with the string, the frequency for a pipe is inversely proportional to the length, which means we can change the note the pipe produces by varying its length. On the other hand, the frequency does not depend on the diameter of the pipe or on anything else about it, such as whether it is square or round or what it is made of. Indeed, other than length the frequency depends only on one quantity, the speed of sound, which is not something we can easily change. This contrasts with a string, for which there are many ways to change the frequency: we can vary the length, diameter, tension, or even the density of the string. For the pipe, there is only one way to do it—changing the length—and so this is the means by which wind instruments produce different notes. A wind instrument is essentially a pipe whose length we vary in some way to produce a scale of notes. Instruments differ in how we vary the length. The trombone uses a sliding section of pipe that adds directly the length. The flute has holes that open up to let the air out and effectively shorten the length. The pipe organ just uses a different pipe with a different length for every note. These choices, along with the means by which vibration is produced, are the main things that distinguish wind instruments from one another.

Instruments do also differ in what the pipe is made of. Clarinets are made of wood, trumpets are made of metal, flutes may be made of either. Even plastics can be used to good effect, as in some recorders and saxophones. However, since the sound is produced by the vibration of the air within the pipe rather than by the pipe itself, the choice of material has little effect on the tonal characteristics of the instrument and none at all on the pitch of the note. Different materials are favored for practical reasons—ease of manufacture, weight, durability, or visual appearance—but they do not play a significant role in the sound. This contrasts with the situation for string instruments. For instance, the material from which strings are made substantially affects both pitch and tone for a string instrument, and the choice and quality of wood used in making a violin or guitar has an enormous impact on tone, since it is the vibration of the wood itself that produces most of the sound of the instrument.

**Example 11.1: An organ pipe**

By longstanding tradition, the lengths of organ pipes are specified in feet and inches. Suppose a certain organ pipe is 2 feet 2 inches long. What note will it produce?

To answer this question we first convert the length into more convenient units: 2 foot 2 inches is 66 cm. Substituting this value into Eq. (11.2) and using the standard value of
343 m/s for the speed of sound we then get

\[ f = \frac{343}{2 \times 0.66} = 260 \text{ Hz}, \]  

which is very close to the frequency of the note C4. So this organ pipe will play a middle C.

**Example 11.2: The lowest note on a tuba**

The lowest note on a (contrabass or BB\textsubscript{b}) tuba is E\textsubscript{0}. How long is the pipe of the tuba?

To answer this question, we first need to calculate the frequency of the note. E\textsubscript{0} is lower than the lowest note on a standard piano and is not included on our chart in Fig. 2.11, but we can calculate its frequency by observing that the note E\textsubscript{1}, an octave higher, has frequency 41.2 Hz, so the frequency of E\textsubscript{0} must be half of this, or 20.6 Hz. Then we rearrange Eq. (11.2) to get \( c = \frac{2\times f}{2} \) and we have:

\[ L = \frac{343}{2 \times 20.6} = 8.33 \text{ meters}. \]  

So the pipe of the tuba is over 8 meters long, or about 27 feet! The piping of a tuba is not straight: it is coiled many times around to make it manageable. (It turns out that this does not significantly effect the sound of the instrument—see Section 11.1.8.) But if it were uncoiled, it would be about as high as a two-story house.

### 11.1.1 Open pipes

Our pipe with closed ends generates vibrations of the air, but it would not make a good musical instrument because there is no way for those vibrations to escape and create sound. With both ends of the pipe closed off (and no other openings) the vibration would be sealed inside. Real wind instruments use open-ended pipes that let the sound out.

But the behavior of an open-ended pipe is harder to understand. Why would the air in an open pipe vibrate at all? One can see why air in a closed pipe vibrates as it pushes off the closed ends of the tube. But in an open-ended pipe, with no ends to push against, why would it not simply escape into the world outside?

And yet the air in an open pipe does vibrate. The explanation is shown in Fig. 11.3. Instead of sloshing back and forth between the two ends of the pipe, the air now vibrates by compressing and expanding in the center of pipe. Suppose for instance that the air starts out bunched together in the center, as shown in the top panel of the figure. (Again, we ignore for the moment the question of how we start the vibration going, although we’ll get to that soon.) This increases the pressure in the center and this pressure pushes the air back out towards the ends of the pipe. And since the ends are open some air will indeed escape from the pipe. This equalizes the pressure along the pipe again.

But things don’t end there. Since the air has inertia, it doesn’t just stop moving when the pressure is equalized. It keeps on going, with more air leaving through the
ends of the pipe and now causing a drop in the pressure in the center of the pipe. Thus a partial vacuum of rarefied air forms in the center, as shown in the second panel of Fig. 11.3, and this now sucks the air back in at the ends of the pipe and back towards the center again, equalizing the pressure once more. Again the motion doesn’t stop when the pressure equalizes. Inertia keeps the air moving towards the center and it bunches up again, pushing up the pressure once more, and so the cycle starts over.

The result is a new kind of vibration where the air is repeatedly compressed and stretched in the middle of the pipe, forcing some air in and out of the ends. It is this air, moving in and out at the ends of the pipe, that produces the sound of a wind instrument.

Now take a look at the final panel of Fig. 11.3, which shows a graph of the pressure inside the pipe. Here an interesting thing happens: the pressure in the middle of the pipe goes up and down as we expect, but the pressure at the ends of the pipe stays fixed—it doesn’t change at all. What is going on here? The answer is that the pressure at the ends of the pipe is necessarily always equal to the prevailing atmospheric pressure in the world outside. The air in the room will be at atmospheric pressure, as air is in any normal space, so we know the pressure right at the ends of the pipe must be equal to that same atmospheric pressure. Strictly speaking the pressure varies a little because of sound. Our pipe is producing sound waves (which is the entire point of a musical instrument) and as we know this causes variation in the
pressure. But we also know that the pressure variation due to sound is minuscule, as small as one part in a million or less (see Example 1.1 on page 5 for instance). So to a very good approximation the pressure at the ends of an open pipe is equal to atmospheric pressure, and hence fixed. The pressure varies in the middle of the pipe, but not at the ends, just as the graph in Fig. 11.3 shows.

It might occur to you that the graph in Fig. 11.3 looks rather similar to the picture of a vibrating string in Fig. 9.1c (page 322), and if so you are not wrong. There is in fact a close analogy between the vibration of a string and the oscillation of the air pressure inside a pipe, which we explore further in Section 11.1.2.

Figure 11.4 shows the equivalent of Fig. 11.2 for the open-ended pipe. Just as with the closed pipe, we can imagine the pressure inside the pipe to be part of a larger pressure wave as shown. Again this is just a thought experiment—the larger wave does not really exist—but as we can see the part of the wave inside the pipe is one half of a wavelength of the full wave. So again the wavelength \( \lambda \) is related to the length \( L \) of the pipe by \( \frac{1}{2} \lambda = L \), or \( \lambda = 2L \), and combining this with Eq. (11.1), we have

\[
f = \frac{c}{2L},
\]

(11.5)

exactly as for the closed pipe.

In other words, the open pipe will produce vibrations of the same frequency as a closed pipe of the same length, and hence it produces the same musical pitch and we can vary the pitch in the same way by changing the length. But the open pipe has the great advantage that, since its ends are open, the sound can escape from the pipe so that we can hear it. The movement of the air in and out of the ends of the pipe pushes on the air immediately adjacent to it and causes that to vibrate too, starting a sound wave that spreads out through the free air and produces a musical note.

This is how a wind instrument works, and arguably it is simpler than the string instruments of Chapter 10. The vibration of the air in a wind instrument directly produces sound, with no need for strings or a soundboard or the complicated paraphernalia of guitars and pianos and violins. Some wind instruments are still quite
complicated—take a look at the system on levers that operates a saxophone, for instance—but others are simple devices consisting of little more than an open-ended tube. Indeed a good trumpet or trombone player can get recognizable, if limited, music from a simple cardboard tube or a garden hose.

11.1.2 Modes of vibration

The story of air vibration in pipes doesn’t end there, however, because it turns out that, like a string, a pipe has more than one mode of vibration. The vibration we have seen so far, depicted in Fig. 11.3, is the fundamental mode, but there are others. Figure 11.5 shows the second and third modes.

In the second mode, shown at the top of the figure, the pipe splits into two halves, with the motion in each half being a smaller copy of the fundamental mode of Fig. 11.3. Crucially, however, notice that the motions in the two halves are mirror images of one another. First the left half is compressed while the right half expands (top panel). Then the left half expands while the right half is compressed (second panel). The third panel shows the variation of pressure along the pipe plotted as a graph.

The third mode is similar, but now the pipe divides into three regions, with the motion in the two outer regions being the same but the one in the middle being opposite. Thus the two outer regions first compress while the middle expands, then the outer ones expand while the middle compresses. The pressure is plotted as a graph in the last panel of Fig. 11.5.

It may again occur to you that the graphs of pressure in Fig. 11.5 look similar to the modes of the vibrating string in Fig. 10.1 on page 335. And indeed there is a close analogy between the two. Note in particular how there are points along the pipe where the pressure doesn’t vary at all, always taking the same value. By analogy with the movement of the string we refer to these points as nodes. In between the nodes there are antinodes, the points at which the pressure variation is greatest. Thus the second mode, for example, has two antinodes, and three nodes, assuming we count the points at the two ends of the pipe as nodes:

\[
\begin{array}{cccccccccccccc}
\text{Node} & \text{Antinode} & \text{Node} & \text{Antinode} & \text{Node} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

Note how the nodes and antinodes alternate along the pipe. There will always be nodes at the ends of the pipe since, as have said, the pressure at the ends is fixed and unchanging—it is always equal to atmospheric pressure.
Figure 11.5: Second and third modes of vibration of a pipe. In the second mode of vibration of the air in an open-ended pipe (top), the pressure changes in the two halves of the pipe are mirror images of each other, one half compressing as the other expands and vice versa. In the third mode (bottom) the pipe has three regions with the two on the outside compressing while the one in the middle expands and vice versa.
We can make an argument similar to the one we made before and imagine the pressure curve for the second mode as a portion of an imaginary wave extending to the left and the right. But notice now that the curve corresponds to a full wavelength and not a half wavelength as before. In other words, the wavelength $\lambda$ for this mode is just equal to the length of the pipe $\lambda = L$. Combining this observation with Eq. (11.1) we then get the frequency $f_2$ of the second mode thus:

\[
f_2 = \frac{c}{L},
\]

(11.6)

which is twice the frequency of the fundamental in Eq. (11.5).

Similarly, looking at the third mode in Fig. 11.5, the pressure curve (bottom of the figure) has three half-wavelengths along the length of the pipe, so we have $3 \times \frac{1}{2} \lambda = L$, or $\lambda = \frac{3}{2} L$, which gives

\[
f_3 = \frac{3c}{2L},
\]

(11.7)

which is three times the fundamental frequency.

We can continue this line of reasoning to higher modes. The fourth mode has two full wavelengths inside the pipe; the fifth mode has $\frac{5}{2}$ of a wavelength, and so on. The general rule is that the $n$th mode has $n$ half-wavelengths inside the pipe, so $n \times \frac{1}{2} \lambda = L$, or $\lambda = \frac{2}{n} L$. Combining this result with Eq. (11.1), the frequency of the $n$th mode is

\[
f_n = \frac{n c}{2L},
\]

(11.8)

which is simply $n$ times the frequency $c/2L$ of the fundamental mode. In other words, all the modes have frequencies that are whole-number multiples of the fundamental frequency, which means they are harmonics of the fundamental. Just as with the vibration of a string discussed in Section 10.1, the modes of vibration of the air in an open-ended pipe have frequencies that are harmonics of the fundamental frequency.

Moreover, as with the string, the air in a pipe doesn’t have to vibrate in only one mode at a time. It is possible for it to vibrate simultaneously in two or more modes, with the vibrations of the different modes getting overlaid one on top of another. If one were to make a graph of the pressure, it would look very similar to the motion of the string depicted in Fig. 10.3 on page 338. The result is that the sound produced by a wind instrument is a mixture of a fundamental frequency and its harmonics. As an example, Fig. 11.6 shows the frequency spectrum of the note A4 played on a flute, and the first six harmonics are clearly visible, plus trace amounts of a few more.

![Figure 11.6: Frequency spectrum of a flute note. The frequency spectrum of the note A4 played on a standard concert flute. Peaks corresponding to several harmonics are visible.](image-url)
As we saw in our discussion in Section 4.3.3, mixtures of harmonics produce periodic waveforms with clear musical pitch and a timbre that depends on the particular balance of harmonics. Mixtures that contain a lot of high harmonics will produce bright timbres, while mixtures that contain mostly lower harmonics will be warmer. The particular mixture we get from an instrument depends on various factors, such as the shape of the pipe, the means by which the air is set in motion, and how the player operates the instrument, such as how hard they blow. We will see a number of examples in this chapter.

11.1.3 Tuning and temperature

A quirk of wind instruments is that the pitch of the notes they produce depends—slightly—on temperature. The figure of \( c = 343 \text{ m/s} \) that we normally use for the speed of sound is correct for air at a standard room temperature of \( 20^\circ \text{C} \), but the value does vary with temperature, being a little faster for higher temperatures and a little slower for lower ones—see Table 1.1 on page 13. At \( 30^\circ \text{C} \), for instance, it is \( 349 \text{ m/s} \), which is not a huge change from \( 343 \text{ m/s} \), but it is enough to make a noticeable difference to the pitch of a wind instrument. Since the fundamental frequency of a pipe of length \( L \) is \( 2c/L \), a higher value of \( c \) will produce a higher pitch, so instruments will be sharp at higher temperatures and flat at lower ones.

Suppose, for example, we play a note at a certain temperature, with frequency \( f_1 \) given by Eq. (11.5):

\[
 f_1 = \frac{c_1}{2L},
\]

where \( c_1 \) is the speed of sound at this temperature and \( L \) is the length of the pipe. If the temperature now changes so that speed of sound has a new value \( c_2 \), while the length of the pipe stays the same, the frequency will change to

\[
 f_2 = \frac{c_2}{2L}.
\]

The number \( n \) of musical cents difference between the two frequencies is given by Eq. (2.35):

\[
 n = \frac{1200}{\log 2} \log \frac{f_1}{f_2} = \frac{1200}{\log 2} \log \frac{c_1/2L}{c_2/2L} = \frac{1200}{\log 2} \log \frac{c_1}{c_2}.
\]

So for instance if our two temperatures are \( 20^\circ \text{C} \) and \( 30^\circ \text{C} \), with speed of sound 343 m/s and 349 m/s respectively, then we have

\[
 n = \frac{1200}{\log 2} \log \frac{349}{343} = 30 \text{ cents},
\]

which is enough to be audible to any listener.

This presents a problem. The temperature in a performance space could easily vary by \( 10^\circ \text{C} \) or more from one day to another, making an instrument that is normally
in tune substantially sharp or flat, which would be very noticeable if it were playing at the same time as other instruments. And even if the temperature doesn’t change, the air inside a wind instrument warms up just because of the player’s breath. When a musician first picks up an instrument like a flute, which hasn’t been played for a while, the air inside it will be at room temperature, perhaps 20°C. But as the player plays, the instrument will warm up significantly—enough to feel warm to the touch—and this warms the air inside too, making the pitch go up. So even if the instrument was in tune to begin with, after a few minutes it will not be.

To get around this problem, most wind instruments include some mechanism for making slight adjustments to the length of the pipe in order to change the tuning and compensate for any changes in temperature. A flute, for instance, is made of two separate pieces of pipe connected together by a joint. The pieces can be pulled apart slightly at the joint to lengthen the pipe and hence tune the pitch of the instrument.

Some wind instruments are harder to tune. A pipe organ, for example, with hundreds or even thousands of pipes, is not easy to tune because each individual pipe has to be adjusted. Pipe organs do usually provide a mechanism for tuning the pipes—for instance there may be a sliding collar at the end of a pipe that can be moved to adjust the length—but tuning an organ is a huge operation that is not undertaken lightly. On the other hand, pipe organs are not warmed by the player’s breath, since they are operated by a mechanical pump and not by a player blowing them, so variation in tuning, at least within a single performance, is not usually a problem. Room temperature can still vary from one day to another and cause a perceptible pitch change, but if there are other instruments playing along with the organ then common practice is to have everyone else tune to match the organ rather than the other way around.

**Example 11.3: Tuning in cold weather**

On a cold day in winter the temperature inside a (poconstantly heated) church is 15°C. Will the church organ be sharp or flat, and by how many musical cents?

The note produced by a pipe is \( f = \frac{c}{2L} \), so the frequency goes up when \( c \) increases and down when it decreases. Table 1.1 on page 13 tells us that the speed of sound at 15°C is 340.3 m/s, a little less than the standard 343 m/s at normal room temperature, so the frequency will go down and the organ will play flat. The amount is given by Eq. (11.11):

\[
 n = \frac{1200 \log \frac{c_1}{c_2}}{\log 2} = \frac{1200 \log \frac{343}{340.3}}{\log 2} = 13.7 \text{ cents},
\]

enough to be noticeably out of tune if there are other instruments playing along at the same time.
11.1 Vibration of air in a pipe

![Diagram of air vibrations in a pipe](image)

**Figure 11.7: Vibration of air in a pipe closed at one end.** In a pipe closed at one end but open at the other a build-up of pressure at the closed end (top panel) forces air towards the open end and out of the pipe (second panel), causing a drop in pressure on the left that then sucks the air back in again. The bottom panel shows the variation of pressure with position along the pipe.

11.1.4 Pipes closed at one end

A variant of the open-ended pipe we have considered so far is a pipe open at one end and closed at the other. In technical discussions of wind instruments such pipes are referred to as *closed pipes*. One should be careful not to confuse these with the fully closed pipes we discussed in Section 11.1, which were closed at both ends. Fully closed pipes are never used as musical instruments, because there is no way for the sound to escape from them, so in practice the nomenclature is usually unambiguous: an "open pipe" is open at both ends and a "closed pipe" is closed at one end and open at the other. We will henceforth use these terms in this way.

A closed pipe can function as a musical instrument—the sound can still escape from the open end—but the air inside it vibrates in a different way from that in an open pipe. The fundamental mode of vibration is illustrated in Fig. 11.7. Suppose the air starts out pushed up against the closed end of the pipe, on the left in the top panel of the figure. (As before, we will for now ignore how it is made to do this, but we will come that soon.) The increased pressure at the left end pushes the air to the right down the pipe and some of it will escape from the open end into the world outside. Eventually, the pressure in the pipe will equalize, but that does not mean the air stops moving. Because it has inertia, the air keeps moving to the right, with
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Figure 11.8: Standing wave in a pipe closed at one end. The wave inside a pipe that is open at one end and closed at the other can be thought of as a quarter wavelength of a larger wave that extends outside the pipe.

more air escaping from the open end and so the air inside the pipe becomes rarefied, forming a partial vacuum (middle panel in the figure). This vacuum then sucks on the air, halting its motion and pulling it back in again through the open end of the pipe, and so the cycle starts again.

The bottom panel of Fig. 11.7 shows how the pressure varies during the vibration and as we can see the graph is now asymmetric. The pressure variation is greatest on the left where the air gets squashed against the closed end of the pipe. This is an antinode of the vibration. At the same time there is no pressure variation at all at the open end on the right. As with the open pipe of Section 11.1.1 (open at both ends, that is) the open end is a node because the pressure there is always equal to atmospheric pressure (apart from the tiny variations due to sound, which are so small that they can be ignored).

As with our other pipes, we can think of the pressure variation in a closed pipe as part of a larger, imaginary wave as shown in Fig. 11.8. But now, instead of occupying a half of a wavelength, the portion inside the pipe occupies only a quarter of a wavelength. This means that the wavelength \( \lambda \) obeys the equation \( \frac{1}{4} \lambda = L \), or \( \lambda = 4L \). Combining this result with Eq. (11.1) we then find that the frequency of the closed pipe is

\[
f = \frac{c}{4L}, \tag{11.14}
\]

which is half the frequency of the open pipe in Eq. (11.5).

In other words, the closed pipe will make a note, but that note is an octave lower than the open pipe of the same length. This can be very useful. Usually if we want to make a note an octave lower, Eq. (11.5) says that we have to double the length of the pipe and the pipe can rapidly become very long and unwieldy. In Example 11.2 for instance we saw that the pipe of a BB♭ tuba is over 8 m long. The closed pipe gives us a different way of getting low notes without making the pipe so long: we close it off at one end. For instance, an open pipe 66 cm long produces the note C4, or middle C, as in Example 11.1. But the same pipe closed at one end produces C3 an octave lower, without being any longer. This is used for instance in the pipe organ, where the lowest notes would require extraordinarily long pipes if those pipes were
open. Some organs go as low as the note C0, which at 16.4 Hz is below the normal limit of human hearing and is less heard than felt, as a rumbling sensation in the front seats. An open pipe that made this note would have length

\[ L = \frac{c}{2f} = \frac{343}{2 \times 16.4} = 10.5 \text{ meters}, \]

or about 34 feet, which would have it sticking out through the roof of many churches. A closed pipe with the same pitch on the other hand would be half the length, making it 5.3 meters or 17 feet—still long but a lot more manageable. In practice, all the lowest notes on a pipe organ are generated using closed pipes for this reason. Closed pipes are also used on some orchestral instruments, such as the clarinet, which as a result can reach notes an octave lower than similarly sized open-pipe instruments like the flute. The pipe of a clarinet is open at the bottom end where the sound emerges but closed off at the top by the mouthpiece. There is a small opening at the top between the mouthpiece and the reed—there has to be for the player to blow into the instrument—so the pipe is not perfectly closed, but to a good approximation the clarinet is a closed pipe.

Like open pipes, closed pipes have multiple possible modes of vibration, but the modes of a closed pipe have some unique features. The vibration depicted in Fig. 11.7 is the fundamental mode. The next two modes after that are shown in Fig. 11.9, along with graphs showing how the pressure changes along the pipe. In each case the closed end of the pipe has maximum variation in pressure where the air squashes up against the end, making this an antinode of the vibration, while the pressure doesn’t change at all at the open end where it’s always equal to atmospheric pressure, making this a node. In between the two ends there can be other nodes and antinodes, but the arrangement always obeys the same rules:

1. We always start with an antinode at the closed end and end with a node at the open end.
2. Nodes and antinodes alternate along the pipe.
3. The distance between a node and an antinode is one quarter of a wavelength of the standing wave.

