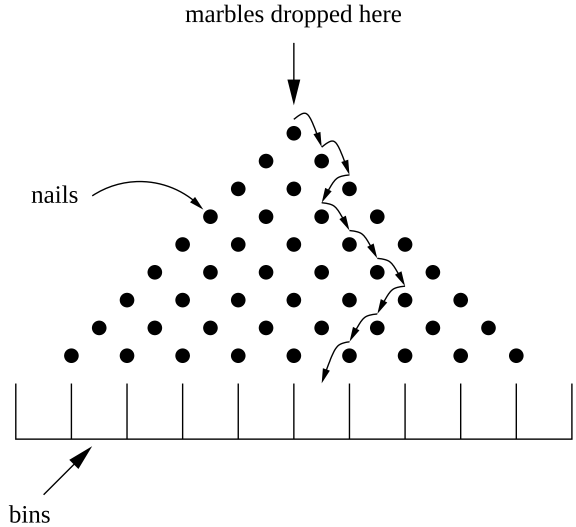
Physics 406: Homework 4

1. **Combinatorics:** A famous carnival toy is a machine that drops a large number of marbles at the top of a pyramid of nails tacked to a board like this:



The marbles fall down and at each step have a 50% chance of going right or left. At the bottom they fall into a set of bins as shown. If there are N rows of nails total, it's pretty easy to convince yourself that there must be $N+1$ bins.

   (a) Write down an expression for the number of paths $g(N,l,r)$ that a marble can take that make a total of l steps to the left and r steps to the right. Eliminate l and r in favor of the distance $x = r - l$ traveled to the right to get the same number in terms of N and x only. (Note that the distance x is measured horizontally from where the marbles start, i.e., from a line down the middle of the picture above.)

   (b) If $N = 10$, how many ways are there of traveling distance $x = 10$ to the right? How many ways are there of traveling distance zero?

   (c) If $N = 10$, about how many marbles will have to be dropped before even a single one of them goes all the way to the right-most bin? And how many if $N = 20$?

   (d) When many marbles are dropped, what is the expected mean distance $\langle x \rangle$ traveled, averaged over all of them? And what is the standard deviation of the distance?

   (e) So if you had to say where a single marble dropped would land, between about which values of x would you feel reasonably confident saying it would end up, if $N = 100$? (If you want to be really precise, you could say which values would you have 90% confidence it would land between, but any sensible answer will do for this question. Saying that x lies between -100 and 100 is not a good answer!)

2. **Entropy of a set of harmonic oscillators:** The quantum simple harmonic oscillator is a quantum system with equally spaced energy levels $\varepsilon = s\hbar\omega$, where $\hbar$ and ω are constants and s is a non-negative integer. If we have N identical such oscillators, their total combined energy can take values $U = n\hbar\omega$, where n is another non-negative integer. In your course pack it is shown that the multiplicity of the state with energy $n\hbar\omega$ is given by the binomial coefficient (or combination)

$$g(N,n) = \binom{N-1+n}{N-1}.$$

(a) Write $g(N,n)$ in a form involving factorials and hence write down the dimensionless entropy σ of the system when in thermal isolation.

(b) When N is large we can, to a good approximation, replace $N-1$ by N. Do this for your expression.

A result which we will use a lot in this course is Sterling's approximation for the logarithm of a factorial, which says that

$$\ln n! \simeq n \ln n - n,$$

where the approximation becomes better and better as $n \to \infty$. We will see how to prove this result in a later lecture. For the moment we just assume it. (If you want to see the proof, it's given in Appendix A of Kittel and Kroemer.) Apply Sterling's approximation to your expression for $G(N,n)$ and derive an approximate expression for σ for large N.

(c) Recalling the definition of the temperature τ in energy units, $\tau = \partial U/\partial\sigma$, differentiate to get an expression for τ in terms of the internal energy. (You have to consider n to be a continuous variable to do this calculation, which is strictly speaking not correct—it is an integer. Later in the course we'll see a better derivation of this result that doesn't require us to do this fix.)

(d) Rearrange to show that

$$U = \frac{N\hbar\omega}{\exp(\hbar\omega/\tau) - 1}.$$

This is the internal energy of a set of N harmonic oscillators and, as we will show, is also the correct expression for the energy of a set of bosons (e.g., photons) in a quantum gas. From this expression we will later derive the famous black-body radiation spectrum of Rayleigh and Planck.

(e) What is the heat capacity of the system?

3. **Partition function of a simple system:** Suppose a simple system has states with three energies, $-\varepsilon$, 0, and $+\varepsilon$. The multiplicities of the states are $g(-\varepsilon) = 1$, $g(0) = 2$, and $g(\varepsilon) = 1$. The system is put in contact with a thermal reservoir at temperature τ (in energy units) and allowed to come to equilibrium.

(a) Calculate the partition function Z of the system.

(b) Calculate the average internal energy of the system as a function of temperature.

(c) Show that the heat capacity is

$$C = \frac{\varepsilon^2}{\tau^2[1+\cosh(\varepsilon/\tau)]}.$$