Physics 406: Homework 7

1. **Entropy of mixing:**

   (a) Suppose we have two boxes with the same volume, with N molecules of a ideal gas in one and N molecules of a different type of ideal gas in the other, both in contact with a thermal reservoir at temperature τ. If the mass of the particles in each gas is m_a and m_b respectively, what is the total entropy of the two boxes together? You can assume that the gases are in the classical regime, where the Gibbs approximation for the partition function of N identical particles works well.

   (b) Now suppose we put our two boxes together and make a hole where the gases can pass from one to the other, so that they mix. When the gases are fully mixed, still in equilibrium at temperature τ, what is the partition function of the mixed system? (Remember that atoms of the same type will be indistinguishable, but atoms of different types are obviously distinguishable from one another.)

   (c) Hence what is the entropy of the whole system after mixing? How much did the entropy change because we mixed the gases? Did any heat flow in or out of the reservoir? If so, which way did it go?

2. **Phonon specific heat above the Debye temperature:** We have shown that the internal energy U of the phonons in a solid is approximately

   $$U = \frac{9N\tau^4}{\theta^3} \int_0^{\theta/\tau} \frac{x^3}{e^x - 1}\, dx,$$

   where the Debye temperature θ is given by

   $$\theta = \sqrt[3]{6\pi^2\hbar^3 v^3 \rho}, \qquad \rho = \frac{N}{V}.$$

   When $\tau \gg \theta$ show by expanding the integrand above that the value of the heat capacity is $C = 3N$ to leading order *and* to second order—i.e., that including the next order term in the expansion makes no difference to the value of the heat capacity at high temperature.

3. **Gibbs distribution for a simple system:** Suppose we have a simple system that has three possible states. Either it can have no particles in it, in which case it has energy 0, or it can have one particle with either energy 0 or energy ε.

   (a) Write down an expression for the grand partition function $\mathcal{Z}$.

   (b) Find the average number of particles in the system as a function of the temperature τ and the chemical potential μ.

   (c) Find the average internal energy of the system.

4. **Classical ideal gas:** We have seen that we can derive the properties of the ideal gas within the grand canonical ensemble by viewing it as the classical limit of a quantum ideal gas. An alternative derivation of the same results is as follows. Recall that the canonical partition function for N particles in a classical ideal gas is

   $$Z_N = \frac{1}{N!} Z_1^N, \tag{1}$$

where the partition function Z_1 for a single particle is

$$Z_1 = \frac{V}{(2\pi\hbar^2/m\tau)^{3/2}}.$$

(a) Write down the general expression for the *grand* partition function $\mathcal{Z}$ of a system with variable particle number and activity λ. Split the sum over states into separate sums over number of particles N and over states s with that number of particles. Hence show that

$$\mathcal{Z} = \mathrm{e}^{\lambda Z_1}$$

for the ideal gas. You will need the result that

$$\mathrm{e}^x = \sum_{N=0}^{\infty} \frac{x^N}{N!}.$$

(b) Hence find an expression for the grand potential Ω of the ideal gas and thus also for the average number of particles $\langle N \rangle$ and the pressure p from derivatives of Ω.