An important consequence of these rules is that the distance from one end of the pipe to the other is always an odd number of quarter-wavelengths: one or three or five, but never two or four. For instance, the mode pictured at the top of Fig. 11.9 has three quarter-wavelengths between one end of the pipe and the other, so \( \frac{3}{4} \lambda = L \) (where \( L \) is the length of the pipe as before), or \( \lambda = \frac{4}{3} L \). Combining this result with Eq. (11.1), we find the frequency of the mode to be

\[ f_3 = \frac{3c}{4\lambda}, \]

which is three times the frequency of the fundamental mode from Eq. (11.14).
Figure 11.9: Higher modes of vibration of a pipe closed at one end. For a closed pipe the next mode of vibration above the fundamental corresponds to the third harmonic. It has an antinode at the closed end of the pipe and a node at the open end, where the pressure is always equal to the prevailing atmospheric pressure. The mode above that corresponds to the fifth harmonic.
More generally, since there are always an odd number of quarter-wavelengths in the pipe, we have \( n \times \frac{1}{4} \lambda = L \), or \( \lambda = 4L/n \), where \( n \) is an odd number, so the corresponding frequency is

\[
f_n = n \cdot \frac{c}{4L}, \quad n = 1, 3, 5, \ldots
\]

(11.17)

Thus, the higher modes of vibration once again have frequencies that are whole-number multiples of the fundamental frequency \( c/4L \), meaning they are harmonics, but now only odd-numbered harmonics occur, never even ones.

This has a big effect on the sound of closed pipes. All the even harmonics—fully a half of all harmonics above the fundamental—are completely missing from the sound, so the timbre will be significantly less bright than an equivalent open-pipe instrument, which has all harmonics present. This goes a long way towards explaining the distinctive tone of an instrument like the clarinet.

Figure 11.10 shows the frequency spectrum of a note played on a clarinet, and we can see that indeed the sound is dominated by the odd harmonics. There is a small residual amount of the even harmonics, which happens because, as we have said, the closed end of the clarinet actually has a small opening, so the instrument is not a perfect closed pipe. The effect is modest, however. To a good approximation, the clarinet has odd harmonics only.

**Example 11.4: Notes from a cardboard tube**

A standard cardboard tube for mailing posters is 36 inches long. You can get a note from such a tube by blowing across the top. What frequency will this tube produce if both ends are open and what will be the frequencies of the next two harmonics above the fundamental? The tube also comes with plastic end-caps for sealing off the ends. If we put one of the end caps on the tube what now will be the frequency of the note the tube produces and what will be the next two harmonics?

Converted into scientific units, 36 inches is equivalent to 91.4 cm. When the tube is open at both ends Eq. (11.5) tells us that the frequency of vibration will be

\[
f = \frac{c}{2L} = \frac{343}{2 \times 0.914} = 187.6 \text{ Hz},
\]

(11.18)

which is close to the note F♯3. The open tube produces all harmonics, so the next two harmonics above the fundamental will be \( 2f \) and \( 3f \), which gives us frequencies 375.3 Hz and 562.9 Hz.

With an end-cap on the tube it becomes a “closed pipe,” with frequency an octave lower:

\[
f = \frac{c}{4L} = 93.8 \text{ Hz},
\]

(11.19)
close to F♯2, and the pipe now produces odd harmonics only, so the next two harmonics are 3f and 5f, which gives us 281.5 Hz and 469.1 Hz.

## Advanced material

### 11.1.5 Standing waves in an open pipe

We have already derived the equation governing air vibrations: they obey the wave equation, Eq. (1.16), which we repeat here for convenience:

\[
\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0, \quad (11.20)
\]

where \( c^2 \) is the speed of sound as usual. We derived this equation for a sound wave moving through open air, but the same equation applies also to vibrations of air inside a pipe.

In a pipe open at both ends we have one additional piece of information, that the total air pressure \( P \) at the ends of the pipe is always equal to the prevailing atmospheric pressure \( P_0 \). This in turn implies that the sound pressure is zero at these points, since from Eq. (1.1) we have \( p = P - P_0 = 0 \).

Suppose then that we have a pipe of length \( L \), running from \( x = 0 \) to \( x = L \). To calculate the sound pressure within the pipe we need to find the solution of Eq. (11.20) subject to the boundary conditions that \( p = 0 \) at \( x = 0 \) and \( x = L \). In fact there is an entire set of solutions that satisfy these conditions, thus:

\[
p(x, t) = A \sin \left( \frac{n \pi x}{L} \right) \sin (2 \pi f_n t), \quad (11.21)
\]

where \( f_n \) is the frequency of vibration and \( A \) is the amplitude. This form does indeed satisfy the boundary conditions at \( x = 0 \) and \( x = L \) provided that \( n \) is an integer \( n = 1, 2, 3 \ldots \), and we can verify that it is a solution to the wave equation by substituting into Eq. (11.20), which gives

\[
\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = -A \left( \frac{\pi n}{L} \right)^2 \sin \left( \frac{\pi n x}{L} \right) \sin (2 \pi f_n t)
\]

\[
+ A \left( \frac{2 \pi f_n}{c} \right)^2 \sin \left( \frac{n \pi x}{L} \right) \sin (2 \pi f_n t)
\]

\[
= A \sin \left( \frac{n \pi x}{L} \right) \sin (2 \pi f_n t) \left[ \left( \frac{2 \pi f_n}{c} \right)^2 - \left( \frac{n \pi}{L} \right)^2 \right],
\]

which is zero if

\[
\left( \frac{2 \pi f_n}{c} \right)^2 - \left( \frac{n \pi}{L} \right)^2 = 0. \quad (11.23)
\]

After rearranging and cancelling some factors, this becomes

\[
f_n = n \frac{c}{2L}. \quad (11.24)
\]

Thus Eq. (11.21) is a solution of the wave equation provided the frequency takes this value.

The solutions (11.21) are precisely the modes of the open pipe that we discussed in Section 11.1.2 and Eq. (11.24) reproduces the frequencies of the modes from Eq. (11.8) that we previously derived less formally.

But how do we know that these are the only solutions for vibrations in the pipe? Could there be others? In fact there are many other solutions, but all the others are just combinations of the ones in Eq. (11.21). To see why this is, note that in terms of \( x \) Eq. (11.21) takes the form of a sine wave. But, as we have seen repeatedly in this book, all functions can be expressed as a sum of an appropriate set of sine waves. Coupled with the fact that these particular sine waves are the only ones that satisfy the boundary conditions \( p = 0 \) at the ends of the pipe, this implies that any distribution of pressure along the pipe that satisfies the boundary conditions can be represented as a combination of the solutions in Eq. (11.21), and hence that we do not need to look any further for other solutions. These ones, plus sums of them, give us all possible solutions that satisfy the boundary conditions.

Equation (11.21) tells us a number of things. First, it gives us a rigorous derivation of the frequencies of the modes, Eq. (11.24), confirming that indeed they are all multiples of the fundamental frequency \( c/2L \) as we thought. Second, it tells us that the shape of the vibration over time is a sine wave \( \sin(2 \pi f_n t) \), and hence that these modes are indeed harmonics: they generate sine-wave pressure variations with frequencies that are integer multiples of the fundamental. Third, it tells us that the spatial shape of the vibration along the pipe is also a sine wave, which is how we made the graphs in Figs. 11.3 and 11.5 for example.
11.1.6 Air Velocity

For some calculations it is useful to know not only the pressure of the air in the pipe but also its velocity. This we can calculate from Eq. (1.25), which says that

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{dp}{dx},$$  (11.25)

where \(\rho\) is the density of air. If we substitute our solution, Eq. (11.21), on the right and perform the derivative with respect to \(x\) we get

$$\frac{du}{dt} = -A \frac{\pi n}{\rho L} \cos \left(\frac{\pi nx}{L}\right) \sin(2\pi f_n t).$$  (11.26)

Integrating both sides with respect to \(t\), we then get

$$u(x, t) = A \frac{n}{2\pi f_n \rho L} \cos \left(\frac{\pi nx}{L}\right) \cos(2\pi f_n t) + F(x),$$  (11.27)

where \(F(x)\) is an arbitrary function of \(x\) that plays the role of an integration constant. We note, however, that the air in the pipe has no overall bulk motion—on average it is not moving either to the left or the right—so its average velocity must be zero at all positions \(x\), which means that \(F(x) = 0\) everywhere. Making use of the value \(f_n = nc/2L\) from Eq. (11.24), we then get

$$u(x, t) = A \frac{1}{\rho c} \cos \left(\frac{\pi nx}{L}\right) \cos(2\pi f_n t).$$  (11.28)

The quantity \(z = \rho c\) is the acoustic impedance of air, which we encountered previously in our discussion of sound waves in Section 1.3.3.

In other words, the velocity in the pipe is essentially equal to the pressure of Eq. (11.21) divided by the acoustic impedance, just as it is in a normal sound wave (see Eq. (1.28)) but with one important difference: the sine functions of Eq. (11.21) have been replaced with cosines. This means that the velocity is not directly proportional to pressure as it is in a traveling sound wave but out of phase with it, both in space and in time.

This is a crucial difference between traveling waves and standing waves. In particular, it means that the points along the pipe where pressure is zero—the nodes of the pressure wave—are velocity antinodes, and the pressure antinodes are velocity nodes. Thus, for instance, the open ends of the pipe are always pressure nodes as we have seen, which means they are velocity antinodes, with maximum velocity value, as we can confirm by setting \(x = 0\) or \(x = L\) in Eq. (11.28) above.

Although pressure and velocity are out of phase, however, their amplitudes are still proportional to one another. From Eq. (11.28) the amplitude of the velocity is \(A/\rho c\), which is just the amplitude of the pressure divided by the acoustic impedance.

11.1.7 Standing Waves in a Pipe Closed at One End

For a pipe closed at one end and open at the other the open end is still a pressure node as before. At the closed end the air cannot move at all since the end wall of the pipe is in the way, so the velocity is zero that this point: the closed end is a velocity node.

Suppose that we have a pipe of length \(L\) again, closed at \(x = 0\) and open at \(x = L\). Then the equivalent of Eq. (11.21), the solution for a standing wave in such a pipe, is

$$p(x, t) = A \cos \left(\frac{\pi nx}{2L}\right) \sin(2\pi f_n t), \quad n = 1, 3, 5 \ldots$$  (11.29)

This makes the open end of the pipe a pressure node provided \(n\) is an odd integer and we can confirm that (11.29) is a solution of the wave equation by substituting into Eq. (11.20) to get

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = -A \left(\frac{\pi n}{2L}\right)^2 \cos \left(\frac{\pi nx}{2L}\right) \sin(2\pi f_n t) + A \left(\frac{2\pi f_n}{c}\right)^2 \cos \left(\frac{\pi nx}{2L}\right) \sin(2\pi f_n t)$$

$$= A \cos \left(\frac{\pi nx}{2L}\right) \sin(2\pi f_n t) \left[ \left(\frac{2\pi f_n}{c}\right)^2 - \left(\frac{\pi n}{2L}\right)^2 \right].$$  (11.30)

which is zero provided that

$$\left(\frac{2\pi f_n}{c}\right)^2 - \left(\frac{\pi n}{2L}\right)^2 = 0,$$  (11.31)

or equivalently,

$$f_n = \frac{n c}{4L}, \quad n = 1, 3, 5 \ldots$$  (11.32)

in agreement with Eq. (11.17).

At the same time we can confirm that the closed end is a velocity node by calculating the velocity from Eq. (11.25) as we did before. Following the same steps once again gives

$$u(x, t) = -A \frac{1}{\rho c} \sin \left(\frac{\pi nx}{2L}\right) \cos(2\pi f_n t).$$  (11.33)
which is indeed zero at $x = 0$ for all times $t$. This equation also tells us that the amplitude of the velocity variation is $A/pc$ as before, but that once again the velocity is out of phase with the pressure, both in space and in time, so that velocity nodes coincide with pressure antinodes and vice versa.

As with the open pipe, other solutions of the wave equation are also possible, but all of them are sums of the solutions in Eq. (11.29). Combined with Eq. (11.32) for the frequencies of vibration, this tells us that all sounds produced by a closed pipe instrument will contain odd harmonics only, an important tonal characteristic of such instruments.

11.1.8 Non-cylindrical pipes

So far in our discussion of wind instruments we have looked at vibrations in cylindrical pipes. It is possible, however, to make instruments with pipes of other shapes. For instance, they do not have to be circular in cross-section. Any pipe that is uniform, meaning it is the same shape and size all the way along, will work fine. All such pipes have vibration modes that are multiples of the fundamental frequency and hence produce a fundamental plus harmonics and clear musical notes. It is common for example to make square organ pipes like this:

Occasionally you also see triangular ones. Pipes of these shapes produce a sound essentially the same as circular pipes. The choice of which shape to use is mostly a matter of convenience.

It is also possible to make instruments with non-uniform pipes—their width varies along their length. The most common case is that of a flared pipe that gets progressively wider from the blowing end to the open end. Flared-pipe instruments, also known as horns, include the trumpet and the saxophone.

The vibration of air in a non-uniform pipe is more complicated than in a uniform one. When a pipe gets wider the velocity of the air inside drops, like a river that flows slowly where it is wide and rapidly where it is narrow. The varying speed of the air can produce a variation in the wavelength of the standing wave, so that the wavelength is different at different points along the pipe. An open end of a pipe is still a pressure node and a closed end is a pressure antinode, but one can no longer say that the pipe length is simply a quarter of a wavelength or three quarters and so forth. The relationship between the frequency of vibration and the pipe length becomes more complicated, depending on the details of how the width of the pipe varies, and though there are still a range of modes of vibration their frequencies are normally not simple multiples of any fundamental. As a result the pipe is inharmonic, meaning that the sounds it produces are not just mixtures of harmonics and consequently the waveform is not periodic and does not sound like a clear musical note. In most cases such pipes are not musically useful.
There are, however, two important exceptions: the conical pipe and the Bessel horn. A conical pipe, as its name implies, is a pipe in the shape of a cone like this:

Despite being non-uniform and closed at one end it turns out that the frequencies of the modes for such a conical pipe are exactly the same as for a uniform open pipe of the same length. The fundamental frequency is \( f = c/2L \) (where \( L \) is the length as before) and the frequencies of the modes are \( f_n = nc/2L \) with \( n = 1, 2, 3 \ldots \) and so on. In other words, a conical pipe plays the same note as an open pipe and it produces all harmonics, even though it is closed at one end. (For the interested reader, a derivation of this result is given in Section 11.1.9.)

This is tremendously useful musically. It allows us to build instruments that have closed pipes, such as reed instruments where the mouthpiece closes off one end of the pipe, but that nonetheless produce a full sound with all harmonics present. An example is the saxophone, which is similar in design to the clarinet but has a conical pipe instead of a cylindrical one. Though this seems like a minor change it makes a big difference: the saxophone plays an octave higher than a clarinet of the same size (because the fundamental frequency is \( c/2L \), rather than \( c/4L \) as on a clarinet) and produces notes that include all harmonics and not just the odd ones, making the sound of the saxophone brighter and completely different from the sound of the clarinet. Figure 11.11 shows the frequency spectrum of the note A3 played on an alto saxophone and, as we can see, the sound does indeed contain both even and odd harmonics. Contrast this with Fig. 11.10 on page 435 which shows the same note played on a clarinet, which has odd harmonics only.

The other musically useful type of flared pipe is the Bessel horn, which is a pipe whose diameter is proportional to a power of distance along its length. It looks like this:
The frequencies of the modes of the Bessel horn are not whole-number multiples of its fundamental frequency and for this reason the Bessel horn is not used on its own as an instrument, but it plays an important role in brass instruments like the trumpet and trombone, which combine a cylindrical pipe for the main part of the instrument with a Bessel horn for the bell (the wide opening at the end). By carefully adjusting the combination of the two parts it is possible to make a pipe that produces correct harmonics, both even and odd, despite being closed at one end and cylindrical for a substantial part of its length. As we have seen, one can also produce such harmonics from a closed pipe by making it conical, but having a pipe that is cylindrical for part of its length turns out to be very useful in the mechanical design of instruments. Unlike woodwind instruments which produce different notes by uncovering tone holes to let air out of the pipe, brass instruments work by splicing additional sections of tubing into the pipe to make it longer and this only works if the pipe is cylindrical. We look at the mechanics of brass instruments in detail in Section 11.4.

Another way in which pipes can differ from the simple cylindrical form is that they may not be straight. As mentioned briefly at the end of Section 11.1, the pipes of wind instruments, and especially brass instruments, may be bent or coiled rather than straight, often for the purely practical reason that it makes them easier to handle. The BB♭ tuba discussed in Example 11.2, for instance, contains over 8 meters of tubing, which would be unmanageable if it were in the form of a single straight pipe but is only mildly unwieldy in its coiled form—see the picture on page 423. This is an extreme example, but many other instruments also use coiled pipes for convenience, including the trumpet, trombone, and bugle. Most woodwind instruments, such as flute, clarinet, and oboe, use straight pipes, but a few such as saxophone and bass clarinet use pipes with bends in them.

By and large, bending or coiling a pipe has little musical effect. The standing wave bends along with the pipe and the modes of vibration are unaffected. In terms of sound, the main effect of bending an instrument pipe is that it allows the bell or opening of the instrument to point in a different direction, often more directly toward the listener, as in a saxophone or sousaphone, and hence to project the sound in a more useful direction. For instance, while the soprano saxophone is small enough that it can be—and often is—made of a single straight pipe, soprano saxophones are also sometimes made with a curved bell in the style of the larger alto and tenor saxophones, in order to better project the sound toward the listener.
11.1.9 Vibration of air in a non-cylindrical pipe

To describe the vibration of air in a pipe of varying width we need a modified version of the wave equation. Let us revisit our derivation of the equation in Section 11.3.2 and as before let \( \xi(x, t) \) be the displacement of the air along the pipe at position \( x \) and time \( t \), but suppose that the cross-sectional area \( A(x) \) of our pipe now varies with position \( x \). Consider a short portion of the pipe from \( x \) to \( x + dx \) thus:

\[
\Delta V = \xi(x + dx, t) A(x + dx) - \xi(x, t) A(x)
\]

Differentiating Eq. (11.35) twice with respect to time, noting that \( A(x) \) is independent of \( t \), and using (11.36), we get

\[
\frac{\partial^2 p}{\partial t^2} = -\frac{B}{A} \frac{\partial^2 \xi}{\partial x^2} = -\frac{B}{A} \frac{\partial}{\partial x} \left( \frac{\rho \frac{\partial^2 \xi}{\partial t^2}}{A} \right)
\]

\[
\frac{B}{\rho A} \frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right) = \frac{B}{\rho A} \frac{\partial^2 p}{\partial x^2}.
\]

Now making use of the fact that \( B/\rho = c^2 \) (see Eq. (1.17)), we can rewrite this equation in the form

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial x^2} = 0.
\]

This modified version of the wave equation is known as the horn equation or Webster’s equation. It describes the motion of the air inside a pipe with varying cross-sectional area.

As an example, let us use the horn equation to calculate the modes of vibration in a pipe that takes the shape of a cone. For such a conical pipe the radius \( r \) increases linearly from one end of the pipe to the other \( r = ax \), where \( a \) is a constant, and hence the area is \( A(x) = \pi r^2 = \pi a^2 x^2 \). Substituting this expression into Eq. (11.38) and cancelling some terms, we get

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial x^2} = 0.
\]

We want a solution to this equation with a pressure node at the open end of the conical pipe at \( x = L \) and a velocity node at the closed end at \( x = 0 \). As with the straight pipe of Section 11.1.7 there is in fact a whole set of such solutions, which take the form

\[
p(x, t) = \frac{C}{x} \sin \left( \frac{\pi n x}{L} \right) \sin(2\pi f_n t),
\]

where \( f_n \) is the frequency and \( C \) is a constant that controls the amplitude. These are similar to the solutions for the straight pipe, Eq. (11.21), but with the crucial addition of the factor of \( 1/x \) at the start.

Equation (11.40) has the required pressure node at the open end of the pipe—we have \( p = 0 \) at \( x = L \) so long as \( n \) is an integer—and we can verify that it is indeed a solution.
by substituting into Eq. (11.39), which gives

\[
\frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 \frac{\partial p}{\partial x} \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = -\frac{C}{x} \left( \frac{\pi n}{L} \right)^2 \sin \left( \frac{\pi nx}{L} \right) \sin (2\pi f_n t)
+ \frac{C}{x} \left( \frac{2\pi f_n}{c} \right)^2 \sin \left( \frac{\pi nx}{L} \right) \sin (2\pi f_n t),
\]

(11.41)

which is zero so long as

\[
\left( \frac{2\pi f_n}{c} \right)^2 = \left( \frac{\pi n}{L} \right)^2 = 0,
\]

(11.42)

or equivalently

\[f_n = n \frac{c}{2L},\]

(11.43)

Thus, as discussed in Section 11.1.8, the modes of a conical pipe closed at its narrow end have the same frequencies as those of a straight pipe open at both ends: the fundamental frequency is \(c/2L\) and the other modes are whole-number multiples with all harmonics being present.

We still have to demonstrate that our solution has the required velocity node at the closed end of the pipe at \(x = 0\), which takes a little more work. First, we note that, despite appearances, the pressure does not go to zero at \(x = 0\) in Eq. (11.40). We might think it does because of the term \(\sin(\pi nx/L)\), which goes to zero, but note also the \(1/x\) term. Recalling that when \(x\) is small \(\sin x \approx x\), we have

\[
\frac{\sin(\pi nx/L)}{x} \approx \frac{\pi nx/L}{x} = \frac{\pi n}{L},
\]

(11.44)

at the closed end of the pipe, which is nonzero. So the closed end is not a pressure node.

We need to demonstrate more than this, however, that the closed end is a velocity node in our solution (which common sense says it must be, since the pipe is closed so the air cannot go anywhere). As in Section 11.1.6 we can calculate the velocity from Eq. (11.25). Substituting our solution for \(p\) into the right-hand side of the equation and performing the derivative gives us

\[
\frac{\partial u}{\partial t} = -\frac{\pi n}{L} \frac{\pi n}{L} \cos \left( \frac{\pi nx}{L} \right) \sin (2\pi f_n t) \left( \frac{1}{x^2} \sin \left( \frac{\pi nx}{L} \right) \sin (2\pi f_n t) + F(x) \right).
\]

(11.45)

Now we integrate with respect to \(t\), which gives

\[
u = \frac{C}{2\pi \rho f_n} \left[ \frac{\pi n}{L} \cos \left( \frac{\pi nx}{L} \right) - \frac{1}{x^2} \sin \left( \frac{\pi nx}{L} \right) \right] \cos (2\pi f_n t) + F(x),
\]

(11.46)

where \(F(x)\) is an arbitrary function of \(x\) that plays the role of an integration constant. The air in our pipe has no overall flow in one direction or the other so the average velocity is zero everywhere, which implies that \(F(x) = 0\) for all \(x\) and hence that

\[
u = \frac{C}{2\pi \rho f_n} \left[ \frac{\pi n}{L} \cos \left( \frac{\pi nx}{L} \right) - \frac{1}{x^2} \sin \left( \frac{\pi nx}{L} \right) \right] \cos (2\pi f_n t).
\]

(11.47)

The velocity at the closed end of the pipe is given by the value of this expression at \(x = 0\), but we need to be careful about setting \(x\) to zero since both terms inside the brackets diverge when we do this. However, recalling that the limits of cosine and sine as \(x\) becomes small are \(\cos x \to 1\) and \(\sin x \to x\), the value of \(u\) for small \(x\) is

\[
u = \frac{C}{2\pi \rho f_n} \left[ \frac{\pi n}{L} \cos \left( \frac{\pi nx}{L} \right) - \frac{1}{x^2} \sin \left( \frac{\pi nx}{L} \right) \right] \cos (2\pi f_n t) = 0,
\]

(11.48)

and hence our solution does indeed have the required velocity node at the closed end of the pipe.

