Physics 406: Homework 9

1. **Variation of the chemical potential with temperature:** We've seen that for fermions the probability of occupation of a single-particle state with energy equal to the chemical potential μ is $\frac{1}{2}$. But remember that μ can depend on temperature, so that the point where $f(\varepsilon) = \frac{1}{2}$ is not always in the same place. Here we calculate how μ varies for the two-dimensional Fermi gas. You can do the same calculation in the three-dimensional case, but the integrals are harder. (In three dimensions they involve the polylogarithm function.)

   (a) The energy states of a particle in a two dimensional box of size $L \times L$ satisfy

   $$\varepsilon = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2),$$

   where m is the mass of the particle. Show that the density of states for spin-$\frac{1}{2}$ fermions, i.e., the number of states per unit interval of energy, is

   $$n(\varepsilon) = \frac{Am}{\pi\hbar^2},$$

   independent of energy, where $A = L^2$ is the area of the box.

   (b) Write an expression for the number N of particles (assumed to be fermions) in the box in terms of the density of states and the Fermi function $f(\varepsilon)$. Plugging in the expressions for $n(\varepsilon)$ and $f(\varepsilon)$, show that

   $$N = \frac{Am}{\pi\hbar^2}\int_0^\infty \frac{\mathrm{d}\varepsilon}{\mathrm{e}^{(\varepsilon-\mu)/\tau}+1},$$

   and hence that the activity is given by

   $$\lambda = \exp\left(\frac{\pi\hbar^2\rho}{m\tau}\right) - 1,$$

   where $\rho = N/A$ is the density of the gas. You will need the result that

   $$\int_0^\infty \frac{\mathrm{d}x}{a\mathrm{e}^x + 1} = \log(1 + a^{-1}).$$

   (c) What value does this give for μ as $\tau \to 0$? And as temperature rises from absolute zero, which way does the chemical potential move—up or down?

2. ^{3}**He as a Fermi gas:** Liquid ^{3}He, the light isotope of helium, has spin $\frac{1}{2}$ and so is a fermion.

   (a) Treating ^{3}He as a perfect gas (which is only approximately right) and noting that its density is $0.081\,\mathrm{g\,cm^{-3}}$, calculate the Fermi energy ε_F. What is the corresponding Fermi temperature (i.e., the Fermi energy expressed in temperature units)? This gives us a rough measure of how cold the ^{3}He has to be before we start seeing quantum effects.

   (b) Calculate the heat capacity of ^{3}He at temperatures much less than the Fermi temperature. The real heat capacity is measured to be $2.89NT$. How close is your estimate to the real figure?

3. **Bose gas at low temperature:**

(a) Write down an expression for the internal energy of a perfect Bose gas in terms of the density of states $n(\varepsilon)$ and the occupation number $f(\varepsilon)$. Assuming the gas is three-dimensional, put in the appropriate expressions for $n(\varepsilon)$ and $f(\varepsilon)$, being careful to include any terms for the Bose condensate of atoms in the ground state.

(b) As we have seen, at low temperatures the activity is very close to 1. Change variables to $x = \varepsilon/\tau$, and so find an expression for the internal energy at low temperatures. Hence find the specific heat at low temperatures. How does this vary with temperature? Would it be possible to distinguish Bose from Fermi gases by the variation of the specific heat?

(c) Find an expression for the entropy of the Bose gas at low temperature, by any means you like.