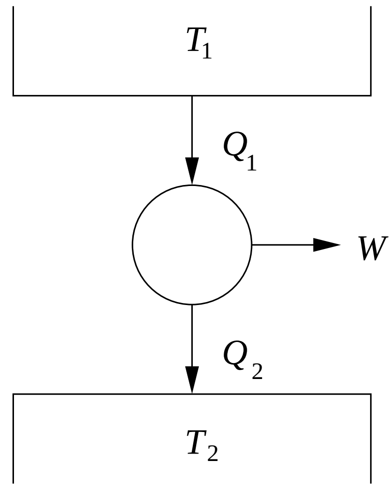
Physics 406: Homework 2

1. **Heat engine:** A perfectly efficient heat engine operates between two very large tanks of water forming thermal reservoirs. One of them is at a high temperature T_1 and the other is at a colder temperature of T_2:



   (a) The efficiency of the engine is defined as the ratio of the work it produces W to the heat that it takes in Q_1 thus: $\eta = W/Q_1$. If the heat engine is reversible, what is the value of η in terms of the heats Q_1 and Q_2 entering and leaving the reservoirs? And what is it in terms of the temperatures of the two reservoirs?

   (b) If the hot reservoir is at 100°C and the cold one at 10°C, what is the value of η? Hence, show that we need to take 82.9 Joules of heat from the hot reservoir to generate just 20 Joules of work. How much heat is rejected into the cold reservoir as a result?

   (c) Now suppose that the tanks of water are not that large after all. In fact, each of them contains just a few liters of water, the same amount in each, with heat capacity C, which we assume to be constant over the temperature range we are looking at. If small amounts of heat $đQ_1$ and $đQ_2$ leave and enter the two tanks at temperatures T_1 and T_2, write down again the expressions for the efficiency η in terms of the temperatures, and in terms of $đQ_1$ and $đQ_2$. Equate these two and prove that

$$\frac{đQ_1}{T_1} = \frac{đQ_2}{T_2}.$$

   When $đQ_1$ leaves the hot tank, what is the change in temperature $\mathrm{d}T_1$ of the hot tank, in terms of $đQ_1$ and the heat capacity? And what is the change $\mathrm{d}T_2$ for the cold tank, in terms of $đQ_2$? (Hint: be careful about the signs of the temperature changes; remember that the hot tank will get colder when heat leaves it.) Combine all these expressions to show that

$$\frac{\mathrm{d}T_1}{T_1} = -\frac{\mathrm{d}T_2}{T_2}. \qquad (1)$$

   (d) Now suppose that the hot tank starts off at temperature $T_1 = T_h$ and the cold one at $T_2 = T_c$. By running our heat engine, we cool the hot one and heat the cold one until they are both at the same final temperature T_f. Integrate Eq. (1) above and derive an expression for T_f in terms of T_h and T_c.

(e) If $T_h = 100°\mathrm{C}$ and $T_c = 10°\mathrm{C}$ as before, show that the final temperature of the system is $51.9°\mathrm{C}$, regardless of the heat capacity of the tanks.

(f) What is the change in entropy of the system from the beginning to the end?

2. **Entropy:** Suppose we take the two tanks from the previous question, again at temperatures T_h and T_c, and simply put them in contact with one another, so that heat can flow from the hot one to the cold one.

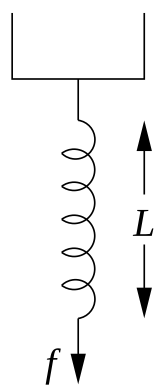
   (a) What are the changes of temperature $\mathrm{d}T_1$ and $\mathrm{d}T_2$ when heat $\text{đ}Q$ flows from hot to cold? Hence show that $\mathrm{d}T_1 = -\mathrm{d}T_2$.

   (b) What is the final temperature T_f' of both tanks when they come to equilibrium? For $T_h = 100°\mathrm{C}$ and $T_c = 10°\mathrm{C}$ as before, what is this final temperature?

   (c) The entropy change of the hot tank is $\mathrm{d}S_1 = -\text{đ}Q/T_1$ and of the cold one is $\mathrm{d}S_2 = \text{đ}Q/T_2$. What is the total entropy change of the hot and cold tanks as they come to equilibrium at temperature T_f'?

   (d) What is the total change in entropy of the whole system? If $T_h = 100°\mathrm{C}$ and $T_c = 10°\mathrm{C}$ and the heat capacity of the tanks are both $C = 10000\,\mathrm{J\,K^{-1}}$, how much does the entropy change?

3. **Thermodynamics of a spring:** A Hooke's law (i.e., linear) spring thus



has a spring constant and resting length that depend on temperature, so that we need to use thermodynamics to calculate its behavior.

   (a) In terms of the length L and the force f on the spring, write down the expression for a small element of work done on the spring $\text{đ}W$. Hence write down the expression for a small change $\mathrm{d}U$ in the internal energy in terms of f, L, and the temperature T and entropy S.

   (b) What is the corresponding expression for a small change $\mathrm{d}F$ in the free energy in terms of T, S, f, and L? From this derive a Maxwell relation for $(\partial S/\partial L)_T$.

   (c) The equation of state of the spring is

$$f = a\frac{L}{T} - b, \tag{2}$$

   where a and b are constants. Recalling that a small amount of heat is $\text{đ}Q = T\,\mathrm{d}S$, combine your Maxwell relation and the equation of state, Eq. (2), to find the amount of heat Q that flows into the spring when we isothermally stretch it from length L_1 to length L_2.