Figure 11.12 shows a plot of the pressure profile along the pipe from Eq. (11.40) for the first few harmonics. These have a very different shape from the corresponding plots for the straight pipe in Figs. 11.3 and 11.5, despite the frequencies being the same.

### 11.2 Flutes

We have looked in some detail that the kinds of vibrations that occur in the pipes of wind instruments, but we have not said how we set the air vibrating in the first place. In practice there are several different techniques, all based on blowing air into the instrument in some manner. Reed instruments such as the clarinet make use of a vibrating reed that opens and closes to allow short bursts of air at higher pressure.
Figure 11.12: Pressure profiles within a conical pipe. All modes of a conical pipe have a pressure antinode at the closed (narrow) end and a pressure node at the open end. The amplitude of the pressure variation decreases as the pipe gets wider, just as a river flows slower where it is wide and faster where it is narrow.

into the instrument. Brass instruments such as the trumpet do the same but use the player’s lips in place of a reed. And flutes, pipe organs, and related instruments use an oscillating jet of air directed at a sharp edge. We will study each of these three approaches in the following sections of the chapter, starting in this section with the oscillating jets of flutes and organ pipes.
Figure 11.13: Tone producing mechanism of a recorder. (a) Close-up of the top of a recorder, showing the mouthpiece where the player blows air in, the gap across which the jet travels, and the sharp edge that it strikes. (b) A cross-section of the arrangement inside the instrument.

11.2.1 The jet-drive mechanism

Flutes, organ pipes, and similar instruments make use of a so-called jet drive to create vibration in the instrument pipe. A simple example of an instrument that does this is the recorder. Figure 11.13 shows the layout of the top of a recorder. There is a mouthpiece or fipple into which the player blows, forcing air through a narrow channel. As it emerges from the channel it creates a jet that crosses a short gap then strikes a sharp edge called the labium (the Latin word for “lip”), which forms a part of the front wall of the instrument.

To understand how the jet-drive mechanism works, take a look at Fig. 11.14, which shows a schematic diagram of the process, with the air channel on the left and the jet emerging from it to strike the labium on the right. First, note that because of

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Before the mid 20th century the physics of wind instruments was less well understood than it is today and it was widely but erroneously believed that the vibration in a flute or recorder was produced by a different phenomenon called an edge-tone. An edge-tone is created when a jet of air hits a sharp edge but without the presence of a resonating pipe as in a recorder or flute. For example, an edge-tone is the sound-producing mechanism at work in a whistle, such as a police whistle or referee’s whistle. The confusion between the jet-drive mechanism of a flute and an edge-tone is understandable, since the combination of jet and sharp edge appears very similar in both, but they are in fact quite distinct phenomena. You may still see edge-tones described as the driving mechanism of flutes in older books, but you should be aware that this is now believed to be incorrect.
the gap between where the jet emerges and the labium, the pipe is effectively open at this end—air can freely pass in and out through the gap. As we have seen, this means that when the air in the pipe vibrates this end will be a pressure node and a velocity antinode, so the motion of the air will be a maximum. (The other end of the instrument may be either open or closed—see below.)

When the instrument is first blown, a portion of the jet finds its way into the interior of the pipe as shown in Fig. 11.14a and pushes on the air in the pipe. Like giving a shove to a pendulum, this starts the air vibrating, with the largest motion being at the open end. This motion drives air in and out of the end of the pipe through the gap as shown in Fig. 11.14b. The moving air pushes against the side of the jet, bending it up and down in time with the vibration and so introducing a slight wave into the jet, as shown in the figure. The wave is initially quite small because the movement of the air in the pipe is small. But it grows as the jet flows away from the mouth of the channel. Think of the stream of water from a hosepipe as you wave the hose back and forth. Your motion may be small, but the swath covered by the water becomes wider and wider as it moves away from the nozzle.

By the time the air jet reaches the labium the wave can be as large as a few millimeters across, causing it to cross repeatedly back and forth across the labium, from the bottom where it blows into the instrument pipe, to the top where it blows out into the world outside. The result is that the fluctuating jet injects a series of
bursts of air into the interior of the pipe at exactly the frequency of the pipe vibration. These bursts push on the air in the pipe and, if the timing is just right, they can amplify the oscillation in the pipe in a kind of positive feedback where the vibration in the pipe creates the wave in the jet, and the fluctuating wave pushes back on the air and keeps it oscillating. Imagine a parent pushing a child on a swing: if they time their pushes just right to sync up with the movement of the swing they can keep the swing going as long as they like.

This is the mechanism by which the air in a flute or recorder is made to vibrate. As long as the player continues to blow and create a jet of air, the oscillation in the pipe will continue and the instrument will produce a sound. Note that while the end of the pipe with the jet is always open to the air via the gap between the jet and the labium, the other end may be either open or closed, so the jet-drive mechanism can be used with either open- or closed-pipe instruments. The flute and recorder are both open-pipe instruments, but certain types of organ pipes have closed pipes, as does the pan flute—see Example 11.6 on 449. In fact the jet drive is the only normal mechanism for wind instruments that can be used with either open or closed pipes. The others—vibrating reeds and the player’s lips in a brass instrument—involves closing off one end of the pipe, so they can only be used with closed-pipe instruments.

In order for the jet-drive mechanism to work, the fluctuations of the air jet have to be timed just right to reinforce the oscillation in the pipe. The burst of air as the jet flips across the labium and into the pipe has to coincide with the part of the cycle where the air in the pipe is moving away from the jet, like the parent pushing the swing at just the right moment to keep it swinging. The specific timing of the jet drive is a result of the interplay between the velocity of the air jet, the distance from the base of the jet to the labium, and the frequency of the oscillation in the pipe.

The velocity of the air jet derives from the blowing pressure applied by the player. Figure 11.15 shows a simple setup for producing a steady jet of air. Air in a tube is at pressure $P$, which is higher than the atmospheric pressure $P_0$ outside. The pressure difference forces air out through a small nozzle or opening, forming a jet. The pressure difference is related to the velocity $v$ of the jet by

$$P - P_0 = \frac{1}{2} \rho v^2,$$  \hspace{1cm} (11.49)
where \( \rho \) is the density of air. This is a version of Bernoulli’s equation, one of the most famous results in the field of fluid dynamics. It can be derived mathematically—see Section 11.2.2 for details—but the basic idea is that the higher pressure inside the tube exerts a force on the air causing it to accelerate as it leaves the tube. The amount of acceleration, and hence the velocity of the jet, depends on the size of the force, which is dictated by the pressure difference between inside and outside, and the mass of the air, which is controlled by the density \( \rho \).

We can rearrange Eq. (11.49) to give the velocity of the air in the jet:

\[
v = \sqrt{\frac{2(P - P_0)}{\rho}},
\]  

(11.50)

and then apply this result to the jet of a flute or recorder. A typical value for the excess pressure \( P - P_0 \) created by a player’s breath in a recorder is about 100 pascals, while the density of air is \( \rho = 1.204 \text{ kg/m}^3 \), so

\[
v = \sqrt{\frac{2 \times 100}{1.204}} = 12.9 \text{ m/s}. 
\]

(11.51)

This is the speed of the air itself, but the crests and troughs of the wave in Fig. 11.14 do not move this fast because they take a snaking, circuitous route and therefore have further to travel, and also because the jet slows down somewhat after it emerges. Experiments have shown that the wave moves at about 40% of the velocity given by Eq. (11.50), or 0.4\( v \).

The other crucial parameter in the timing of the jet drive is the distance \( d \) between the start of the jet and the labium—see Fig. 11.16. The time for the wave to cover this distance is

\[
\frac{d}{0.4v} = \frac{5d}{2v},
\]

(11.52)

The ideal value of \( d \) turns out to be about one half of a cycle of the wave of the jet. But the time to travel a half cycle is, by definition, also the time for the wave to go
through one half of its period of oscillation. Since the oscillation of the wave is driven by the oscillation in the pipe it must have the same frequency \( f \), so the time for one period of the oscillation is \( 1/f \) and the time for half a period is \( 1/2f \). Thus we have
\[
5d/2v = 1/2f,
\]
or
\[
d = \frac{v}{5f}.
\]
(11.53)

We can calculate the velocity \( v \) from Eq. (11.50) and, given the frequency of the note being played, Eq. (11.53) then tells us where the labium should be placed in order to produce a sound. There is some wiggle-room—the labium doesn’t need to be at exactly the distance given by the equation—but if we stray too far the instrument will not work. We will get only the rush of air from the jet and no clear note.

As an example, consider the alto recorder and let us take the figure of \( v = 12.9 \) m/s from Eq. (11.51). A typical note in the middle of the range of the instrument would be C5 with a frequency of 523 Hz. From Eq. (11.53) we then have
\[
d = \frac{12.9}{5 \times 523} = 4.9 \text{ mm},
\]
so the gap should be around 5 millimeters, which is close to the size of the gap on an actual alto recorder.

Equation (11.53) also tells us that for a specified position of the labium the ideal velocity of the air jet is given by \( v = 5fd \), which means that a lower velocity—and hence lower breath pressure—is needed to get low notes and a higher velocity for high ones. This fact is well known to players of the recorder: you have to blow softly to reach a low note. Blowing softly means that the note will also be quiet, so playing a low note loudly is not possible on a recorder, which is a weakness of the instrument. On a flute, by contrast, where the air jet is produced directly from the player’s lips, the flutist has the option to increase the distance \( d \)—they do this by rolling the instrument away from them, which moves the labium further from their lips—thereby increasing the required velocity \( v = 5fd \) and allowing them to blow harder on low notes, hence producing a louder note. This is one of several ways in which the flute allows musical possibilities not available on the recorder.

**Example 11.5: An organ pipe**

Figure 11.17 shows a diagram of an organ pipe, of the type known as a principal or “diapason” pipe, which is a cylindrical metal pipe, open at the top. (We discuss the various types of organ pipes in Section 11.2.4.) Air enters at the bottom of the pipe and passes through the triangular chamber called the “toe,” then through a narrow opening that generates the jet, which in turn strikes the labium. The toe plays no part in the actual generation of the note. Only the top part of the pipe from the jet upward sounds the note.

Suppose this sounding portion of the pipe is 117 cm long and the distance from the opening of the jet to the labium is 4.5 cm. What jet velocity \( v \) would be needed to make this pipe work? And how much air pressure would the organ pump need to provide?
Recalling that the air-jet end of the pipe behaves like an open end, the pipe in this case is open at both ends, so its frequency is given by

\[ f = \frac{c}{2L} = \frac{343}{2 \times 1.17} = 146.6 \text{ Hz}, \]  

which is the note D3. At the same time, we can rearrange Eq. (11.53) to give the jet velocity thus:

\[ v = 5fd = 5 \times 146.6 \times 0.045 = 33 \text{ m/s}, \]  

which is a typical air velocity for a pipe organ. The velocity is related to the pressure provided by the organ pump via Eq. (11.49), which tells us that the difference between the pump pressure and the prevailing atmospheric pressure (sometimes call the “over-pressure”) should be

\[ P - P_0 = \frac{1}{2} \rho v^2, \]

where \( \rho = 1.204 \text{ kg/m}^3 \) is the density of air. So in this case we need

\[ P - P_0 = \frac{1}{2} \times 1.204 \times 33^2 = 655 \text{ Pa}. \]  

**Example 11.6: The pan flute**

One of the earliest known examples of a flute-like instrument is the ancient Greek instrument called the pan flute or syrinx. Similar instruments have also been in use for centuries in the Andes. A pan flute consists of a set of short pipes, each a few centimeters long, with one pipe for each pitch to be played. The pipes are normally arranged side by side from shortest to longest, in a single row or sometimes two parallel rows, and are blown across the top to produce a note, as one might blow across the top of a bottle, with the player’s lips resting against the edge of the pipes. The player produces the air jet directly with their lips (as they do on a standard concert flute also) and the open end of the pipe provides the sharp edge. The pipes are closed at the bottom and hence produce a note an octave lower than an open pipe of the same length.

Suppose a particular pan flute has pipes ranging in length from 10 cm to 25 cm and the player’s breath has an air velocity of 15 m/s. What should the diameter of the pipes be to put the player’s lips the correct distance from the labium?

The frequency of the note produced by a closed pipe is given by Eq. (11.14) to be \( f = c/4L \). For pipes of length 10 cm and 25 cm, this gives frequencies of

\[ \frac{343}{4 \times 0.1} = 858 \text{ Hz}, \quad \frac{343}{4 \times 0.25} = 343 \text{ Hz}, \]  

which are approximately the notes A5 and F4. Then Eq. (11.53) says that for our two pipes the distance \( d \) from the player’s lips to the sharp edge of the pipe should be

\[ \frac{15}{5 \times 858} = 3.5 \text{ mm}, \quad \frac{15}{5 \times 343} = 8.7 \text{ mm}. \]  

So the shorter pipe should be about 3.5 mm in diameter and the longer one about 9 mm.
Chapter 11 | Wind and brass instruments

Advanced material

11.2.2 Bernoulli’s equation

In Section 11.2.1 we used Bernoulli’s equation, which relates the pressure and velocity of moving air, to calculate the velocity of the air jet in a wind instrument. Bernoulli’s equation can be derived from Newton’s second law of motion. In Section 1.3.3 we showed that the second law implies that the velocity \( v \) and pressure \( p \) are related by

\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x}.
\]  

(11.61)

(See Eq. (1.25) on page 13.) Note that we previously used \( v \) for velocity but we use \( v \) here because it will be helpful to distinguish between the velocity of the air in the jet of an instrument like a recorder or flute and the velocity \( u \) of air in the main instrument pipe.

We previously applied Eq. (11.61) to the velocity of air in normal sound but it applies in any situation, including to a jet of air from a tube as in Fig. 11.15. Imagine a small packet of air in such a situation, moving along the tube and then out through the opening to form a jet, with the packet’s velocity \( v(x) \) depending on its position \( x \) along the tube. Initially it is moving slowly, then faster as it leaves the tube. Over a short time interval \( dt \) the packet travels a distance \( dx = v(x) \, dt \) and its change in velocity over the same interval will be

\[
\frac{dv}{dt} = \frac{dv}{dx} \, dx = \frac{dv}{dx} \, v \, dt,
\]  

(11.62)

and hence the rate of change of velocity is

\[
\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial v}{\partial x} \frac{1}{2} \left( \frac{1}{2} \rho v^2 + p \right),
\]  

(11.63)

Substituting this result into Eq. (11.61) and rearranging, we get

\[
\frac{\partial}{\partial x} \left( \frac{1}{2} \rho v^2 + p \right) = 0.
\]  

(11.64)

Now we integrate with respect to \( x \) to get

\[
\frac{1}{2} \rho v^2 + p = C,
\]  

(11.65)

where \( C \) is an integration constant. This is Bernoulli’s equation. It tells us that the velocity \( v \) and the pressure \( p \) can both vary, but the particular combination \( \frac{1}{2} \rho v^2 + p \) stays constant. This means that as the pressure goes up the velocity must go down, and vice versa.

Applied to the situation of the tube in Fig. 11.15, Eq. (11.65) implies that the value of \( \frac{1}{2} \rho v^2 + p \) inside the tube and outside in the jet must be the same. Recall that \( p \) is defined to be the difference between the total air pressure \( P \) and atmospheric pressure, \( p = P - P_0 \). Outside the tube \( P \) is just equal to atmospheric pressure \( P_0 \), so \( p = 0 \) and \( \frac{1}{2} \rho v^2 + p = \frac{1}{2} \rho v^2 \). At the same time, in the situations we are interested in with musical instruments, the velocity of the air in the instrumentalist’s lungs and mouth is essentially zero—the air there moves very little compared to the rapid flow in the jet—so \( v = 0 \) and \( \frac{1}{2} \rho v^2 + p = P = P_0 \). Equating our expressions outside and inside, we then get

\[
P - P_0 = \frac{1}{2} \rho v^2.
\]  

(11.66)

We used this equation in Section 11.2.1 to calculate the velocity of the jet from the pressure difference between the two sides.

We can think of Eq. (11.66) as a type of energy conservation equation. The left-hand side \( \frac{1}{2} \rho v^2 \) is the kinetic energy density of the air in the jet—the amount of kinetic energy per unit volume—and the right-hand side \( P - P_0 \) is the change in potential energy density due to the change in air pressure when the air moves through the opening. The equation says that the increase in kinetic energy is equal to the decrease in potential energy.

11.2.3 Harmonics of flutes and organs

The particular mix of harmonics present in a note played on a pipe driven by an air jet depends on the details of how the jet oscillates across the labium. As it moves in and out of the pipe the jet injects a series of bursts of air into the pipe and if one could measure the amount of air being injected it would look something like Fig. 11.18. Physicists have built complex mathematical models of the dynamics of these air bursts, but the central elements are straightforward to understand. First, the shape of the curve in Fig. 11.18 is similar to a square wave, but rounded at the corners because the jet has nonzero width, so it doesn’t move instantaneously in and out of the pipe, but rather transitions gradually. Second, the amount of time spent in and out of the pipe—the relative size of the “on” and “off” regions in the figure—does not have to be the same and depends on whether the jet is aimed at the labium head-on or slightly to one side.
As we have seen, the jet provides the driving force that pushes on the air inside the pipe. The precise connection between the velocity in the pipe and the velocity of the jet is a complicated one which depends on how the jet dissipates into the larger space inside the pipe. Again this has been the subject of considerable research including the creation of sophisticated mathematical and computer models. But for the moment let us make the simple assumption that the velocity at the end of the pipe is just proportional to the driving velocity of the jet.

For air in an open pipe vibrating in its nth mode, the velocity at the end is given by Eq. (11.28) with x set to zero:

\[
u(0, t) = \frac{A}{\rho c} \cos(2\pi f_n t),
\]

(11.67)

which is just a simple cosine wave. Real pipes, as we have seen, vibrate in a combination of modes, so the velocity can be written as a sum of terms

\[
u(0, t) = \sum_{n=0}^{\infty} U_n \cos(2\pi f_n t),
\]

(11.68)

where \(U_n = A/\rho c\) is the amount of the nth harmonic.

Suppose that the velocity generated by the jet—the function shown in Fig. 11.18—is \(v_e(t)\). The mathematical form for this function, derived by W. G. Bickley in 1937, is

\[
v_e(t) \propto 1 - \tanh \frac{a \cos(2\pi f t) - h}{b},
\]

(11.69)

where \(a\) is the amplitude of the wavering of the jet, \(b\) is the radius or half-width of the jet, and \(h\) is an offset that measures how far off-center the jet hits the labium.

We want Eq. (11.68) to be equal to this velocity \(v_e\). As we know from Fourier analysis, any periodic function can be written as an appropriate combination of sine (or cosine) waves like Eq. (11.68) provided we get the coefficients \(U_n\) right. The correct values of the coefficients are given by the Fourier formula

\[
U_n = \frac{2}{T} \int_0^T v_e(t) \cos(2\pi f_n t) \, dt = \frac{2}{T} \int_0^T v_e(t) \cos\left(\frac{2\pi n t}{T}\right) \, dt,
\]

(11.70)

where \(T = 1/f_1 = 2L/c\) is the period of the note being played. We cannot perform the integral in Eq. (11.70) exactly, but it can be done numerically on a computer. Some typical results are given in Fig. 11.19, which shows the amplitudes of the first four harmonics as a function of \(h/b\), the ratio of the jet offset to the width of the jet. Note for instance that when the offset is zero and the jet is exactly centered on the labium the amplitude of the second and fourth harmonics is zero. This turns out to be true for all even harmonics, so a pipe played in this way will produce quite a mellow tone, perhaps similar to a clarinet, which also lacks even harmonics. As the jet moves off-center the even harmonics reappear and the relative strength of the different harmonics varies. When \(h/b = 1\), for instance, we hear the first, second, and fourth harmonics but not the...
third harmonic. And around $h/b = 1.5$ there is a point where the fourth harmonic vanishes leaving only harmonics $1$, $2$, and $3$ (and perhaps some higher ones not plotted in the figure). Note also how the whole figure is symmetric from left to right, meaning that it does not matter which way the jet is off-center—the harmonics will be the same either way. And note how all harmonics die off for large values of $h/b$: beyond about $h/b = 2$ the entire sound fades away because the jet misses the labium altogether and is either completely inside the pipe or completely outside it. In either case it will not produce any vibration of the air in the pipe.

The variation in the relative strengths of the harmonics means that by varying the offset of the jet relative to the labium a musician or instrument maker can change the timbre of the instrument. This phenomenon is well known, for instance, to flutists, who use it to produce brighter or mellower tones depending on how they direct their breath. It is also used in the “voicing” of pipe organs to produce particular timbres by adjusting the aim of the air jet. In this context the position of the jet plays a role somewhat analogous to the location of the plucking or striking point of a string on a string instrument, which also affects the balance of harmonics, as we saw in Section 10.2.

As we’ve said, this calculation assumes that the velocity of vibration of the air in the pipe is proportional to the velocity of air due to the jet, but the real situation is more complicated. For instance, the width of the pipe has a noticeable effect, with narrower pipes emphasizing the higher harmonics and producing a brighter tone while wider ones have less higher harmonics and sound warmer. The broad contours of our calculation do carry through however. In particular, if there are harmonics missing or weak in the air velocity pattern of the jet then they will be missing or weak in the final sound too, so the basic picture shown in Fig. 11.19 is still a reasonable guide. Again this situation is somewhat analogous to a string instrument, where the vibration of the string dictates which harmonics

Figure 11.19: Harmonics generated by an oscillating air jet. Amplitudes of the first four harmonics of an open pipe driven by an air jet with the shape given in Eq. (11.69), as a function of $h/b$, the ratio of the distance $h$ by which the jet is off-center to the half-width $b$ of the jet.
are present, but the specific balance of harmonics is also affected by the soundboard and body of the instrument, whose behavior can influence the timbre in complex ways.

A similar calculation also applies for pipes closed at the end away from the air jet and labium, except that these pipes can only produce odd harmonics, as discussed in Section 11.1.4, so the amplitudes of all even harmonics would automatically be set to zero, producing a warmer, mellow tone than an open pipe. One example of such an instrument is the pan flute, discussed in Example 11.6. Another is a "stopped" organ pipe, as described in the following section.

