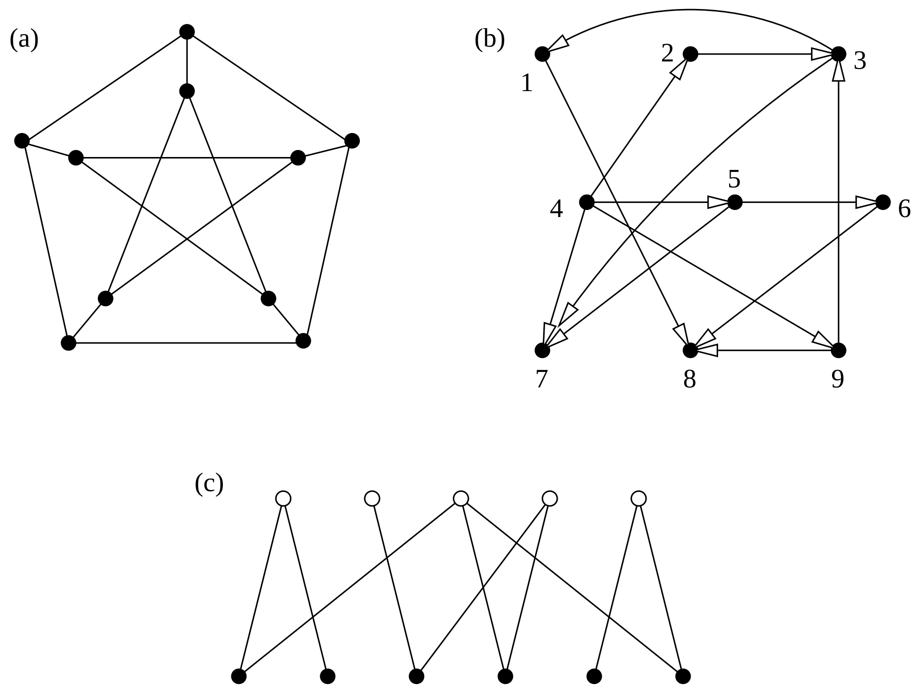
Complex Systems 535/Physics 508: Homework 2

1. **Planar and acyclic graphs:** Here are three small graphs, one undirected, another directed, and the third bipartite:



Draw figures showing the following:

   (i) that graph (a) is not planar, using Kuratowski's theorem. (Hint: Kuratowski's theorem says a graph must contain an expansion of K_5 or UG. In the past, people have come up with random other rules to answer this question. I really do want to see a proof using Kuratowski's theorem.)

   (ii) that graph (b) is not cyclic;

   (iii) the bibliographic coupling graph of (b);

   (iv) the two one-mode projections of (c).

2. **Eigenvalues of an acyclic digraph:** Consider an acyclic directed graph with no self-edges (i.e., no edges connecting vertices to themselves). We showed in class that the adjacency matrix of such a graph can be written in upper triangular form by a suitable labeling of the vertices.

   (i) Show that all eigenvalues of the adjacency matrix of such an acyclic digraph are zero.

   (ii) Write down an expression for the number of closed cycles of length r in a graph in terms of the eigenvalues of the adjacency matrix.

   (iii) Hence, or otherwise, show (conversely) that if all eigenvalues of the adjacency matrix are zero the graph must be acyclic.

3. **Lowest eigenvalue of the Laplacian:** Consider an undirected graph (i.e., every vertex is reachable from every other) with Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

   (i) What is the lowest eigenvalue λ_1 of $\mathbf{L}$ and what is the corresponding eigenvector?

   (ii) Show that if the graph were not connected (i.e., has more than one component) then $\lambda_2 = 0$.

   The eigenvalue λ_2 is called the *algebraic connectivity* of the graph, and will come up repeatedly in our study of networks.

4. **A graph with a specified degree sequence:** Let $\{k_i\}$ be the degree sequence of a large graph, and suppose that, subject to this degree sequence, vertices are connected at random.

   (i) Show that the expected number of edges between vertex s and vertex t is $k_s k_t / 2m$, where $m = \frac{1}{2}\sum_i k_i$ is the total number of edges in the graph.

   (ii) Hence show that the expected mean degree of the neighbors of a vertex is $\langle k^2 \rangle / \langle k \rangle$.

   (iii) Prove thereby that "your friends have more friends than you do." That is, that the expected mean degree of the neighbors of a vertex is never less than the expect mean degree of the vertex itself, no matter what the degree sequence is.

5. **Extra credit: Geodesic paths and the adjacency matrix:** Consider the set of all paths from vertex s to vertex t on an undirected graph with adjacency matrix $\mathbf{A}$. Let us give each path a weight equal to α^r, where r is the length of the path.

   (i) Show that the sum of the weights of all the paths from s to t is given by Z_{st}, which is the st element of the matrix $\mathbf{Z} = (\mathbf{I} - \alpha\mathbf{A})^{-1}$.

   (ii) What condition must α satisfy for the sum to converge?

   (iii) Hence, or otherwise, show that the length ℓ_{st} of a geodesic path from s to t, if there is one, is

$$\ell_{st} = \lim_{\alpha \to 0} \frac{\partial \log Z_{st}}{\partial \log \alpha}.$$