11.2.4 The pipe organ

A pipe organ is a formidable complex machine composed of thousands of moving parts, but in terms of sound generation it is actually one of the simpler wind instruments because it has a separate pipe for each note to be played. Where orchestral wind instruments like the clarinet are laden with complicated levers and pads and tone holes that allow a single pipe to play many notes, each pipe on an organ is just a simple tube that plays one note only, making the organ a good first example of the principles we have been discussing.

Organ pipes come in a range of different designs that produce correspondingly different timbres. A small selection of the many possibilities is shown in Fig. 11.20. Pipes that produce sound using a jet drive—which is most pipes in a typical organ—are called flue pipes. There are some pipes that use a reed instead of a jet drive, but we defer discussion of these until later. Even among flue pipes, however, there is still a wide range of types. The most common, known as a principal or diapason pipe, is a straight, cylindrical, metal pipe of medium width with the jet-drive mechanism at the bottom and an open end at the top. A number of such pipes can be seen for instance in Fig. 11.21, which shows a mid-sized three-manual organ. Only a few of the more than 2000 pipes on this organ are visible in the picture—most are hidden behind the facade on the front of the instrument.

Principal pipes are traditionally made from an alloy of tin and lead, although thorough experimentation in the first half of the 20th century revealed little effect on tone from the choice of material, as is often the case with wind instruments—see Section 11.1. With its open end, a principal pipe produces a fundamental of frequency of \( c/2L \) and

![Figure 11.20: Types of organ pipes.](image)

There are numerous styles of organ pipes. Principals are open-topped metal pipes of medium width that form the foundation of the traditional organ sound. Flutes are wider and produce a softer sound with fewer high harmonics. Strings are narrow with a brighter sound. Stopped pipes, such as stopped principals, are closed at the top and hence sound an octave lower that their open counterparts. Pipes may also be made of wood, in which case they are normally square rather than round, but neither the shape nor the construction material has much effect on their sound.
Figure 11.21: A pipe organ. A modern medium-sized chapel organ in England. Built in 2013, this organ has three manuals, 44 ranks, and over 2000 pipes. Most of the pipes are hidden behind the decorative facade of the instrument. The visible ones are principal pipes. Also visible behind the railing at the bottom of the picture is the organ console which houses the manuals, pedals, and stops. Organ by Dobson Pipe Organ Builders, Lake City, Iowa. Photo: David Iliff.

all harmonics. Note that the triangular part of the pipe at the bottom, called the toe, is closed off and does not play a part in sound production. Only the straight part of the pipe contributes to the sound, so the pipe is indeed cylindrical, as it must be to produce evenly spaced harmonics and a periodic waveform.

*Flute* and *string* pipes are similar to principals but wider (flutes) or narrower (strings). The wider flute pipes produce fewer high harmonics than principals and hence have a warmer tone that is said to be similar to a flute, although the re-
semblance is rather remote. Conversely the narrower string pipes produce a larger amount of high harmonics and a correspondingly brighter tone that allegedly sounds like a string orchestra. Pipes may also be made of wood rather than metal. Wooden pipes are normally rectangular in cross-section rather than round, but this is mainly a matter of convenience of manufacture and does not have any significant impact on timbre.

Other pipes are closed at the top so that they sound an octave lower, with fundamental frequency $c/4L$, and produce odd harmonics only. Such pipes are said to be stopped. Thus one might talk about a stopped flute or stopped diapason. Because they lack even harmonics, stopped pipes generally have a softer timbre than unstopped ones. The timbre of pipes can also be adjusted by changing the direction of the air jet so that it strikes the labium off-center. As discussed in Section 11.2.3, a jet centered exactly on the labium also produces odd harmonics only, while an off-center jet will produce both odd and even harmonics and hence a brighter sound.

Organ pipes are divided into ranks—sets of pipes of the same type but different lengths, so that they play a range of different notes. Thus you might have a principal rank or a stopped flute rank. The pipes of a rank get both shorter and narrower as they progress from low pitch to high, which allows them to maintain a consistent timbre across the musical range. In fact, it is common that the width does not decrease quite as fast as the length, so that higher-pitched pipes are wider relative to their length than lower-pitched ones. This causes the amount of higher harmonics to decrease slowly as we go to higher pitches, making the timbre less shrill in the upper registers, which is considered desirable.

The ranks of pipes in a pipe organ are controlled by one or more keyboards, similar in appearance to a piano keyboard and known in the trade as manuals—see Fig. 11.22. A large church or concert organ will normally have three or four manuals, traditionally called choir, great, swell, and (if there is a fourth) solo. There is also a set of pedals, again in the shape of a piano keyboard, that allow the organist to play additional notes using his or her feet.

By convention an organ manual spans five octaves, nominally from C2 to C7, although this can be changed as described below. The specific sound produced when you play a manual is selected using stops, small knobs that can be pulled out to activate specific ranks of pipes. Pulling out a stop labeled “Great 8 principal,” for instance, would connect the great manual to a rank of principal pipes, allowing the organist to play the principal sound on that manual.

What does the “8” mean on the stop? This is an indication of the pitch of the notes that will be produced. The 8 is short for “8 feet,” which is (roughly) the length of the pipe for the lowest note on the manual. 8 feet is equivalent to 2.44 meters and an open pipe of this length will produce a note with frequency $f = c/2L = 70.3 \text{ Hz}$, which is C♯2. In fact the lowest note for an 8-foot stop is actually C2—the 8 is only
approximate, more convenient than saying “8 feet and 7 1\frac{1}{4} inches” which is the actual length of a pipe that plays C2. So the 8 is telling us that this stop will produce notes starting at C2 on the lowest key of the manual and going up five octaves from there to C7.

An 8-foot stop like this, covering C2 to C7, is considered the nominal pitch for an organ manual. But there will also be another stop, probably right next to the 8-foot stop, labeled “4 principal.” When pulled out, this stop also connects the manual to the principal pipes, but the connection is different now, so that the lowest sounding pipe on the manual is a 4-foot pipe, which produces the note C3, and the other keys go upward from there, again covering five octaves and so extending now to C8. So by selecting which stop we use we can control the pitch range covered by the manual.

The real power of the pipe organ, however, is that we can pull out both of these stops at the same time and then each key on the manual will produce two notes simultaneously. With both the 8- and 4-foot stops pulled out, the lowest key on the manual will produce both C2 and C3, and every other key will similarly produce two notes an octave apart. Playing a melody on such a manual will produce the sound of that melody played in octaves. An organ will normally also have 2-foot and 1-foot stops that produce the next two octaves up, and 16- and maybe even 32-foot stops
that produce the next two octaves down. Pull out all six stops from “32 principal” to “1 principal” and every key on the manual will produce six notes simultaneously in six different octaves. Even a small three-node chord, such as a major triad, will generate no less than 18 notes and produce a thunderous sound.

But it does not stop there. Other stops such as “8 strings” or “4 stopped flute” allow the organist to select different ranks of pipes to be controlled by the manual and produce different timbres of sound, and each of these comes in a selection of different pitches as well. There is nothing (other than good taste) to prevent the player from pulling out as many different stops as they like, so that a simple chord played on the manual produces strings, flutes, and all kinds of other sounds in multiple octaves all at the same time. This is the meaning of the phrase “pulling out all the stops”—an organ with all the stops pulled out will produce a prodigious sound indeed.

The other manuals of the organ as well as the floor pedals all have their own stops to select which sounds they produce, so that one can set up different combinations of sounds on different manuals. The options available vary between manuals, with different manuals being intended for different musical uses. The pipes for the swell manual, for example, are normally enclosed within a box with mechanically operated shutters on the outside that can be opened and closed by the organist to vary the amount of sound that comes out. This allows the player fine control over the volume level for these pipes, something that is otherwise missing from the pipe organ. The pipes for the choir manual are often similarly enclosed in a box for volume control, and typically lean toward softer timbres such as flutes, which are used for accompanying singers or instrumentalists. The pipes controlled by the foot pedals are primarily used for playing bass parts and consist typically of 16- and 32-foot stops, which produce some of the lowest notes on the organ.

Pipe organs vary greatly in size and capability, from small chamber and practice instruments with a single manual and two or three ranks of pipes to spectacular concert organs the size of entire houses. The world’s largest currently functional organ, the Wanamaker Grand Court Organ, was built for the 1904 World’s Fair in St. Louis, Missouri and was later sold to a department store in Philadelphia, where it resides to this day. It has been added to over the years and now has six manuals, 464 ranks, and 28 750 pipes.

**Example 11.7: The highest and lowest notes on the organ**

Suppose that a particular five-octave organ manual has stops ranging from 1 foot to 32 foot. Let us calculate the highest and lowest notes it can play. An 8-foot stop, as we have said, plays a lowest note of C2. Similarly a 4-foot stop plays C3, a 2-foot stop plays C4, and a 1-foot stop plays C5. But if the lowest key on the manual is C5 then the highest is five octaves above that, which is C10, a very high note indeed. Recall that the highest note on the piano is C8 with a frequency of 4186 Hz, so C10, two octaves higher, has frequency $4186 \times 4 = 16744$ Hz, barely within the limit of audibility for most people.
Meanwhile a 16-foot stop would be an octave below the 8-foot one, with lowest note C1, and a 32-foot stop would be an octave below that at C0, well below the lowest note on the piano (normally A0) and having a frequency 16.4 Hz that is below the limit of audibility for human listeners. Some organs even have a 64-foot stop with lowest note of C−1 (pronounced “C minus-one”) with frequency 8.2 Hz. The primary effect of such notes is a palpable shaking of the seats in the choir stalls (as this author can attest from childhood experience). Note that for practical reasons a 64-foot stop is normally implemented on stopped (i.e., closed) 32-foot pipes that sound an octave lower than their length would suggest. A true 64-foot pipe would extend through the roof of most churches and concert halls.

11.2.5 Overblowing

As we have said, in the jet-drive mechanism for wind instruments it is ideal if there is about a half period of the jet wave from the opening where the jet emerges to where it hits the labium. If this is not the case then the timing will be off and the jet will deliver a weaker boost to the pipe. Thinking again of the analogy to pushing a swing, it would be as if we repeatedly pushed the swing at the wrong moment, either too early or too late. This can happen if the labium is in the wrong place, but also if the velocity of the air jet is too high or too low.

For example, you might blow a recorder too weakly, so the velocity of the air jet is low, in which case the instrument may fail to make a clear note at all. If you blow it too hard then the situation is more complicated because although the timing of the air jet will be off, which makes it less effective, the air jet is also faster and therefore pushes the air in the tube harder, which compensates for the poor timing so that under normal circumstances the instrument will actually make a louder note when blown harder.

However, something else interesting happens when you blow a recorder too hard. Recall the there is more than one mode of oscillation of the air in a pipe. For a pipe open at both ends as in a recorder, there are modes at every multiple of the fundamental frequency: 2f, 3f, 4f, and so forth. If we blow too hard, so that the air jet is poorly timed for the fundamental frequency, it can still be well timed for one of the higher modes. For example, the second harmonic has frequency 2f, so its period is 1/2f and a half of that period is 1/4f. Following our earlier calculation in Section 11.2.1, we equate this with the time 5d/2v that the wave takes to travel to the labium and find that the jet will be perfectly timed for the second harmonic when 1/4f = 5d/2v or

\[ v = 10fd, \]

which is exactly twice the velocity given by Eq. (11.53).

This means that as we blow the instrument harder there will come a point—roughly when the velocity is twice as great—where the jet is better timed to drive
the second harmonic than the fundamental, and at this point the sound will suddenly jump up from the fundamental to the second harmonic. From the point of view of the player, if you blow harder, you get a note an octave higher. This is called \textit{overblowing} and it is an important musical technique used to get high notes out of wind instruments. Higher notes on the flute and the recorder, for instance, are reached by deliberately overblowing to make the sound jump up an octave.

Indeed it can sometimes be difficult \textit{not} to overblow. When $f$ is small Eq. (11.71) tells us that the velocity required to overblow the instrument is also small, meaning that it is quite easy to overblow low notes by mistake. This behavior is particularly noticeable on the recorder, where blowing even a little too hard in the lower register of the instrument causes the note to jump up the octave. The situation is better on the flute, where the player can move the labium away from their lips and hence increase the distance $d$, allowing a larger air velocity without overblowing.

Overblowing doesn’t stop at the second harmonic. If you blow harder still the jet will become poorly timed for the second harmonic, but can be well timed for the third harmonic. Following the same calculation, a half period of the third harmonic is $1/6f$, so the ideal air-jet velocity to drive oscillation in the third mode of the pipe occurs when $1/6f = 5d/2v$, or $v = 15fd$. If we blow the instrument this hard, we should be able to get it vibrate in its third mode, which would produce a note an octave and a fifth above the fundamental.

More generally a half period of the $n$th harmonic is $1/2nf$, so the ideal velocity to drive the $n$th harmonic is given by $1/2nf = 5d/2v$, or

$$v = 5nf.$$

In other words, to produce the $n$th harmonic we need the air-jet velocity to be $n$ times the velocity for the fundamental. On the flute, the player controls air velocity by pursing their lips to narrow the gap through which the air is forced, increasing the pressure, which in turn increases the velocity. On a recorder one simply blows harder, which again increases the air pressure. Equation (11.49) on page 446 tells us exactly how much you have to increase the pressure to produce a given increase in velocity. One can think of the difference $P - P_0$ as an "overpressure"—the amount of extra pressure that the player has to produce with their lungs. To double the velocity and reach the second harmonic, for instance, Eq. (11.49) tells us that you would have to quadruple this overpressure. This would require some effort, although usually a somewhat smaller pressure increase will suffice, and playing is helped further by the use of a "register hole"—see Section 11.2.6. In practice, the second harmonic is quite easy to produce on both flutes and recorders, and skilled players can reach higher harmonics as well, although it becomes progressively more difficult as one goes up the harmonic series. A good flute player can reach the first six harmonics.

Overblowing also works for pipes that are closed at one end, except that such pipes can only vibrate at odd multiplies of their fundamental frequency, and hence
the next available mode after the fundamental is the third harmonic with frequency $3f$. 
Equation (11.72) still applies and one should be able to overblow such a pipe to pro-
duce the third harmonic by blowing with about three times the velocity, giving a note
an octave and a fifth above the fundamental. Overblowing on closed-pipe jet-drive inst-
struments is, however, uncommon. The pan flute (see Example 11.6) is sometimes
overblown, and there are certain rare types of stopped organ pipes that are designed
to be overblown, but these are the only examples this author is aware of.

Increasing the jet velocity is not the only way to produce higher harmonics. An
alternative is to change the distance $d$ that the jet travels to reach the labium. If we
decrease this distance then we decrease the time for the jet wave to reach the labium
and hence favor higher harmonics over lower ones. The ideal distance to produce
the $n$th harmonic can be calculated by rearranging Eq. (11.72) thus:

$$d = \frac{v}{5nf}.$$  (11.73)

For many instruments, including the recorder and pipe organ, there is no easy way to
change the distance to the labium, but it can be done on flutes and similar instruments
where the air jet is produced directly from the player’s lips. On an orchestral concert
flute the player can rotate the flute upward to bring the labium closer to their lips
and hence push the instrument to jump to a higher harmonic. In practice, flutists use
a combination of rotating and pursing of the lips (which increases the air pressure)
to overblow the instrument.

**Example 11.8: Overblowing the recorder**

A typical lung pressure (“overpressure”) when playing the recorder is 100 Pa. What would
be a typical pressure when overblowing to the second harmonic? To answer this question,
we observe that Eq. (11.72) says that the velocity needed to play the first harmonic is
$v = 5fd$ but the velocity for the second harmonic is $10fd$, twice as high. Bernoulli’s
equation, Eq. (11.49) on page 446, tells us that the overpressure $P - P_0$ is proportional to $v^2$,
so when we double the velocity we quadruple the pressure. So an overpressure of about
$4 \times 100 = 400$ Pa would be need to play the second harmonic. This is a considerable increase
on the 100 Pa for the first harmonic. Moreover, the situation becomes worse as we go to
higher notes because the velocity $10fd$ also increases with frequency. Things can be made
easier, however, through the use of a “register hole,” as described in the next section.

### 11.2.6 Register holes

On an instrument like the recorder, where the only thing the player can change about
their breath is the air pressure, it can be difficult to overblow and reach the second
harmonic, particularly on the higher notes where the required air velocity becomes
very high—see Eq. (11.71). To help the situation, the recorder has a *register hole*, a
small hole on the back of the instrument, about half wave up and operated by the player’s left thumb. (The recorder also has many “tone holes” on the front of the instrument that are used for creating different notes—we discuss how these work in the next section.) The purpose of the register hole is to discourage the instrument from vibrating in its first mode. The hole is normally covered up by the player’s thumb, but if the thumb is removed and the hole opened, it allows a small amount of air to enter or leave the pipe at this point. In the second mode of vibration, the center point of the pipe is a pressure node—see Fig. 11.5—meaning the sound pressure \( p \) at this point is zero, so the total pressure \( P = P_0 + p \) is just equal to atmospheric pressure \( P_0 \). Opening the register hole half way along the pipe will thus have no effect at all on this second mode: the air will be at atmospheric pressure on either side of the open hole, no different inside the instrument from out, so no air will flow either into the hole or out of it.

If the pipe is vibrating in its fundamental mode, on the other hand, then the center point of the pipe is a pressure antinode—see Fig. 11.3—meaning the pressure has maximum variation at this point. When the register hole is open air will leak out of it when the pressure inside the pipe is high and flow in when the pressure is low. This leakage reduces the amount of pressure variation in the pipe: whenever the pressure tries to build up, air leaks out and deflates it again. The result is that the amplitude of the pressure variation is reduced significantly and the strength of the fundamental mode decreased, while the second mode is still unaffected.

By tilting the playing field in this way, the register hole makes it easier to play the second harmonic and hence get higher notes on the recorder. A combination of overblowing and opening the register hole allows the player to jump up the octave even on higher notes that normally would require dauntingly high air jet velocity.

In principle, one could use a register hole to hit higher harmonics as well. The rule is that the hole should be placed at a pressure node of the harmonic you want to hit, so that it has no effect on that harmonic, but saps pressure from the others. Thus a register hole a third of the way along the pipe would aid in hitting the third harmonic, and one a quarter of the way along would aid in hitting the fourth harmonic. The recorder only has one register hole, for the second harmonic, but some reed instruments, such as the oboe, have more than one—see Section 11.3.4.

A register hole is also used on the specialized organ pipe known as a flute harmonique. The flute harmonique is a metal flute-style pipe (i.e., a wider cylindrical flue pipe—see Section 11.2.4) that uses a combination of higher air velocity, shorter distance to the labium, and a small register hole half way along the pipe to overblow to the second harmonic and produce a note an octave higher than normal. The flute harmonique produces a distinctive whistle, somewhat piercing and useful for melodic solo lines.

The standard concert flute does not have a register hole as such. The combination
of changing the air velocity and changing the distance to the labium is enough on its own to overblow the instrument in most cases. However, for some of the highest notes, which involve hitting the fourth, fifth, or sixth harmonics, players will use one of the tone holes as an ersatz register hole—if the hole falls in the right place then opening it up will have the same effect as a register hole and aid in hitting the appropriate harmonic. This trick is known as venting, and it is also used on some reed instruments such as the clarinet.

11.2.7 Tone holes

So far, we have considered mainly pipes that produce just a single note, such as organ pipes, but many wind instruments, like the flute and recorder, produce a whole range of notes from a single pipe. They do this by using tone holes, which are holes arranged at intervals along the pipe to let air out. The principle is straightforward. Tone holes can be closed by the player by covering them with a finger as in the recorder or a mechanically operated pad as in the flute, or they can be left open. With all tone holes closed the instrument behaves as a single length of pipe and produces its lowest note. Open a tone hole, however, and we shorten the pipe—the pipe effectively ends at the open hole. In terms of the physics, the open hole connects the pipe to the air outside, so the pressure at that point is equal to atmospheric pressure. Hence the pressure is fixed at the location of the hole making it a pressure node that behaves the same as the open end of any normal pipe. And because the effective length of the pipe is now shortened, the pitch of the note rises.

By placing a selection of tone holes at various points along the pipe, one can create a range of notes and hence a complete scale. The position of the tone hole needed to create a particular note is given by the standard formula

\[ f = \frac{c}{2L} \]

for an open pipe rearranged thus:

\[ L = \frac{c}{2f}. \] (11.74)

Recall that the opening by the labium acts as one end of the pipe. The tone hole should be placed a distance \( L \) down the pipe, measured from the labium.

The simplest version of this idea is perhaps the concert flute. The flute has 11 tone holes that when opened allow it to play 11 notes in half-step intervals. Along with the note obtained when all tone holes are closed, this gives us a full 12-tone octave scale. To get higher notes one uses overblowing. For instance, one can overblow to get the second harmonic, an octave up from the fundamental, then use the tone holes to get 12 notes in that octave as well. We describe the flute in more detail in Section 11.2.9.

The recorder, despite its apparent simplicity, is actually a bit more complicated than the flute because its tone holes are smaller and because there are only seven of
them. Let us look at the recorder in more detail.

11.2.8 **The Recorder**

The recorder is a *duct flute*, a jet-driven, cylindrical, open-pipe wind instrument in which a mouthpiece or fipple, blown by the player, directs a jet onto a fixed labium. Different notes are produced by a combination of tone holes covered directly by the player’s fingers and overblowing to sound the second harmonic and occasionally the third, aided by the use of a register hole. The recorder comes in various sizes, the most common being the soprano (with a lowest note of C5), the alto (F4), and the tenor (C4)—see Fig. 11.23.

Although simple in principle, there are some subtleties to the recorder. In particular, because the tone holes are covered by the player’s fingers, they have to be quite small—no larger than the size of a typical fingertip. This means that they are not large enough to completely open the pipe to the air outside. When a hole is opened, air will flow in and out of it, but not fast enough to completely equalize the air pressure inside and outside the pipe, so the pressure variation at the hole is reduced but not eliminated altogether and the hole is not a perfect node of the standing wave in the pipe. This complicates things: achieving different notes on the recorder is not simply a matter of opening a hole at the appropriate point and changing the effective length of the pipe.

Figure 11.24 illustrates the difference between large and small tone holes. A large hole effectively acts as the end of the pipe. For sound producing purposes the pipe stops at the hole and there is little or no vibration in the remaining portion of the pipe, so the standing wave looks like Fig. 11.24a. There may be other tone holes in the portion of the pipe to the right where there is no wave, but it makes little difference whether these are open or closed, though usually one opens them, both to let out as much sound as possible, and because the instrument is easier to play this way.

Now contrast this with the behavior of a smaller hole. A smaller hole causes only partial equalization of the pressure inside and outside the pipe, so there is still some vibration of the air at the hole as well as further down the pipe, albeit at a diminished level. The result is a pressure profile like that of Fig. 11.24b. The effective length of the standing wave, indicated by the dotted line, now extends a little further along the pipe, past the open tone hole. In essence, a small tone hole acts like a larger hole a bit further down the pipe. How much further depends on the size of the hole.

Although the behavior of a smaller tone hole is more complicated, it does not in practice present a problem for the design of the instrument. We just have to place the tone holes a bit further up the pipe than we normally would to produce a particular note. Exactly where is usually determined by trial and error. The mechanics of a smaller tone hole also opens up some new possibilities for creating notes on the recorder. Because there is still a partial standing wave in the section of the pipe...
Figure 11.24: Sound pressure in a pipe with a tone hole. (a) A large tone hole at the position indicated creates a true pressure node, with zero sound pressure, so the standing wave terminates at this point, making the effective length of the pipe equal to the distance to the tone hole. (b) Making the tone hole smaller limits the amount of air that can pass through it, so the pressure does not completely equalize inside and outside the pipe. As a result the sound pressure continues to vary inside the pipe at the position of the hole, although at a reduced level that is equivalent to a standing wave a little longer than the distance to the hole.

Pipe beyond the tone hole, we can open further holes in this section and affect the shape of this standing wave and hence the frequency of the overall vibration. This is called cross-fingering—the use of additional holes further along the pipe to make small adjustments to the effect of the main hole. With cross-fingering the shape of the standing wave can become quite complex and there is no longer a simple formula that tells us the frequency of the note, but useful configurations can again be found by trial and error. The modern recorder uses cross-fingering extensively to allow it to play a complete 12-tone scale even though there are only seven tone holes. The seven holes basically produce the seven notes of a major scale, but with the help of cross-fingerings one can obtain the other five notes to complete the 12-tone scale. The fingerings for each note are shown in Fig. 11.25.
The use of tone holes complicates the situation with the register hole of the recorder. As discussed in Section 11.2.6, the register hole should be placed half way along the pipe for overblowing the second harmonic. But when the effective length of the pipe changes because of the tone holes the half-way position also changes, so the ideal position of the register hole is higher up the pipe for higher notes, as high as a quarter of the way from the top in the most extreme cases. The actual position of the register hole on a recorder is about a third of the way from the top, which is a compromise, a bit too high up for the lowest notes and a bit too low for the highest ones. This is not perfect, but in practice it’s good enough.

11.2.9 The concert flute

Perhaps the preeminent example of an instrument in the flute family is the concert flute, one of the primary wind instruments of the orchestra and a mainstay of many musical genres. The flute comes in a range of sizes, from the diminutive piccolo to the impressive bass flute, but the common orchestral version is a middle-sized one about 67 cm long. Most modern flutes are made of metal, usually silver or a silver alloy, although one does occasionally encounter wooden flutes. As with most
wind instruments, construction material has little effect on the sound—listening tests reveal only modest tonal differences between metal and wood flutes, or between flutes made of different types of metal.

Unlike most jet-drive instruments, the flute is blown transversely, with the player holding the instrument up to their mouth sideways. There is no mouthpiece: the player uses their own lips to blow across a hole in the pipe and the edge of the hole acts as the sharp edge or labium. In practice, blowing transversely does not make a great deal of difference to the physics of the instrument: the resulting jet still oscillates in and out of the pipe and excites vibration by injecting repeated bursts of air. The end of the pipe nearest where the player blows is closed off and plays little part in sound production. The blowing hole itself acts as the open end of the pipe, with the resonating part of the pipe extending from the blowing hole to the open other end, a distance of about 60 cm, so that the frequency of the lowest note on the flute is around

\[ f = \frac{c}{2L} = \frac{343}{2 \times 0.60} = 286 \text{ Hz}, \]  

which is about C♯4. In fact the lowest note on the concert flute is C4—there is a small “end correction” because the pressure node at the end of the pipe actually occurs a short distance past the end, rather than right where the pipe stops.

The tone holes on the flute are not closed directly by the player’s fingers, but by pads controlled using a set of levers. This has a number of advantages. First, it means that the spacing of the tone holes is not limited by the size of the player’s hands, which allows the pipe to be longer and hence allows the instrument to reach lower notes. Second, the system of pads and levers allows tone holes to be either normally open or normally closed. Most are normally open, the pads held up by springs until the player operates the appropriate mechanism. A few, however, such as trill pads, are normally closed and open only when a lever is pressed.

However, the main advantage of using pads instead of fingers to close the tone holes is that the holes can be made significantly larger than on an instrument like the recorder, and this means that when a hole is opened it lets all the air out of the pipe, effectively terminating the pipe at that point. This makes the flute a rather straightforward instrument in many ways. It simply has one tone hole at the appropriate place for each note and there is no need for cross-fingerings. The basic concert flute produces the note C4 when all the tone holes are closed and has 11 holes that produce the next 11 half-steps up to B4—see Fig. 11.26. Note that by convention the tone holes are named after the note produced when they are closed, not open, so the lowest tone hole is C and the highest is B. To get B you open all 11 holes.

Beyond the first octave, notes are reached by overblowing to hit higher harmonics, which in the flutist’s vocabulary are called registers. As mentioned in Section 11.2.6, the flute does not have a register hole, relying instead on changes in blowing velocity and position to hit the various harmonics. Jumping up to the second
harmonic (or second register) gives you the next octave up, which, combined with
the use of the tone holes, produces the notes C5 to B5. Further notes are reached
using the third, fourth, fifth, and sixth harmonics, which is as far the flute goes. The
seventh harmonic, as discussed in Section 4.3.4, does not coincide with a note of the
scale and hence is not musically useful (at least in Western music) and the eighth and
higher harmonics are difficult to reach.

The use of harmonics and tone holes in combination means that some registers
of the flute overlap with each other, giving the flutist more than one way to play
some notes. For instance, the note G5 can be reached either by playing the second
harmonic and opening the tone holes up to G or by playing the third harmonic and
closing all tone holes. This ability to play the same note in different ways is common
to many wind instruments and adds a useful level of flexibility. Different fingerings
may be more or less easy to reach in different contexts and the availability of alternate
choices for a note can make certain musical passages much easier to play.

The flute has a range of about 3 1/2 octaves from its lowest note to its highest.
The highest note is produced by playing the sixth harmonic and opening all the tone
holes. The sixth harmonic is two octaves and a fifth above the C4 fundamental, which
puts it at G6, and the additional 11 half-steps from the tone holes give us a highest
note of F♯7. In fact, the conventional flute repertoire only uses the notes up to D7, the
remaining notes up to F♯7 being quite difficult to play, though they can be produced
by a skilled player.

At the low end of the range, some professional flutes are made slightly longer to
allow them to reach a half-step lower to the note B3, with the addition of a 12th tone
hole for playing C4. Modern flutes also have a few additional holes that, while not
strictly necessary, provide easier ways to hit some of the higher notes and to perform
trills (rapid alternation between two adjacent notes)—see Fig. 11.26 again.

One disadvantage of using 11 separate tone holes on the flute is that the player
only has ten fingers, so they cannot simply use one finger for each hole as on the
recorder. The solution is a rather complicated mechanism, originally invented by
Theobald Boehm in 1847, that allows all of the tone holes to be operated by depressing
suitable combinations of levers and pads.

Because of the one-hole-per-note design of the flute, it is relatively straightforward
to work out where the tone holes should be placed. For example, we have said
that the length of the sounding part of the pipe is \( L = 60 \) cm and to go up a half-step
we want to the frequency to increase by a factor of \( 2^{1/12} \), which means that the length
should decrease by the same factor. So to go from the lowest note of C4 to C♯4 we
need to the pipe length to become

\[
\frac{60}{2^{1/12}} = 56.6 \text{ cm},
\]

and hence the first tone hole should be \( 60 - 56.6 = 3.4 \) cm from the end of the pipe.
Similarly, the second tone hole increases the frequency by two half-steps or a factor of $2^{2/12}$, so the pipe length should be $60/2^{2/12} = 53.5$ cm and the second hole should be 6.5 cm from the end. By applying the same method we can calculate the positions of all 11 tone holes on the flute.

**Example 11.9: Position of the highest tone hole on the flute**

Where should the highest tone hole of the flute be located? We can answer this question using the same method as above. The highest tone hole increases the pitch by 11 half-steps and, given that the total length of the sounding part of the flute pipe is 60 cm, a note 11 half-steps higher requires a pipe of length $60/2^{11/12} = 31.8$ cm, so the 11th tone hole should be $60 - 31.8 = 28.2$ cm from the end of the pipe.

This is almost half way along the pipe, which makes good sense: 11 half-steps is almost an octave, and a pipe that sounds an octave higher would be half as long. Looking at Fig. 11.26, we can see that indeed the B♭ tone hole is very close to being half way between the blowing hole and the lower end of the flute.

### 11.3 Reed instruments

Next in our examination of wind instruments we turn to the reed instruments, such as clarinet, oboe, and saxophone. A reed instrument makes use of a vibrating reed—a sturdy yet flexible sliver of cane or sometimes plastic—that opens and closes a small gap at the end of the instrument, allowing bursts of air into the pipe which drive vibration in a manner somewhat analogous to the bursts of air from the jet drive of a flute (Section 11.2.1).

Figure 11.27 shows a diagram of the mouthpiece of a clarinet—the part where the player blows at the top of the instrument. The mouthpiece consists of a hollow tube, usually made of plastic, fitted over the end of the instrument pipe, with a curved opening on the bottom. The instrument reed is clamped to the mouthpiece so that it covers this opening as shown in the figure, leaving just a small gap between the tip of the reed and the plastic of the mouthpiece where air can enter. The player holds the mouthpiece in their mouth and blows, and the pressure created by their lungs forces air through the gap and into the instrument pipe.

A crucial difference between this arrangement and the jet drive of a flute is that the jet drive has an opening in the pipe next to the labium (see Fig. 11.14) that allows air to flow easily in and out of the pipe. In the clarinet there is no equivalent opening. There is a small gap where the player blows into the pipe, but the amount of air that flows through this gap is modest, and moreover it only flows inward, so there is no way for air to leave the pipe other than at the far end of the instrument (or through open tone holes). As a result the blown end of a reed instrument is essentially closed. This is central to the sound of these instruments since, as we have seen, a pipe closed at one end behaves quite differently from a pipe open at both ends. It also means
that when the air inside the pipe vibrates, the blown end is a pressure antinode, the pressure varying widely as the vibrating air pushes up against the closed end of the pipe. This variation in pressure is crucial for the operation of the reed.

In normal operation, the reed is pushed upward against the underside of the mouthpiece by a combination of the pressure of the player’s breath and the grip of their teeth and lips. This closes off the gap through which air would normally flow, either partially or completely, and hence little or no air flows into the pipe even though the player is applying breath pressure. As the oscillating pressure inside the pipe builds up, however—because the mouthpiece end is a pressure antinode—it pushes on the inside of the reed and opens up the gap, allowing a burst of air to flow in. When the pressure drops again the gap closes. The result is a series of bursts of air exactly in time with the varying pressure inside the instrument. As in a jet drive, this series of bursts, if timed just right, can amplify the oscillation inside the pipe and produce an extended vibration, like pushing a swing at exactly the right moment each time it swings across. All that’s needed is an initial puff of breath from the player to set the air moving back and forth in the first place and then the reed does the rest.

In Section 11.2.1 we saw that the velocity \( v \) of air flowing through a gap or opening is given by

\[
v = \sqrt{\frac{2(P - P_0)}{\rho}},
\]  

and this equation applies to the flow of air past the reed of a reed instrument as well, with \( P \) being the air pressure from the player’s breath, \( P_0 \) being the lower pressure inside the instrument pipe, and \( \rho \) being the density of air. Note that the velocity depends only on the pressure difference \( P - P_0 \) and the density. You might imagine it also depends on how narrow the gap is, that squeezing through a smaller opening
would produce a higher velocity, but this is not the case. The velocity is the same whether the opening is large or small. This means that the total amount of airflow goes up with the size of the gap between reed and mouthpiece. A larger gap with the same air velocity means more flow, just as a wider hosepipe carries more water than a narrow one if the speed of the water is the same.

The size of the gap, however, depends on the air pressure. The pressure \( P \) from the player’s breath pushes on the outside of the reed, while on the inside there is only the lower pressure \( P_0 \), which means there is a net force pushing the reed inward. It is this force that is responsible for bending the reed inward, reducing the size of the gap through which the air is flowing, or closing it off completely if the force is large enough. (As mentioned above, the bending of the reed is also helped by the force of the player’s teeth and lips as they grip the mouthpiece, and the player can vary this force to change the characteristics of the instrument somewhat.)

Now consider what happens if the player starts blowing gently, but then slowly increases their breath pressure \( P \). At low pressure the reed will not get pushed in much and the gap between reed and mouthpiece will be wide enough to let a significant amount of air through. As the pressure increases, so does the velocity \( v \) (see Eq. (11.77)) and so the airflow will go up. But as the pressure increases further, the reed starts to close off the gap. For a while the increasing velocity might be enough to compensate for the decreasing size of the gap so that airflow continues to increase, but eventually the closing gap wins out and the airflow goes down. Ultimately, when the gap closes completely, airflow stops.

Thus, if we were to plot a graph of the rate of airflow against the pressure difference \( P - P_0 \) it would look something like Fig. 11.28. When no pressure is applied the airflow is zero. As pressure rises the airflow initially goes up, but at some point it turns around and goes back down again, reaching zero at the point where the reed closes the gap entirely.

A reed instrument operates in the latter part of this curve, the shaded region in Fig. 11.28, where the airflow is decreasing. In this region the airflow has the counterintuitive property that the harder the player blows the less air flows. In technical language we say that the mouthpiece has negative conductance.

Now we throw in a third element: the pressure \( P_0 \) inside the pipe is not actually constant, but varies because the air in the pipe is oscillating. Moreover the pressure is varying particularly strongly inside the mouthpiece because the mouthpiece end of the instrument is a pressure antinode. When the inside pressure \( P_0 \) goes up, it decreases the pressure difference \( P - P_0 \) (the horizontal axis in Fig. 11.28) and hence increases the airflow into the pipe. In plain terms, the increased pressure inside the pipe pushes on the reed and opens it a little wider, allowing more air to flow in. This is what produces the bursts of air that keep the oscillation in the pipe going: each oscillation of pressure produces a burst through the mouthpiece, resulting in a set
of bursts at exactly the frequency of the vibration in the pipe. In order to reinforce the vibration these bursts have to be timed right, just like the bursts in the jet-drive mechanism of Section 11.2.1, and this requires careful design of the shape and size of the mouthpiece, something that has evolved by trial and error over centuries to create today’s agile and refined wind instruments.

The requirement that we fall in the “negative conductance” portion of the curve in Fig. 11.28 gives us a minimum and maximum working pressure as shown by the dotted lines in the figure. A reed instrument will not work if the player blows too softly, a fact well known to any clarinet or saxophone player, and it also will not work if the pressure is too high, since the reed will close off against the mouthpiece and no air will flow at all. We can run into issues before this point, however. Too high a breath pressure can produce a high velocity \( v \) that throws off the timing of the air bursts so that they no longer push the vibrating air in the pipe at the ideal moment to keep it moving. On the other hand, faster bursts of air can be well suited to a vibration at a higher frequency, such as one of the higher harmonics of the pipe, and hence with higher breath pressure one can cause the instrument to overblow and jump to the second or third harmonic or even higher. This can be a good thing: as on the flute and recorder, overblowing is an important technique for reaching the highest notes on reed instruments.

A crucial requirement for a usable reed instrument is that the reed be able to
vibrate at a wide range of frequencies in order to cover all the notes we want to play. Whether the reed is able to do this depends on its design and the material from which it is made. Reeds made of some materials, such as steel, would not work well because they would be able to vibrate at only one very specific frequency—imagine plucking a thin metal reed and think of the sound it would make, a bright metallic ping with a clear note. Such a reed has a well-defined natural frequency at which it wants to vibrate and it would be difficult to make it vibrate at any other frequency. A clarinet reed, on the other hand, vibrates readily at a wide range of frequencies. Reeds with this property are called soft reeds, though the name is slightly misleading since a soft reed need not actually be soft. A soft reed just means that it is not fussy about what frequency it vibrates at. Clarinet reeds are normally made of the stem of the giant cane plant (also known as elephant grass) and they are in fact quite hard and woody. But they are “soft” in the sense that they are able to generate good notes across several octaves. The same cane material is used for reeds of other instruments as well.

Reeds that vibrate only at one specific frequency, such as metal reeds, are called hard reeds, and are not useful for traditional wind instruments, though they do have their place, in instruments like the harmonica and accordion, which we discuss in Section 11.5.

**Advanced Material**

### 11.3.1 Airflow past a reed

Consider the clarinet mouthpiece and reed and suppose the air pressure applied by the player on the outside is $P$, while the pressure inside is $P_0$:

The velocity of the air going into the mouthpiece is

$$v = \sqrt{\frac{2(P - P_0)}{\rho}}.$$

(See Eq. (11.77)). Essentially all of this air flows directly past the tip of the reed and into the mouthpiece. There are small openings along the sides of the reed, but there is little airflow through these. If the width of the reed is $w$ and the distance between the reed and the edge of the mouthpiece is $x$ as shown above, then the area of the opening through which the air passes is $A = wx$ and the airflow—the volume of air per second—is

$$U = Av = wx \sqrt{\frac{2(P - P_0)}{\rho}}.\quad (11.79)$$

At the same time, the reed is bent inward by the air pressure on the outside as described in Section 11.3. If the area of the free part of the reed (called the “facing area”) is $S$ then the force on the outside is $SP$, but there is also an opposing force $SP_0$ from the air on the inside, so the net inward force is $SP - SP_0 = S(P - P_0)$. If the value of $x$ when the reed is at rest is $x_0$, then the distance it is bent inward is $x_0 - x$. The reed behaves like a spring obeying Hooke’s law, meaning the force required to move it distance $x_0 - x$ is $k(x_0 - x)$, where $k$ is the spring constant. Equating this with the applied force from the pressure, we have

$$k(x_0 - x) = S(P - P_0). \quad (11.80)$$
which can be rearranged to give the value of $x$ thus:

$$x = x_0 - \frac{S}{k}(P - P_0).$$  \hfill (11.81)

Substituting this expression into Eq. (11.79), we find that the airflow is

$$U = w\left[x_0 - \frac{S}{k}(P - P_0)\right] \frac{2(P - P_0)}{p} = ap^{1/2} - \beta p^{3/2},$$  \hfill (11.82)

where $p = P - P_0$ and

$$\alpha = wx_0\sqrt{2/p}, \quad \beta = \frac{wS}{k}\sqrt{2/p}.$$  \hfill (11.83)

Equation (11.82) is the curve plotted in Fig. 11.28 on page 471. It tells us that $U = 0$ when $p = 0$, which is natural since we expect there to be no airflow when we are not blowing the instrument, and that the curve rises initially but then turns over and falls off. As discussed in Section 11.3, the working regime of a reed instrument is in the decreasing portion of the curve, the shaded region in Fig. 11.28. We can use Eq. (11.82) to calculate the limits of this region. The maximum working pressure $p_{\text{max}}$ falls at the point where the airflow $U$ goes to zero, which gives us

$$\alpha p^{1/2} - \beta p^{3/2} = 0.$$  \hfill (11.84)

Cancelling a factor of $p^{1/2}$ and rearranging, we get

$$p_{\text{max}} = \frac{\alpha}{\beta} = \frac{kx_0}{S}.$$  \hfill (11.85)

In plain English, this says that the net air-pressure force $Sp_{\text{max}}$ on the reed when the channel is completely closed must be equal to the force $kx_0$ needed to bend the reed all the way to the closed position.

Meanwhile the minimum working pressure $p_{\text{min}}$ coincides with the highest point of the curve in Fig. 11.28, which is where the derivative of Eq. (11.82) is zero, which means

$$\frac{1}{2} \alpha p^{-1/2} - \frac{3}{2} \beta p^{1/2} = 0.$$  \hfill (11.86)

Multiplying throughout by $2p^{1/2}$ and rearranging, we find that

$$p_{\text{min}} = \frac{\alpha}{3\beta} = \frac{1}{3}p_{\text{max}}.$$  \hfill (11.87)

Thus, regardless of the value of $p_{\text{max}}$, the minimum working pressure is always one third of the maximum, which gives the musician a conveniently wide range of working pressures, allowing them room to vary the pressure to control volume level, tone, and overblowing.

Taking as a typical working pressure a point half way between the minimum and maximum, the applied pressure difference when playing a reed instrument will under normal circumstances be about

$$p = \frac{2kx_0}{3S}.$$  \hfill (11.88)

Clarinet and saxophone reeds come in a range of sizes (which affects $S$) and thicknesses (which affects $k$). The value of $x_0$ depends on the design of the mouthpiece and mouthpieces come in a range of styles with different $x_0$. Hence the reed player has considerable ability to choose their desired working value of $p$.

Higher pressure produces a louder tone and also a brighter one with more high harmonics, so players looking for a brighter, stronger tone will select stiffer reeds or use a mouthpiece with greater curvature that increases the gap size $x_0$. Players looking for a mellower, warmer tone, will select a softer reed or a mouthpiece with less curvature.

As a general guide, a typical value for the force $kx_0$ required to push the reed all the way up against the mouthpiece is about 100 grams or 1 newton, while the area $S$ of the moving portion of the reed is about 2 square centimeters or $2 \times 10^{-4} \text{ m}^2$. Plugging these values into Eq. (11.88), we find that a typical breath pressure for a clarinet-type reed instrument is

$$p = \frac{2 \times 1}{\frac{3}{3} \times 2 \times 10^{-4}} = 3300 \text{ pascals}.$$  \hfill (11.89)

This figure is much higher than the breath pressure needed to play, for instance, a recorder, which is in the region of only 100 pascals. This is one of the reasons why the recorder is often given to young children as a first wind instrument, while an instrument like the clarinet is more appropriate for older children and adults who have stronger lung pressure.
11.3.2 The clarinet

Clarinets come in a range of sizes but the most common is the soprano clarinet, which is about 66 cm long. The clarinet has a cylindrical pipe and is usually made of wood, although some inexpensive clarinets are made of hard plastic. As with other wind instruments, the material of the pipe does not have a large influence on the sound.

Because the pipe is cylindrical and because it is effectively closed at the blowing end while open at the other end, the fundamental frequency of the clarinet is \( f = \frac{c}{4L} \), an octave lower than an equivalent open-pipe instrument. For a length of 66 cm this gives a frequency of

\[
f = \frac{c}{4L} = \frac{343}{4 \times 0.66} = 129.9 \text{ Hz}
\]

for the lowest note on the clarinet, which is close to the note C3. In fact the lowest note is a little higher at D3—the flared opening at the bottom of the instrument means that the effective length of the pipe for acoustic purposes is a little shorter than 66 cm.3

Thus, despite being about the same length as the flute, the clarinet can play about an octave lower. On the other hand, as we have seen, closed-pipe instruments only produce odd harmonics. Figure 11.10 on page 435 shows the frequency spectrum of a note played on a clarinet, which has strong odd harmonics but only small amounts of the even ones. (The strength of the even harmonics is not exactly zero because the mouthpiece end of a clarinet is not perfectly closed, on account of the small opening between the reed and the mouthpiece where air enters the instrument.)

The clarinet uses tone holes to produce different notes, mainly operated by pads and levers as on the flute. To reach higher notes one overblows the instrument as described in Section 11.3, but an important difference arises between the clarinet and other wind instruments when overblowing. Because the clarinet produces only odd harmonics, the next harmonic you hit when you overblow is the not the second but the third. This means that instead of jumping up an octave you jump the larger interval of an octave and a fifth, or 19 half-steps. If you overblow still more you can reach the fifth harmonic, two octaves and a third above the fundamental, but this is as far as it goes—the clarinet uses just the first, third, and fifth harmonics, which clarinetists call the chalumeau, clarion, and altissimo registers.

The clarinet makes use of a register hole to help with hitting the third harmonic or clarion register, but the register hole works slightly differently from that on, say, a recorder because we are aiming for the third harmonic and not the second. Recall that the register hole on a recorder is ideally placed in the middle of the pipe, since this is a node of the second harmonic. Opening this hole has no effect if the pipe is

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3Music for soprano clarinet is written two half-steps higher than it is actually played, so the lowest note D3 is written as E3. We say that the clarinet is a “transposing instrument.”
vibrating in its second mode, but it does have an effect on the first harmonic by letting air in and out of the pipe and reducing the amplitude of vibration at the center point. Thus opening the hole favors the second harmonic over the first. A similar thing happens on the clarinet, except that since we are aiming to hit the third harmonic we need to place the register hole at a node of this harmonic. Looking at Fig. 11.9 on page 434, we see that there is a node of the third harmonic one third of the way along the pipe from the closed end, so the ideal placement of the register hole is a third of the way along. As on the recorder, however, this changes depending on what note one is playing. If we open one or more tone holes, it changes the effective length of the pipe and hence changes the ideal position of the register hole. For the lowest notes in the clarion register a position one third of the way along is indeed perfect, but for higher notes the hole wants to be closer to the mouthpiece. In practice, the location of the register hole is a compromise, a little too close to the end of the pipe for some notes and a little too far for others. On a modern clarinet the register hole is about a fifth of the way down the pipe. There is no separate register hole for playing the altissimo register (the fifth harmonic): this is reached simply by blowing harder, as well as by changes in how the player grips the reed with their teeth and lips.

The fact that the clarinet has only odd harmonics makes a difference to the tone holes and fingering of the instrument too. The lowest note, as we have said, is D\textsuperscript{3} and if you overblow this note to hit the third harmonic you get the note 19 half-steps higher, which is A\textsuperscript{4}. All the notes in between—there are 18 of them—have to be produced using tone holes, which means you need 18 tone holes on the clarinet. On an instrument like the flute, by contrast, which overblows an octave, you only need 11 tone holes.

This difference makes the fingering more complex on the clarinet than on the flute. Eleven tone holes is already a lot to manage with only ten fingers, but 18 requires a complicated arrangement of keys and levers to make them all playable. The positioning of the tone holes is different as well. As we saw in Example 11.9 on page 468, the tone holes of the flute go about halfway up the instrument, since this is what is needed to cover a factor of two in frequency. The tone holes of the clarinet, however, need to cover factor of three in frequency, which means that the pipe length for the highest note will be a third of that for the lowest note and hence the highest tone hole is two-thirds of the way up the pipe—see the picture of a clarinet on page 474.

Although it makes the clarinet more complicated to play, this arrangement also gives the clarinet an exceptionally wide range. The altissimo register, which corresponds to the fifth harmonic, is two octaves and a major third above the fundamental, so overblowing the lowest note of D\textsuperscript{3} in the altissimo register produces F\textsuperscript{♯5}. From there one can then use the tone holes to go up a further 18 half-steps, putting the highest note on the instrument at C\textsuperscript{7}. This gives the clarinet almost a four-octave
range, the largest of any of the orchestral wind instruments. Skilled clarinetists can go even higher by making use of higher harmonics, but this is not considered part of the standard clarinet range.

Note that the clarion and altissimo registers overlap on the clarinet, meaning that, as on the flute, some notes can be played in more than one way. For instance, the note F♯5 can be played as the tenth note in the clarion register or as the lowest note in the altissimo register. Having alternate fingerings for the same note can be useful in making certain passages easier to play.

Because it lacks even harmonics, the clarinet has a relatively soft, hollow sound with less upper-harmonic brightness than other instruments, although with its ability to reach very high notes in the altissimo register the clarinet can cut through dense musical textures and finds use for instance in jazz and marching bands. The particular pattern of levers and fingerings used on the clarinet also lends itself to the playing of rapid arpeggios and much clarinet music has been written to exploit this capability of the instrument.

**Example 11.10: Bass clarinet**

The bass clarinet is a larger cousin of the more common soprano clarinet, with a cylindrical pipe and a total pipe length of 1.08 meters. What are the lowest notes in the chalumeau, clarion, and altissimo registers of the bass clarinet?

The lowest notes in each register are obtained with all tone holes closed, so that the sound comes from the entire length of the pipe. As a cylindrical, closed-pipe instrument, the fundamental frequency of the bass clarinet is given by Eq. (11.14):

$$ f = \frac{c}{4L} = \frac{343}{4 \times 1.08} = 79.4 \text{ Hz}, \quad (11.91) $$

which is close to the note Eb2. In fact, the true lowest note on the bass clarinet is a half-step lower than this, at D2, because of the effect of the flared opening at the end of the pipe. (As with the soprano clarinet, music for bass clarinet is written two half-steps higher than it sounds, so the lowest note is written as E2 but has the pitch of D2.)

The clarion and altissimo registers correspond to the third and fifth harmonics, an octave plus a fifth and two octaves plus a major third higher than the fundamental. So the lowest notes in these registers are A3 and F♯4. Thus, the bass clarinet is exactly an octave lower than the soprano clarinet.

**11.3.3 The saxophone**

The saxophone (or sax for short) is one of the more recent additions to the pantheon of orchestral instruments. Rather than evolving over the centuries like most instruments, it was invented, essentially by a single person, Adolphe Sax, in the 1840s. Saxophones come in various sizes, the most common being the soprano, alto, and tenor varieties. Superficially the saxophone is similar to the clarinet, since they are
both reed instruments, played with a very similar mouthpiece that allows musicians to switch easily back and forth between them, and both have pipes with tone holes operated with pads and levers. There are some easily spotted differences, such as the fact that the saxophone is usually made of metal whereas the clarinet is usually wood. As we have said, however, the material of the pipe does not have much effect on tone, since it is the air in the pipe that vibrates, not the pipe itself. Saxophones also differ from clarinets in that they often have a curved shape, with a bell that bends upward, helping to project the sound toward the listener. But this too is mostly a technical difference, and some saxophones, such as certain types of soprano, are straight like a clarinet.

There is, however, a more important difference between the saxophone and the clarinet, although one that is harder to spot: where the clarinet has a cylindrical pipe, the pipe of the saxophone is conical. As discussed in Section 11.1.8, a conical pipe, even though it is closed at one end, has the same fundamental frequency as an open pipe of the same length and produces all harmonics. This makes the sound of the saxophone entirely different from that of the clarinet. A saxophone will sound an octave higher than a clarinet of the same size, and produces a brighter sound because of the presence of many harmonics that are missing on the clarinet—compare Figs. 11.10 and 11.11 (pages 435 and 439). The conical shape of the saxophone pipe is a little hard to see for saxophones with a curved bell, but it can be seen quite clearly on the soprano sax in Fig. 11.30. The use a conical pipe in order to produce all harmonics is common to many reed instruments. In addition to the saxophone, this strategy is adopted in the oboe, cor anglais, and bassoon, and is central to the sound of each of these instruments—see the discussion in Section 11.3.4.

Once we understand that the saxophone has a conical shape, its other features follow naturally. For a pipe of length $L$, the fundamental frequency, with all tone holes closed, will be $c/2L$. Thus, for example, the soprano saxophone is about 71 cm long, giving it a lowest frequency of

$$\frac{343}{2 \times 0.71} = 242 \text{ Hz},$$

which is close to the note B3. The actual lowest note is a little lower, at A♭3, because of effects of the mouthpiece and bell.

When overblown the saxophone jump to any harmonic including even ones (and not just the odd ones as with the clarinet). The first, second, and third harmonics are used in the standard saxophone repertoire. The saxophone has a register hole to assist in hitting the second harmonic, operated by a lever commonly referred to as the "octave key," since it causes the instrument to play an octave higher.

Because it overblows an octave above the fundamental, the saxophone, like the flute, requires 11 tone holes, so it can play all notes in the lowest octave of the instrument, with the highest hole being half way up the instrument, then higher notes are
Figure 11.31: The double reed of an oboe. The double reed consists of two separate cane reeds bound to a short tube, which in turn connects to the pipe of the instrument. (a) At rest the reeds sit slightly apart, allowing air to pass between them when the instrument is blown. (b) As blowing pressure is increased, the reeds are pushed together, eventually cutting off the airflow.

achieved by overblowing. In fact the saxophone uses essentially the same arrangement and fingering of tone holes as the flute, and because of this it is common for saxophone players to also play flute and vice versa. Conversely, saxophone fingering is very different from that of the clarinet, with its 18 tone holes, making it considerably more challenging for players to switch between sax and clarinet, despite the superficial similarity of the two instruments.

The highest note in the standard saxophone range is achieved by playing the third harmonic and then opening all the tone holes. On soprano saxophone, for instance, the fundamental is at A with all tone holes closed, so the third harmonic is E (an octave and a fifth higher), and opening the tone holes gives an additional 11 half-steps for a highest note of D#. In fact, the instrument can go slightly higher to E by the use of additional holes, and skilled saxophonists can go higher still by making use of the fourth and higher harmonics, something that is common in pop and jazz saxophone playing.

11.3.4 Double reeds and the oboe

Instruments such as the clarinet and saxophone employ a single reed that vibrates against the edge of the mouthpiece to allow pulses of air into the instrument. An alternative is a double reed, as used for instance in the oboe and bassoon, in which two separate reeds move together and apart to open and close the airway and there
is no mouthpiece at all. Figure 11.31 shows a diagram of the setup.

The two reeds of a double reed are normally made of cane, the same material used for clarinet and saxophone reeds, and like a clarinet reed they are thin, flexible, and springy while still being quite sturdy. At rest they sit slightly apart with a small gap between them that allows air into the top of the instrument (Fig. 11.31a). The player takes both reeds in their mouth and blows, forcing a jet of air between the reeds. At the same time the reeds are pushed together by a combination of air pressure and the force of the player’s lips gripping the reeds. This creates a “negative conductance” similar to that seen in the clarinet: as blowing pressure increases the airflow decreases. Oscillation of the pressure at the top of the pipe can then drive the reeds to open and close, letting bursts of air into the instrument in time with the natural vibration frequency of the pipe. By careful design of the shape of the instrument, the timing of these bursts can be made to reinforce the oscillating air in the pipe and hence produce a note.

Despite the different mechanics of the double reed, the end result is closely similar to the single reed and there is little difference in sound between single and double reeds. The choice between the two is mostly a matter of convenience and tradition. In fact, it is possible (though rare) to play an oboe with a single-reed mouthpiece and it sounds essentially the same as with the more standard double reed.

On the other hand, an oboe does sound very different from a clarinet or saxophone, so what is responsible for the difference? An oboe is about the same length as a soprano saxophone—around 66 cm, versus 71 cm for the saxophone. And like the saxophone it has a conical pipe, with the reed end being effectively closed and the other end open, so it produces all harmonics and a similar lowest frequency of

\[ f = \frac{c}{2L} = \frac{343}{2 \times 0.66} = 260 \text{ Hz}, \]  

which is the note C4, or middle C. In fact the lowest note on the oboe is B♭3—like the saxophone the lowest note is a little lower than the length would suggest because of effects produced by the shape of the ends of the pipe.

The oboe differs from the saxophone in being made of wood rather than metal, although this has little effect on tone. It also has a significantly more complicated system of tone holes, pads, and levers. Since it plays all harmonics, the oboe overblows an octave above the fundamental, and hence one might imagine that, like the saxophone and the flute, it would use only an octave’s worth of tone holes, extending half way up the pipe. In fact, however, the

Figure 11.32: Frequency spectra of oboe and saxophone. The frequency spectra of an oboe and a soprano saxophone playing the same note A4. Note how the oboe has more strength in the higher harmonics, particularly the third and the fourth, compared to the saxophone, for which the first two harmonics are strongest.
tone holes extend further than this, about two thirds of the way up, and produce 14 half-steps instead of 11 as on the saxophone. Although this is not strictly necessary to produce a scale, it does give the instrument some additional fingering options that make for greater playability.

In normal practice an oboe player uses the first three harmonics (or registers) of the pipe to produce notes, so that the highest register is an octave and a fifth above the fundamental. An octave and a fifth above the lowest note of B♭3 gives us F5, and adding the 14 extra half-steps from the tone holes produces a highest note of G6, almost a three-octave range. The oboe makes use of register holes to aid in hitting the higher harmonics, with three different register holes to allow for the fact that the ideal placement of the register hole depends on what note you are playing—see the discussion in Section 11.2.8.

However, the most important difference between the saxophone and the oboe is in the shape of the pipe. While both have flared conical-shaped pipes, the oboe has a much narrower cone than the saxophone. The saxophone pipe widens at an angle of about 3° as one goes down the pipe, whereas the oboe widens at less than 1°. This has the effect of accentuating the higher harmonics of the oboe, in a manner similar to the way the narrow “string” pipes of a pipe organ accentuate high harmonics. This difference in width is the primary reason for the different tone of the oboe and saxophone. Figure 11.32 shows a comparison between the frequency spectra of an oboe and a soprano saxophone playing the same note A4, and the increased amplitude of the higher harmonics in the oboe is clear. The result is a bright, piercing tone on the oboe that is quite different from the sound of the saxophone. Because of its penetrating sound the oboe is frequently used as a solo instrument in the orchestra and is also traditionally the instrument to which other instruments tune before an orchestral performance: the oboe will play the note A4, loudly and clearly, so that all instruments can tune to the same pitch.

Two other instruments, the cor anglais or English horn and the bassoon, operate on similar principles to the oboe. The oboe, cor anglais, and bassoon can be thought of as the soprano, alto, and bass members of a single musical family—similar in design, operation, and timbre, but differing in size and pitch. Where the oboe spans the range B♭3 to G6, the larger cor anglais spans E3 to C6 and the bassoon spans B♭1 to about C5, although a skilled bassoon player can go considerably higher. There is no widely used tenor member of the oboe family, although there does exist a rare instrument, confusingly called the bass oboe, which has a range B♭2 to G5 that falls in between the cor anglais and bassoon.

11.3.5 Other reed instruments

We have discussed single-reed instruments with both straight pipes (clarinets) and conical pipes (saxophones) and double-reed instruments with conical pipes (oboes
and bassoons). What about the fourth combination, a double reed with a straight pipe? There is no modern example of such an instrument in the Western musical tradition, but during the Renaissance era there was an instrument called the crumhorn that used this arrangement. Only occasionally played today, the crumhorn is a J-shaped instrument that comes in a range of sizes and uses finger holes like a recorder to select different notes. It produces a rather nasal buzzing sound reminiscent of a kazoo and has only a limited pitch range of a little over an octave. In principle it is possible to overblow the crumhorn to reach higher notes—having a cylindrical pipe we would expect it to overblow to the third harmonic like a clarinet—but in practice this is difficult to do and players normally use only the fundamental mode, making the crumhorn a rather limited instrument.

Other early reed instruments include the chalumeau, a predecessor of the clarinet, and the shawm, a predecessor of the oboe. A little further afield are the bagpipes, an unusual instrument that consists of an airtight bag, filled with air from the player’s breath through a tube, then squeezed under the player’s arm to feed one or more pipes equipped with double reeds that produce the actual sound. In the best known version, the Scottish Highland bagpipes, there are four pipes. Three produce continuous drones on the notes B♭₂, B♭₃, and B♭₃ (so two are the same note), while the fourth, called the chanter, has finger holes that allow the player to play a nine-note scale in E♭ major. (The instrument thus plays in one key only.) An interesting technical detail of the Highland bagpipes is that the note A♭ is tuned to a frequency ratio of 7/4 above B♭—which is a multiple of the seventh harmonic—rather than to the more conventional minor seventh, which has a ratio of 9/5. As discussed in Section 4.3.4, the seventh harmonic is not normally used in Western music, since it is not close to any note of the 12-tone equal temperament scale. If one is not attached to the 12-tone scale, however, there is no reason why the seventh harmonic should not sound pleasing, though the bagpipes are one of the very few instances of its use. (It is also used in the Vietnamese string instrument called the dan bau and occasionally in barbershop-style close-harmony singing.)

Reed instruments also appear in a range of non-classical and non-Western musical traditions. Instruments include the Greek aulos and sipsi, Chinese guan and suona, the zurna of North Africa and Western Asia, the Welsh pibgorn, and the Armenian duduk. Both single and double reeds are used, typically with finger holes similar to those of a recorder. The principles described in this chapter can be readily applied to these instruments. The guan, for instance, is a double-reed instrument made from a cylindrical bamboo pipe with seven tone holes. This immediately tells us that the instrument will produce odd harmonics only, and hence a softer tone, and that when overblown it will sound an octave and a fifth above its fundamental like a clarinet. The suona, on the other hand, has a conical pipe and hence produces a brighter sound with all harmonics, somewhat similar to an oboe, and overblows an
Figure 11.33: A straight bugle. The straight bugle consists of a cylindrical pipe with a mouthpiece, joined to a flaring Bessel horn. The dashed line marks the join between the two parts. There are no tone holes or valves, so the bugle can only play open harmonics.

octave above the fundamental.

11.4 Brass instruments

Along with flutes and reeds, the third main class of wind instruments are the brass instruments, such as trumpet and trombone. Brass instruments are in some ways similar to reed instruments, in that they consist of a pipe that is effectively closed at the blowing end, in which vibrations are produced by periodic bursts of air from the player’s breath. Brass instruments are made of metal rather than wood like a clarinet, but this is mostly a matter of convenience of manufacture. As we have said, the material of a wind instrument does not have a great effect on its tone. There are however several more substantial ways in which brass instruments differ from reeds. First, where reed instruments use either a cylindrical or conical pipe to produce sound the brass instruments use a different shape, the “Bessel horn.” As we will shortly see, this allows them to produce all harmonics despite being closed pipe instruments.

A second difference between brass and reed instruments is in the means by which the air is set vibrating. In place of the reed of a clarinet, a trumpet or trombone uses the player’s lips to inject bursts of air into the instrument. This mechanism is sometimes called a lip reed, to draw attention to similarities with reed instruments, but there are also substantial differences between the workings of a clarinet reed and the player’s lips on a trumpet.

Third, where instruments like clarinet and saxophone produce different notes by opening tone holes and effectively making the pipe shorter, brass instruments take the opposite approach and make the pipe longer, either by stretching it as in the slide trombone or by splicing in extra lengths of pipe as in the trumpet.
11.4.1 Vibration modes of a brass instrument

The simplest of brass instruments, and a good one to start our discussion, is the straight bugle or fanfare horn (Fig. 11.33). The bugle consists of a simple metal horn with no keys or tone holes. Most modern bugles are coiled for more convenient handling—see the picture on page 440—but in earlier times bugles were straight, and straight bugles are still used occasionally for their theatrical flair. Having no tone holes or other means of changing the length of the pipe, bugles can only play the fundamental frequency of the pipe plus its overtones via overblowing. As a result, bugle music tends toward simple arpeggios and fanfares such as classic military bugle calls like the *Reveille* or *Taps*.

Figure 11.33 shows the basic form of the straight bugle, which has the classic flared “horn” shape of all brass instruments, particularly noticeable in this straight format. There is a cup-shaped mouthpiece into which the player blows, connected to a narrow pipe, cylindrical in shape, that then flares out into a broad bell. The bell is crucial to the working of the instrument, serving two purposes. One is to allow more sound out of the end of the pipe than would emerge if there were only a small opening with no bell. The other, as we now discuss, is to adjust the overtones of the instrument to form a series with both even and odd harmonics.

The main pipe of the bugle is cylindrical and, like a reed instrument, the blowing end of the pipe is effectively closed, sealed off by the player’s lips and mouth. A small amount of air enters when the player blows, but the flow is modest, and no air can leave. So the pipe is essentially closed at this end and open at the other and hence we would expect it to produce odd harmonics only—see Section 11.1.4.

However, it is, as we will see, central to the way brass instruments are played that they produce even harmonics as well. One way to achieve this would be to make the instrument conical, and in fact there are conical bugles that do exactly this. However, for mechanical reasons most brass instruments require a pipe that is cylindrical or widens only slightly along its length. As an example, the slide trombone, which we study in Section 11.4.4, employs a sliding section of tubing to lengthen or shorten the pipe and produce different notes. Such a slide will only work if the pipe is cylindrical—the pipe must be the same width all the way along for the slide to move freely while still maintaining an airtight seal. Similarly, the trumpet (Section 11.4.5) employs a system of valves to splice additional lengths of tubing into the middle of the pipe, but there is no way to do this and have the pipe be conical—if the spliced section were conical it would have different diameter at its two ends and the diameters would not match where it joins the rest of the pipe.

So we have a problem. How can we use a cylindrical pipe but still produce both odd and even harmonics? The solution is the bell. The bugle and other brass instruments consist of two parts: a cylindrical pipe with a strongly flared “Bessel horn” attached to its end. We encountered the Bessel horn previously in Section 11.1.8.
Technically it is defined as a horn with radius \( r \) that varies with the distance \( x \) from the opening according to the formula

\[
r = \frac{r_0}{(x/x_0 + 1)^{\alpha}},
\]

(11.94)

where \( r_0 \) is the width of the bell and \( x_0 \) and \( \alpha \) are parameters that control the shape and the rate of the flare. Figure 11.34 shows some examples of Bessel horn shapes with various parameter values.

The Bessel horn is attached to the end of the cylindrical part of the tube to make the complete bugle and, by a rather complicated calculation, it can be shown that the frequency \( f_n \) of the \( n \)th mode of the resulting compound instrument is approximately

\[
f_n \approx \frac{c}{4L} \left[ 2n - 1 + 0.7\sqrt{\alpha(\alpha + 1)} \right],
\]

(11.95)

where \( L \) is the length of the complete pipe and \( c \) is the speed of sound as usual. Normally this formula does not give a proper harmonic series, meaning that the frequencies are not integer multiplies of any fundamental. However, if we make the particular choice \( \alpha = 1 \) we get

\[
f_n = \frac{c}{4L} (2n - 1 + 0.9899),
\]

(11.96)

which is a correct harmonic series, and indeed is exactly the same as the series produced by an open pipe of the same length. In other words, we have found another way to produce a correct harmonic series from a closed pipe—in addition to using a conical pipe we can also use a cylindrical pipe with a bell.
There are a couple of catches. One is that Eq. (11.95) is only approximate. The actual frequencies differ very slightly from the formula, and hence notes will be slightly out of tune. This can be fixed to some extent by tiny adjustments to the shape of the instrument, something manufacturers have become very good at with centuries of practice, and a good bugle or trumpet player can correct any residual imperfections of tuning by small changes in the position of their lips, mouth, and tongue—the so-called embouchure.

A more significant problem is that Eq. (11.95) is only correct for \( n = 2 \) and above. For \( n = 1 \) it gives the wrong answer. The frequency of the first mode (with \( \alpha = 1 \)) turns out to be about 0.35\( c/L \) which is completely different from the \( c/2L \) that we would expect for the fundamental of our harmonic series. In practice, this means that the first mode is not usable for musical purposes: it produces a badly out-of-tune, unpleasant sounding note. When playing brass instruments, therefore, the lowest mode one uses in normal performance is the second mode. Players use a large number of modes to get the full range of the instrument. The standard repertoire of the trumpet and trombone goes up to the 8th harmonic, the tuba goes to the 12th, and the French horn can go as high as the 16th.

**Example 11.11: Notes on the bugle**

A standard straight bugle is about 1.5 meters long. What are the lowest five notes it produces?

We can answer this question by using Eq. (11.96). The lowest normal note on a bugle, or any brass instrument, corresponds to the second mode or harmonic, with \( n = 2 \), so it has frequency

\[
f_2 = 2 \times \frac{c}{2L} = \frac{343}{1.5} = 229 \text{ Hz},
\]

which is the note B♭3. If this is the second note of the harmonic series then the next four notes are the third, fourth, fifth, and sixth harmonics, which are F4, B♭4, D5, and F5:

The bugle plays no other notes between these notes: it only plays the harmonic series and nothing else. Thus, for instance, the bugle call *Taps* goes like this:
11.4.2 Sound production

As with other wind instruments, the sound vibration in a brass instrument is driven by a series of bursts of air from the player’s breath. By contrast with the jet drive of a flute or the reed of a clarinet, however, a brass player uses their own lips to produce the bursts. They purse their lips together and stretch them across the mouthpiece of the instrument, then blow to produce a sound. Though sometimes called a “lip reed,” this mechanism works in a manner distinct from a reed, differing in two important ways. First, where breath pressure acts on a reed to bend it inward and close off the air channel, with the lips air pressure forces them apart and opens up the channel. Try closing your own lips, then blowing air out of your mouth. Once the pressure inside your mouth builds up enough it will push your lips apart. Second, when at rest a reed is in the “open” position, held there by the springiness of the reed material and allowing air to flow past it. If you push a reed closed it will spring open again when you release it. A brass player’s lips work in the opposite direction, their tension holding them closed. They can be pushed apart by the flow of air, but when released they will close again.

In short, a reed is pushed closed by air pressure but pulled open by its own springiness, while lips are the other way around, pushed open by air pressure but pulled closed by their tension. As a result the mechanism of sound production in a brass instrument is quite different from a reed instrument. In particular, there is no longer a region of “negative conductance” where increasing breath pressure produces decreasing airflow. The lips move further apart with increasing pressure and the velocity of flow also increases, so greater pressure always means more airflow, never less. Instead, therefore, the mechanism for producing air bursts relies on a different process, which involves vibration of the lips in a circular motion as shown in Fig. 11.35.

In effect, the lips behave something like a pair of rubber bands, which part to let air through but also vibrate in their own right. Stretching the lips across the mouthpiece allows them to vibrate as a rubber band would if you plucked it, though rather than being plucked the motion is driven by the passage of air. When the player blows, the mounting air pressure in their mouth first pushes their lips forward (Fig. 11.35a) then apart (Fig. 11.35b), allowing air to escape. No longer holding in the air pressure, the parted lips fly backward (Fig. 11.35c), then close again (Fig. 11.35d), cutting off the air. The closing action is a part of the natural rubber-band vibration of the lips—push them apart and in time they will snap back together again.

Like the string of a string instrument, the lips have a natural frequency of vibration that depends on their length, mass, and tension. In order for a brass instrument to work, the player has to tune this frequency quite closely to the desired note, usually within a few half-steps, which they do primarily by adjusting lip tension. If they can get the frequency reasonably close to its target then the rest of the work is done
Figure 11.35: Circular motion of the lips when playing a brass instrument. (a) Air pressure inside the mouth pushes the closed lips forward and eventually (b) apart, allowing a burst of air to escape. (c) The lips relax backward and then (d) close once more, and the cycle starts again.

for them by the vibration of the air in the instrument pipe. As we have seen, the blowing end of a brass instrument is effectively closed and hence a pressure antinode—the pressure variation is at a maximum. This pressure variation pushes on the lips in time with the frequency of the note being played and can cause the lips to sync up and vibrate at the same frequency. The result is a series of bursts of air through the lips that are in time with the vibration in the pipe and hence can drive that vibration.

The interplay between the vibration in the pipe and the vibration of the lips is central to the way a brass instrument is played. It allows the instrument pipe to control the pitch of the note while at the same time allowing the player to select between various modes of the pipe by changing their lip tension and hence hit a wide range of different harmonics—wider than is possible on woodwind instruments. So long as the player can tune their lip vibration within a few half-steps of the desired note, the pipe will do the rest. Thus the process of hitting harmonics is different on a brass instrument from a flute or a clarinet. It is not really correct to call it “overblowing” (and that word is not normally used of brass instruments), since it doesn’t necessarily involve blowing harder (although it may). It is about tuning the vibration of the lips to match the desired note.

Though the note produced is primarily determined by the natural frequency of vibration of the instrument pipe, there is a little room for variation. If the player aims the frequency of their lip vibration lower than the intended note, which they would
do by using a looser lip, then it will pull the pitch of the note down slightly. This is known in the business as “lipping down.” Similarly, if the player aims high they can “lip up” and pull the pitch of the note up slightly. As we will see shortly, for the skilled brass player who has mastered the art of making such fine pitch adjustments, these are invaluable techniques. Because of intrinsic limitations in the way they work, brass instruments can have difficulty hitting note pitches accurately, and the ability to adjust the tuning by lipping up and down is crucial if one wants to play these instruments at a high level.

11.4.3 Playing different notes

One can play different notes on a brass instrument by hitting different harmonics. As we have seen, a brass instrument starts at the second harmonic and can go up to the 8th or higher, so on a B♭ trumpet for instance one gets these notes:

But what do we do if we want notes other than these “open harmonics”? To get notes in between, instruments like the trumpet and trombone use an approach different to that of other wind instruments. Where instruments like the flute and the clarinet use tone holes to shorten the effective length of the pipe, brass instruments lengthen the pipe by adding extra sections to it, which means you get lower notes and not higher ones as you do with tone holes. Since there are no holes in the instrument other than the bell, it also means that almost all of the sound from a brass instrument emerges from the bell. This is also different from other wind instruments, where a significant portion of the sound comes from the tone holes. Among other things, this means that brass instruments are highly directional: they project a great deal of sound in the direction where the bell is pointing, but relatively little in other directions. Anyone who has ever stood or sat in front of a trumpet when it was being played can attest to this.

There are two approaches used for lengthening the pipe of a brass instrument. Most brass instruments, including the trumpet, switch sections of tubing in and out of the pipe using a system of valves. The exception is the trombone, which uses a slide. We will look at both approaches, starting with the trombone, since it is simpler.

11.4.4 The slide trombone

Figure 11.36 shows a drawing of a slide trombone. Trombones come in a range of different sizes, but the most common by far is the tenor trombone pictured here. The
The slide trombone is a relatively simple instrument, not much more complex than the bugle of Section 11.4.1, with a cup-shaped mouthpiece, a cylindrical pipe, and a flaring bell. The pipe is bent into an S-shape, which makes it easier to handle but doesn’t significantly impact the sound. The crucial element that makes the trombone different from the bugle is the slide, a U-shaped section of metal tube that the player can move back and forth to lengthen or shorten the pipe. The arms of the slide sit over two straight sections of pipe of slightly smaller caliber, and are lubricated with a light oil to help them move freely.

When pulled as far as it will go towards the player, the slide is said to be in first position. With the slide in first position, the pipe of a tenor trombone has length 2.69 meters, so that the second harmonic—the lowest one used in normal performance—produces a frequency of

\[ f = 2 \times \frac{c}{2L} = \frac{c}{L} = \frac{343}{2.69} = 127.5 \text{ Hz}, \]  

which is the note B2. In fact, the actual note is B♭2—like many of the wind instruments we have studied, the calculation is slightly off because of effects produced by the shape of the pipe ends.

Now, while still playing the same harmonic, the player can produce other notes by moving the slide. Moving the slide away from the mouthpiece lengthens the pipe and hence lowers the note it produces. We can calculate how much the slide needs to move for a particular note from Eq. (11.98). Suppose that we have two different notes with frequencies \( f_1 \) and \( f_2 \) and corresponding pipe lengths \( L_1 \) and \( L_2 \), so that

\[ f_1 = \frac{c}{L_1}, \quad f_2 = \frac{c}{L_2}. \]  

The slide trombone is often just called a “trombone,” but the full name is useful to distinguish between this instrument and the rarer valve trombone, which has the same pitch and general shape as a slide trombone but uses trumpet-style valves instead of a slide.
Dividing one equation by the other and rearranging, we then have

\[ L_2 = L_1 \frac{f_1}{f_2} \]  

(11.100)

For instance, say we want to play the note A2, a half-step lower than the first-position B♭2. Then \( f_2 = f_1 / 2^{1/12} \) and \( L_1 = 2.69 \) m, so

\[ L_2 = 2.69 \times 2^{1/12} = 2.85 \text{ meters.} \]  

(11.101)

In other words, to go down the half-step we just multiply the length of pipe by \( 2^{1/12} \). The new length of 2.85 meters is 16 cm longer than the 2.69 meters we started with, so to play A2 we want to make the pipe 16 cm longer, which means we need to move the slide a distance of 8 cm. It is 8 cm not 16 cm because the air has to travel up one arm of the U-shaped slide and back down the other, so it will travel 8 cm further on the way out and 8 cm further on the way back for a total of 16 cm. This position of the slide is called second position.

Now we can repeat the calculation again. To go down by another half-step to Ab2 we again need to multiply the length by \( 2^{1/12} \), which gives us \( 2.85 \times 2^{1/12} = 3.02 \) meters, which is 17 cm longer than 2.85 meters. So we need to move the slide out by an extra 8.5 cm to reach this note, slightly further than before. This gives us third position.

And we can continue down the same path, to fourth position, fifth position, and so forth. Each position gives us a note a half-step lower, and each time we need to move the slide a little further than before so that the positions get increasingly far apart, as shown in Fig. 11.36.

In normal use, there are seven positions of the trombone slide, giving us seven notes a half-step apart. Starting from B♭2 in first position on the second harmonic, for instance, the 2nd through 7th positions give us A2, Ab2, G2, F♯2, F2, and E2. E2 is the lowest note on the tenor trombone, since there are no more slide positions beyond the seventh and no lower harmonics because brass instruments do not play the first harmonic.

One can get higher notes, however. By jumping up to the third harmonic we get the note F3 in first position, then the other six slide positions again give us half-steps below that: E3, Eb3, D3, C♯3, C3, and B3. And then we can move to the fourth harmonic, and so on. Figure 11.37 shows the notes produced in each slide position for harmonics 2 to 6 on the tenor trombone.

One interesting detail to notice is that there is more than one way to play some notes. For instance, the note E3 can be played on the third harmonic with the slide in second position or on the fourth harmonic in seventh position. This is a useful feature, since some positions may be easier to reach than others in the context of a particular musical passage.
Figure 11.37: Notes on the tenor trombone. Each row of this chart shows the notes played by the tenor trombone for one harmonic and in each of the seven positions. Note that, by tradition, music for the trombone is written an octave higher than it actually sounds and this chart follows that practice. Thus, for instance, the first-position note for the second harmonic is written as B♭₃ but is actually B♭₂.

The trombone slide is in many ways an elegant and effective solution to the problem of changing the pipe length on a brass instrument. It is simple in construction and operation and allows the instrument to play any note of the 12-tone scale over a wide range. It also allows the trombone to perform glissandi—sliding between notes instead of jumping from one to another—something that finds occasional use as a special effect. Moreover, the trombone slide is the only mechanism that allows for perfect tuning of notes on brass instruments, the valve mechanisms of the other instruments having significant tuning problems as we will shortly see.

However, the slide trombone also has some substantial disadvantages. For one thing, it is difficult to play, requiring very accurate positioning of the slide in order to play in tune. Recall that the smallest pitch interval perceptible to the human ear
is about five musical cents, a frequency ratio of $2^{5/1200}$ (see Section 2.4.1). Repeating the calculation of Eq. (11.101) for this frequency ratio, we find that a pitch change of five cents from first position corresponds to increasing the length of the pipe from 2.69 meters to $2.69 \times 2^{5/1200} = 2.6978$ meters, meaning the length would change by just 7.8 mm and the slide would move a half of that distance or 3.9 mm. So perfect tuning, to the limit of human hearing, would require the trombonist to hit the correct slide position to within about 4 mm. They also need to be able to do this quickly. The fastest musical passages can have eight or ten notes per second, and it is a substantial challenge to move a trombone slide that fast. A standard tenor trombone slide weighs about 200 grams (slightly less than half a pound) and, unlike the modest movements required to play piano, violin, or clarinet, rapid movement of such a large weight requires significant strength.

However, perhaps the biggest disadvantage of the trombone is the difficulty of playing legato, meaning the creation of smooth, uninterrupted transitions between consecutive notes. One cannot jump the slide of a trombone from one position to another instantaneously. One must move between positions usually over the course of a fraction of a second—longer if one is moving between positions far apart—and this generates a glissando or slide between one note and the next, an effect that can occasionally be musically useful but more often is distracting and undesirable. One way to avoid glissandi is for the player to cut off their breath and stop the note while the slide is moving, then start again once they reach the new position, but this has its own problems, giving a broken or staccato sound that is also considered undesirable.

Trombone players practice long and hard to overcome these issues. Smooth legato transitions between notes are one of the abiding preoccupations of the dedicated trombonist, requiring split-second breath control and rapid but accurate movement between slide positions. But in the end there are limitations to what even the best player can achieve because of the very nature of the instrument, and it is primarily for this reason that most brass instruments take a different approach to changing pipe length, making use of valves.

11.4.5 Valves and the trumpet

The trumpet is in some ways like a smaller version of the trombone. It has the same cylindrical-pipe-plus-bell shape that produces a good approximation to a full harmonic series (starting at the second harmonic), but with an overall pipe length that is half that of the trombone, so it plays an octave higher. Thus the lowest (second) harmonic is B♭3, and succeeding harmonics play F4, B♭4, D5, F5, and so forth. Like the trombone, the trumpet plays a full scale by a combination of these harmonics and lengthening the pipe to make notes lower. The important difference, however, is that instead of a slide, the trumpet lengthens the pipe by splicing in extra sections of tubing using a system of valves.
Figure 11.38: The B♭ trumpet. The standard trumpet has three valves, numbered as shown, each of which operates a U-shaped crook. The second valve has the shortest crook, then the first valve, then the third. The third valve is equipped with a slide for precise tuning.

Figure 11.38 shows a diagram of a standard three-valve B♭ trumpet, the most common kind of trumpet, as used in orchestras, jazz, and marching bands. The pipe starts at the mouthpiece, at the top left in the figure, passes along the top of the instrument, then wraps around to the bottom, where it passes through three valves—the vertical cylinders in the middle—before curving back up again and continuing to the bell. Each valve has a button or key at the top which is normally held up by a spring, but can be pressed down by the player. When the key is in the up position the air in the pipe passes straight through the valve, but when the key is pressed down the air is rerouted through a short U-shaped section of tubing called a crook, before rejoining the main pipe again. The result is that the pipe gets a little longer when a key is pressed, which lowers the note the instrument plays.

Each of the three valves has a crook of different length. Traditionally the three valves are numbered 1, 2, and 3 as shown in Fig. 11.38 and you might imagine that the lengths of the crooks would go from shortest to longest in the same order, but they do not. In fact valve 2 has the shortest crook, followed by valve 1, then valve 3. This slightly illogical arrangement is primarily for practical convenience. The crooks of valves 1 and 3 can be tucked inside the curve of the instrument, parallel to the main pipe, but the crook for valve 2 has nowhere to go and ends up sticking out of the side of the valve casing as shown in Fig. 11.38. Because of this, it makes sense for this to be the shortest crook—a longer one would stick out further and make the instrument unwieldy.

When the second valve is depressed and the shortest crook is added to the pipe, it lowers the pitch of the instrument by one half-step. Thus if one were playing the second harmonic, depressing the second valve would lower the note from B♭3 to A3.
Figure 11.39: Notes on the B♭ trumpet. Each row of this chart shows the notes played by the trumpet for one harmonic and in each of the seven valve combination positions. Filled circles denote depressed valves and the circles are in numerical order of the valves. Note that trumpet music is normally written two half-steps higher than it actually sounds, but the notes shown here are the actual pitches (so-called concert pitch).

Similarly the first valve crook lowers the note by two half-steps, which gives A♭3. And by pressing both valves 1 and 2 at the same time we add both of the first two crooks to the pipe which lowers the note by three half-steps, giving G3. Meanwhile, the third valve crook lowers the pitch by three-half steps, giving us a second way to play G3 (although this one is not normally used), and in combination with the other two valves gives us four half-steps (valves 2 and 3 together), five half-steps (valves 1 and 3), or six half-steps (all three valves at the same time). Combining these valve positions with the harmonics then gives you a complete scale of notes, as shown in Fig. 11.39. As with the trombone some notes can be played in more than one way, which can be convenient for playing certain fast or tricky passages.
11.4.6 Tuning on the trumpet

Let us calculate how long the crooks need to be in order to produce the right notes. The length of the main pipe of a trumpet is about 1.48 meters, giving a frequency for the second harmonic of

\[
    f = 2 \times \frac{c}{2L} = \frac{343}{1.48} = 232 \text{ Hz},
\]

which is the note \(B_3\) as we have said. If we want to play the note one half-step lower than this, which is \(A_3\), the length of pipe we would need is given by Eq. (11.100):

\[
    L_2 = L_1 \frac{f_1}{f_2} = 1.48 \times 2^{1/12} = 1.568 \text{ meters}.
\]

So the second valve crook, which lowers pitch by one half-step, must add a total of 1.568 – 1.480 = 0.088 meters or 8.8 cm to the length of the pipe.

Similarly, for the first valve, which lowers the pitch by two half-steps, the required length of pipe would be

\[
    1.48 \times 2^{2/12} = 1.661 \text{ meters},
\]

so the first valve crook has to add 18.1 cm to the pipe. And for the third valve we want three half-steps, which gives a length of

\[
    1.48 \times 2^{3/12} = 1.760 \text{ meters},
\]

so the third valve crook has to add 28.0 cm to the pipe.

These calculations, however, hide a serious problem with this system that arises when we start combining valves. Consider, for example, what happens when we press the first and second valves at the same time, which is supposed to lower the pitch by three half-steps. If we do this we add 8.8 + 18.1 = 26.9 cm to the length of the pipe. But this is the wrong answer. As we saw in Eq. (11.105), we actually need to add 28.0 cm to go down three half-steps in pitch, so we are more than a centimeter short of our goal. In practice, this means that the note produced by pressing the first two valves will be slightly sharp.

How much sharp? The pipe will end up being only 1.480 + 0.269 = 1.749 m long instead of the required 1.760 m. Rearranging Eq. (11.100), the ratio of the frequency of the note we play and the true note we are aiming for is \(f_1/f_2 = L_2/L_1\) and, using Eq. (2.35) from page 36, we get the number \(n\) of musical cents difference thus:

\[
    n = 1200 \log_2 \frac{f_1}{f_2} = 1200 \log_2 \frac{L_2}{L_1} = 1200 \log_2 \frac{1.760}{1.749} = 10.6 \text{ cents}.
\]

This is a modest pitch difference, but large enough that the note will sound audibly out of tune to an attentive listener. Recall that the smallest pitch difference detectable
by human ears is about five musical cents (see Section 2.4.1), so ten cents is well above the threshold of audibility.

The root cause of this problem is that when we depress valves 1 and 2 at the same time the lengths of their crooks get added together. But this is not how musical pitch works. Changes in pitch involve multiplying the frequency by an appropriate number, which, as Eqs. (11.103) to (11.105) show, also translates to multiplying the length. So adding together the lengths for one half-step and two half-steps is the wrong operation and was never going to give the correct length for three half-steps.

And the problem gets worse for other valve combinations. For instance, when we combine valves 2 and 3 to get four half-steps we end out of tune by 15.5 cents and when we combine all three valves to get six half-steps the result is off by a massive 53.6 cents, large enough to be very apparent to any listener.

This is a fundamental problem with the way the trumpet and similar instruments operate. The approach of combining crooks to get a desired note is at odds with the way musical frequencies work and is always going to produce notes that are out of tune. There are, however, some things we can do to improve the situation, the main one being to adjust the length of the crooks to improve the tuning of the worst notes, at the expense of some of the others. This approach is illustrated in Fig. 11.40.

The vertical bars in Fig. 11.40a show the decrease in pitch that derives from depressing the valves in each of the six conventional combinations. The dashed lines show the ideal pitches of the notes, labeled as if one were playing the second harmonic, and it is clear that as the notes get lower we consistently produce pitches that are too high, to the point where the last note, the one produced by depressing all three valves at once, ends up actually being closer to F than the intended E. Only the A♭ and A produced by valves 1 and 2 alone are precisely in tune, which is by design, since the lengths of the crooks were chosen to create pitch drops of exactly one and two half-steps.

The trick to improving the tuning is to sacrifice some of the accuracy of these two intervals to improve the tuning of the others. Figure 11.40b shows one version of this approach. The second valve crook is still the same length as before at 8.8 cm so that the A is in tune, but the first valve crook is now slightly longer. This makes the A♭ a little flat, but also pushes down the pitch of the G so that it is less sharp than it was before. A natural choice is to make the A♭ and the G equally out of tune (although in opposite directions), which results in them each being 5.5 cents off. Given that the most sensitive ear can just distinguish a pitch difference of five musical cents, 5.5 cents is right at the limit of audibility and probably not noticeable to most listeners. Moreover, a skilled trumpet player can correct the remaining pitch difference by “lipping” up or down, as described in Section 11.4.2—adjusting the tension of their lips to push the pitch a little one way or the other and produce a note that is truly in tune.

Take a look at Section 11.4.7 if you are interested in the details of these pitch calculations.
Figure 11.40: Tuning on the trumpet. The vertical bars represent the drop in pitch introduced by each of the three crooks for each of the fingerings of the valves, starting from the B♭ of the second harmonic. Fingerings are denoted by the small circles at the top, with the circles arranged in numerical order of the valves: 1 2 3. (a) If the three crooks are chosen so that when used alone they lower the pitch by exactly 1, 2, and 3 half-steps, then combinations of crooks become increasingly out of tune as we go to lower notes. (b) By making the first and third valve crooks a little longer the tuning is improved, though still not perfect.

We also make the third valve crook longer than before, which pushes both the F and the F♯ closer to their ideal pitches, and again we split the difference, making them both the same amount out of tune, which turns out to be 5.6 cents, also near the limit of audibility, and small enough to be fixed by lipping up or down.

The result, summarized in Table 11.1, is that, while the tuning is still not perfect, we can get all the notes from B♭ down to F close enough that most ears will hear nothing wrong and any remaining imperfections can be fixed by the player. This does not completely solve the problem because the last note, the E, is still badly off. Even with longer crooks on the first and third valves, the E is still sharp by 29.9 cents—not as bad as the 53.6 cents we had before, but nonetheless audibly out of tune to anyone paying attention and too far off to be corrected by lipping down.

The solution to this problem is the third valve slide, a small U-shaped movable section of piping, like a miniature version of a trombone slide, that fits over the end of the third valve crook and can be moved back and forth by the player using the ring finger of their left hand though the small hoop shown in Fig. 11.38. When playing
Table 11.1: Pitch errors on the trumpet. The number of musical cents by which each valve combination on the trumpet is out of tune, for the original crook lengths of Fig. 11.40a and the improved ones of Fig. 11.40b. Positive numbers indicate notes that are sharp; negative ones indicate notes that are flat.

<table>
<thead>
<tr>
<th>Fingering</th>
<th>Half-steps</th>
<th>Pitch error (cents)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Uncorrected</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>10.6</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>15.5</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>30.3</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>53.6</td>
</tr>
</tbody>
</table>

a note such as E3 that requires all three valves to be depressed, the player extends the slide, making the total length of the third valve crook longer. This pushes the note back in tune, at the expense of making the instrument a little more complicated to play. In practice, a good trumpet player will use the slide on every applicable note, except in rapidly moving passages where the note is very short and its poor intonation will not be noticed.

Looking at Fig. 11.39, however, we notice that every note that requires all three valves—the notes in the last column of the chart—can be played in an alternate way, except for E3 and B3. For instance the E4 on the fourth harmonic can also be played using the second valve on the third harmonic, and the Ab4 on the fifth harmonic can be played using the first valve on the fourth harmonic. By careful use of alternate fingerings, therefore, a trumpeter can avoid having to use all three valves and the third valve slide except on two specific notes, E3 and B3.

**Example 11.12: Lengths of the first and third valve crooks**

At the start of Section 11.4.6 we calculated the lengths of the crooks on the trumpet but, as described above, the first and third valve crooks on a real trumpet are actually made slightly longer in order to improve tuning. Based on the information given in Section 11.4.6 what are the lengths of the first and third valve crooks on a standard B♭ trumpet?

When we press the first valve, the trumpet plays a note two half-steps lower, or 200 musical cents. But we are told in Section 11.4.6 that because of adjustments to the length of the crook this note comes out 5.5 cents flat, meaning it is actually 205.5 cents lower. Given that the starting pipe length on the B♭ trumpet is 1.48 meters, this means that the total length of pipe with the first valve depressed must be

\[
1.48 \times 2^{\frac{205.5}{1200}} = 1.667 \text{ meters.} \quad (11.107)
\]

So the crook itself must be \(1.667 - 1.48 = 0.187\) meters long or 18.7 cm, which is 6 mm longer than the 18.1 cm we calculated in Eq. (11.104).
Turning to the third valve, when the first and third valves are depressed at the same time the trumpet plays a note that is ostensibly five half-steps lower, or 500 cents, but we are told that in fact the note comes out 5.6 cents sharp, meaning that in reality it is only 494.4 cents lower. Thus the total length of pipe with first and third valves depressed must be

\[ 1.48 \times 2^{494.4/1200} = 1.969 \text{ meters,} \tag{11.108} \]

so the two crooks together have length 1.969 - 1.48 = 0.489 meters. Given that the first valve crook has length 0.187 meters as above, this means the third valve crook must have length 0.489 - 0.187 = 0.302 meters, or 30.2 cm, which is significantly longer than the 28 cm we calculated in Eq. (11.105).

### Advanced Material

#### 11.4.7 Crook lengths on the trumpet

As described in the previous section, the crook lengths on the trumpet and other valve instruments are a compromise designed so that the tuning, while not perfect, is good enough. Here we show how to calculate the ideal lengths of the crooks.

Suppose the length of the instrument pipe with no valves depressed is \( L \), and let the extra length added by the crooks of valves 1, 2, and 3 be \( x_1L \), \( x_2L \), and \( x_3L \) respectively, where \( x_1 \), \( x_2 \), and \( x_3 \) are positive numbers. Thus for instance the total length with valve 2 depressed will be \((1 + x_2)L\), with both valves 1 and 2 it will be \((1 + x_1 + x_2)L\), and so forth.

Starting with valve 2, we choose \( x_2 \) so that depressing the valve lowers the pitch by exactly one half-step, which means that \( 1 + x_2 = 2^{1/12} \), or

\[ x_2 = 2^{1/12} - 1 = 0.05946. \tag{11.109} \]

For the first valve we would like to choose \( x_1 \) so that it produces a note two half-steps down, meaning \( 1 + x_1 = 2^{5/12} \). But we would also like the next note down the scale, which we get by pressing valves 1 and 2 at the same time, to be three half-steps down, meaning that \( 1 + x_1 + x_2 = 2^{3/12} \). Unfortunately, there is no one value of \( x_1 \) that will achieve both of these goals, no way for the instrument to be perfectly in tune on both notes. Instead therefore, we compromise. We choose \( x_1 \) to overshoot the two half-step mark slightly, making the tuning of that note a little flat. The number of cents it is flat is given by

\[ n_1 = \frac{1200}{\log 2} \log \frac{1 + x_1}{2^{5/12}}. \tag{11.110} \]

At the same time the next note is sharp by a number of cents

\[ n_{1+2} = \frac{1200}{\log 2} \log \frac{2^{5/12}}{1 + x_1 + x_2}. \tag{11.111} \]

The optimal choice of \( x_1 \) is the one that makes both of these notes out of tune by the same amount (albeit in opposite directions). Setting \( n_1 = n_{1+2} \) and taking exponentials of both sides, we find that

\[ \frac{1 + x_1}{2^{5/12}} = \frac{2^{3/12}}{1 + x_1 + x_2}. \tag{11.112} \]

This can be rearranged into the quadratic equation

\[ x_1^2 + (2 + x_2)x_1 + 1 + x_2 - 2^{5/12} = 0, \tag{11.113} \]

which has one positive solution

\[ x_1 = \frac{-2 + x_2 + 2^{5/12} - x_2}{2} = 0.1260, \tag{11.114} \]

where we have used the value for \( x_2 \) from Eq. (11.109) above. Substituting this result back into Eqs. (11.110) and (11.111) it is straightforward to confirm that the two notes are indeed out of tune by the same amount

\[ \frac{1200}{\log 2} \log \frac{1 + x_1}{2^{5/12}} = 5.45 \text{ musical cents}. \tag{11.115} \]

This fixes the lengths of the first two crooks. The third is chosen in a similar manner. We would like \( 1 + x_2 + x_3 = 2^{4/12} \) and \( 1 + x_1 + x_3 = 2^{5/12} \) but there is no single value that will achieve both. So instead we overshoot a little on
the first, undershoot on the second, and split the difference so that the two notes are equally out of tune, which gives
\[
\frac{1 + x_2 + x_3}{2^{9/12}} = \frac{2^{5/12}}{1 + x_1 + x_3}.
\] (11.116)
This can be rearranged to give a quadratic equation for \(x_3\):
\[
x_3^2 + (2 + x_1 + x_2)x_3 + (1 + x_1)(1 + x_2) - 2^{9/12} = 0,
\] (11.117)
whose sole positive solution is
\[
x_3 = \sqrt{(x_1 - x_2)^2 + 2^{33/12} - x_1 - x_2} - 1 = 0.2045.
\] (11.118)

Again it is straightforward to confirm that this makes both notes out of tune by the same amount
\[
\frac{1200}{\log 2} \log \frac{1 + x_2 + x_3}{2^{24/12}} = 5.59 \text{ musical cents.} \tag{11.119}
\]

With the limit of human pitch sensitivity falling around five musical cents, all of these notes are close enough to being in tune that most listeners will not notice the difference. The catch, however, is that the last note in the scale, the note given by depressing all three valves together, is badly out of tune. For this note to be in tune we would need \(1 + x_1 + x_2 + x_3 = 2^{6/12}\) but in practice it is sharp by
\[
\frac{1200}{\log 2} \log \frac{2^{6/12}}{1 + x_1 + x_2 + x_3} = 29.9 \text{ musical cents,} \tag{11.120}
\]
which will be clearly audible to any listener. This note therefore requires the use of the third valve slide to play in tune.

One might ask whether the scheme described here is the best one can do. Perhaps there is some other variant of the same idea that gets closer to the correct pitches. In fact one cannot do much better, though there is one alternative that offers a marginal improvement. If one is willing to sacrifice the perfect tuning of the second valve crook, Eq. (11.109), then one can make the imperfections of the next four notes uniform, which is arguably better. Specifically, adjusting the value of \(x_2\) slightly to 0.05938 makes the second valve note (the note one half-step down) out of tune by 0.13 cents, and the next four are then all equally out of tune by 5.51 cents. This is a small improvement from the previous value of 5.59 for the third valve notes but slightly worse than the previous 5.45 for the second valve notes. Still, one could argue that there is merit to making all the notes equally out of tune (except for the final note with all three valves depressed, which is unchanged at 29.9 cents sharp).

These, however, are very small changes, smaller than would be audible to any ear. In practice, therefore, the choice defined by Eqs. (11.109), (11.114), and (11.118) is as good as it gets.

### 11.4.8 Other brass instruments

We have discussed the trumpet, trombone, and bugle in some detail, but there are a great number of other brass instruments as well, including the flugelhorn, cornet, French horn, tenor horn, baritone horn, euphonium, tuba, and sousaphone. All operate on essentially the same principles, which is what makes them brass instruments: a flared horn shape with a wide bell that produces a close approximation to the harmonic series, use of the player’s lips to inject bursts of air into the pipe and produce sound, and (except in the case of the bugle) some sort of mechanism for lengthening the pipe to change notes. The slide trombone uses a slide to lengthen the pipe; all other brass instruments use a system of valves and crooks similar to the trumpet.

There are several ways in which the various brass instruments differ from one another. Chief among them is size, and particularly the total length of pipe, which affects overall pitch. Brass instruments range from the tiny piccolo trumpet, which
11.4 Brass instruments

Brass instruments also vary in the precise shape of the pipe. As we have seen, brass instruments consist of a cylindrical pipe attached to a flaring bell, but they vary in where the flare starts and how rapidly it widens. The trumpet and trombone have a relatively long cylindrical section followed by a short, rapidly flaring bell. Other instruments such as the flugelhorn, French horn, and tuba have a flare that sets in sooner but widens more slowly into the bell. Instruments of the latter style are sometimes described as having “conical bore,” but these are not true conical pipes in the sense of Section 11.1.8 that produce a complete harmonic series. They still produce only an approximation to the harmonic series starting from the second harmonic. The wider pipe does however give rise to a mellower tone with fewer high harmonics, similar to the way wide flute-style organ pipes are mellower than narrow string-style pipes (see Section 11.2.4). The flugelhorn, for example, is essentially a trumpet with a “conical bore” and hence produces a softer, warmer version of a trumpet sound.

Valve instruments also vary in the number and function of their valves. The trumpet-style three-valve system is the most common, but some instruments have four or more valves, usually organized as some extension of the trumpet system. The tuba, for instance, typically has five valves (though there are some variants with more), including three trumpet-style valves, one which adds an extra crook that improves the tuning of certain notes, and one which allows the instrument to play the fundamental of the harmonic series (something that is not possible on other brass instruments). The French horn has three main valves plus an additional valve operated with the thumb that shifts the entire instrument up a fourth to allow it to play high parts with greater ease.

In addition to the traditional brass instruments, there are a few other instruments that, while not strictly brass in the conventional sense, can arguably be included in the category since they work in essentially the same way. Two examples are the alphorn and the didgeridoo. The alphorn comes from the alpine region of Switzerland and Austria and consists of a straight horn about 2.5 meters in length carved from one or more pieces of wood. Despite being made of wood, it is played in the same way as a brass instrument and produces a similar sound. It has no mechanism for lengthening (or shortening) the pipe and hence produces only notes of the harmonic series, starting from the 2nd and going as high as the 16th harmonic, three octaves lower. Instruments also vary in their overall shape and playing style, although this is mostly a matter of convenience and does not have a great effect on sound. The trumpet and trombone, for instance, are held straight out in front of the player, while the tuba and euphonium are coiled up and played with the instrument on the seated player’s lap. The sousaphone is unusual among the brass instruments in being hung around the player’s neck.

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higher.

The didgeridoo is an Aboriginal Australian instrument traditionally made from the trunk or branch of a eucalyptus tree that has been hollowed out by termites. Typically about 1.5 meters long, the didgeridoo is, like the alphorn, played in the manner of a brass instrument and also lacks any means for changing the pipe length. In normal use it plays only the fundamental of the pipe, and hence cannot play a melody, functioning instead as a combination of drone and rhythm instrument. Characteristic punctuations in the sound are introduced by varying the position of the tongue and by performing vocalizations while playing. The instrument is unusual in being played continuously by using “circular breathing,” a technique that involves drawing breath in through the nose while simultaneously expelling it through the mouth in order to play without interruption for long periods of time—sometimes minutes on end.

11.5 HARD REED INSTRUMENTS

Finally in this chapter let us touch briefly on one further, relatively rare category of wind instruments, the **hard reed** instruments, which include the harmonica and accordion. As mentioned in Section 11.3, a “hard reed” means one with only a very narrow range of vibration frequency, unlike the broad range of frequencies over which a “soft” clarinet or oboe reed will vibrate. Hard reeds need not actually be hard in the mechanical sense, although they often are. Harmonica and accordion reeds are made of metal, for instance.

Because a hard reed vibrates at essentially only one frequency, the pitch of the note it produces is dictated by the reed itself and not by the instrument pipe. Indeed most hard-reed instruments do not have a pipe at all. On the other hand, they require a separate reed for each note they play, tuned to the frequency of that note. A standard blues harmonica, for instance, has 20 different reeds (although it plays only 19 different notes because two of the reeds are tuned to the same note). The 20 reeds are divided into ten pairs, with one reed in each pair arranged so that it sounds when the player blows air out of their mouth and other when they draw air in. This arrangement allows the player to blow or draw on adjacent sets and produce two- or three-note chords, in addition to single notes.

On the accordion air is provided by a hand-operated bellows and the instrument is played using a piano-like keyboard and/or rows of buttons that play either individual notes or complete chords. A typical accordion can have a range of over three octaves, plus additional bass notes, for a total of more than 40 reeds.

However, the most important hard reed instrument by far, and also the most unusual, is the human voice. The vocal chords, though quite soft in texture, function musically as a hard reed, producing a frequency that is independent of the resonating
pipe formed by the airway. The human voice is sufficiently important and unique to merit an entire chapter on its own. We study it next.

Chapter summary:

- Most wind instruments make use of **vibrating air in a pipe** to produce sound. Air in a cylindrical pipe will vibrate when set in motion. In a pipe of length $L$, **open at both ends**, the air vibrates with fundamental frequency $c/2L$, where $c$ is the speed of sound. An example of an instrument with an open-ended cylindrical pipe is the flute.

- Like a stretched string, air in a pipe also has higher modes of vibration. In a pipe open at both ends the air can vibrate at any whole-number multiple of the fundamental frequency, i.e., any **harmonic**, and the sounds of wind instruments are made up of a mixture of vibrations at different harmonics, the timbre of the instrument being a result of the particular mixture produced.

- Cylindrical pipes that are **closed at one end** have a fundamental frequency $c/4L$, half that of an open pipe, and other modes whose frequencies are odd multiples of the fundamental—three, five, seven times the fundamental, and so on. Hence such pipes produce odd harmonics only, which gives them a softer, warmer sound than open-ended pipes. The clarinet is an example of such an instrument. (Pipes closed at both ends are not useful as musical instruments, since there is no way for the sound to escape from the pipe.)

- Instrument pipes can also be non-cylindrical in shape and two shapes in particular are common: conical pipes and Bessel horns. A **conical pipe** is by definition closed at one end, but nonetheless produces the same fundamental frequency $c/2L$ as an open cylindrical pipe and all harmonics. The saxophone and oboe are examples of instruments with conical pipes.

- **Brass instruments** have a cylindrical pipe attached to a Bessel horn that takes the form of a wide flaring bell. Examples include the trumpet and trombone. Like conical pipes, brass instruments produce the same frequencies as an open cylindrical pipe despite being closed at one end, except that the fundamental is very flat relative to the other harmonics and is therefore not normally used. Only the higher harmonics are used.

- There are three main mechanisms used to set air vibrating in the pipe of a wind instrument. The first is the **jet drive**, as used in the flute and recorder, in which a jet of air oscillates back and forth, driven by the movement of the air in the
pipe, and the jet in turn reinforces that movement, creating a positive feedback that keeps the oscillation going.

- The second mechanism is a reed (or double reed), as used in the clarinet, saxophone, and oboe, where oscillation of the pressure inside the pipe pushes on the flexible reed, opening an aperture and allowing bursts of air into the pipe, which again reinforce the oscillation.

- The third mechanism, used in the trumpet and other brass instruments, makes use of the player’s lips and is sometimes called a lip reed. Drawn tightly across the mouthpiece, the lips vibrate in a circular motion and admit a string of air bursts into the instrument pipe, setting the air there vibrating.

- The pitch produced by a wind instrument can be varied in several ways to produce a scale. The simplest is to use a separate pipe of a different length for every note, as in a pipe organ or pan flute. Another is to use tone holes, holes along the pipe that are opened and closed either by the player’s fingers or by mechanically operated pads. When open, a tone hole allows air to escape and effectively shortens the pipe, raising the pitch of the note. A third mechanism for changing pitch is to lengthen the pipe, either by using a slide, as in the trombone, or by using a system of valves to splice in additional sections of tubing, called crooks, as in a trumpet. The valve-and-crook system has intrinsic tuning problems that require the player to make additional small adjustments to pitch by varying their lip tension.

- Hard reed instruments, such as the harmonica and accordion, do not normally use a pipe at all. Instead the note is determined by the natural frequency of vibration of the reed, and a different reed is used for every note.

### Exercises

11.1 Here are some properties of air and helium gas:

<table>
<thead>
<tr>
<th>Gas</th>
<th>Density (kg/m³)</th>
<th>Bulk modulus (N/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>air</td>
<td>1.23</td>
<td>1.42 × 10⁵</td>
</tr>
<tr>
<td>helium</td>
<td>0.179</td>
<td>1.01 × 10⁵</td>
</tr>
</tbody>
</table>

a) Based on these figures and Eq. (1.17) on page 12, calculate the speed of sound in air. How does your result compare with the known speed of sound, 343 m/s?
b) What would the speed of sound be on a planet with a helium atmosphere?
c) An organ pipe plays the note C4 on Earth. What note would it play on the planet with the helium atmosphere?
11.2 The speed of sound is 343 m/s at 20°C but it increases with temperature by 0.61 m/s for every one degree Celsius.
   a) On a warm day the temperature is 25°C. What will the speed of sound be?
   b) A particular flute plays A4 at a perfect 440 Hz at 20°C. What will the frequency be when the temperature is 25°C?
   c) Is this sharp or flat, and by how many musical cents?
   d) To compensate for this problem, the flute has a joint in the middle that can be pulled out or pushed in to lengthen or shorten the instrument. Would one have to lengthen or shorten the flute to get it to play A4 at 440 Hz on a 25°C day?
   e) If the flute is 60 cm long before adjusting its length, by how much would you have to change the length to get it to play in tune?

11.3 The excess air pressure $p = P - P_0$ produced by the pump of a pipe organ is 900 pascals.
   a) What is the velocity of the air jets in the flue pipes?
   b) The length of the sounding portion of a certain principal pipe is 87.5 cm. What note does it play?
   c) At what distance from the mouth of the air jet should the labium be placed?

11.4 These two plots show the frequency spectrum of the same note played on a soprano saxophone and on a soprano clarinet:

![Graphs of frequency spectra](image)

   a) Both instruments are playing the same note. Approximately what note is it?
   b) Which instrument is which? Explain what the giveaway features are that allow you to tell.
   c) Briefly explain the scientific reason for the difference.
   d) Both instruments are about the same length, but one can play lower notes than the other. Which one can play lower, and why?
   e) Which instrument will have the brighter tone? Explain how you can tell.

11.5 With no valves depressed, the lowest normal note on a trumpet is B♭3, the B♭ below middle C.
   a) If you use the valves, what is the lowest note you can get?
   b) The trombone is an octave lower. What is the lowest normal note you can get in the first slide position?
   c) And what is it in seventh position?
d) The tuba is an octave lower still, but the tuba can play the fundamental of the harmonic series, whereas the trumpet and trombone normally do not. What is the lowest note on a tuba?
e) What is the frequency of this note in hertz?
f) Would this be audible to a human being?

11.6 Recall that one does not normally play the first mode on a trombone. In normal use the lowest mode one plays is the second mode. On a standard (tenor) trombone, with the slide all the way in (first position), this mode produces the note B♭2.

a) What are the next four “open notes” above this that one can play without moving the slide from first position? Give the full note names including the octave number (e.g., C3).
b) With the slide in first position, the length of the trombone pipe is 2.75 meters. To produce the note A2, one moves the slide out to “second position”. How much longer would the pipe have to get to produce this note?
c) Hence, how far would one have to move the slide out?
d) How much would you have to move the slide out to reach 7th position, which is 6 half-steps down from first position?