

A Timoshenko beam theory with pressure corrections for layered orthotropic beams

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Abstract

A Timoshenko beam theory for layered orthotropic beams is presented. The theory consists of a novel combination of three key components: average displacement and rotation variables that provide the kinematic description of the beam, stress and strain moments used to represent the average stress and strain state in the beam, and the use of exact axially-invariant plane stress solutions to calibrate the relationships between all these quantities. These axially-invariant solutions, which we call the fundamental states, are also used to determine a shear strain correction factor as well as corrections to account for effects produced by externally-applied loads. The shear strain correction factor and the external load corrections are computed for a beam composed of isotropic layers. The proposed theory yields Cowper's shear correction for a single isotropic layer, while for multiple layers new expressions for the shear correction factor are obtained. A body-force correction is shown to account for the difference between Cowper's shear correction and the factor originally proposed by Timoshenko. Numerical comparisons between the theory and finite-elements results show good agreement.

Keywords: Timoshenko beam theory, shear correction factor

1. Introduction

The equations of motion for a deep beam that include the effects of shear deformation and rotary inertia were first derived by Timoshenko (1921, 1922). Two essential aspects of Timoshenko's beam theory are the treatment of shear deformation by the introduction of a mid-plane rotation variable, and the use of a shear correction factor. The definition and value of the shear correction factor have been the subject of numerous research papers, some of which are discussed below. Shames and Dym (1985, Ch. 4, pg. 197) provide

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an excellent overview of the classical approach to Timoshenko beam theory. This paper however, draws primarily from research and theories which refine Timoshenko's original approximations.

[Prescott \(1942\)](#) derived the equations of vibration for thin rods using average through-thickness displacement and average rotation variables. He introduced a shear correction factor to account for the difference between the average shear on a cross section and the expected quadratic distribution of shear.

[Cowper \(1966\)](#) presented a revised derivation of Timoshenko's beam theory starting from the equations of linear elasticity for a prismatic, isotropic beam in static equilibrium. Cowper introduced residual displacement terms that he defined as the difference between the actual displacement in the beam and the average displacement representation. These residual displacements account for the difference between the average shear strain and the shear strain distribution. Cowper introduced a correction factor to account for this difference and computed its value based on the three-dimensional solution of a cantilever beam subjected to a tip load.

[Stephen and Levinson \(1979\)](#), developed a beam theory along the lines of Cowper's, but recognized that the variation in shear along the length of the beam would lead to a modification of the relationship between bending moment and rotation. This variation had been neglected by Cowper.

Following the work of [Cowper \(1966\)](#) and [Stephen and Levinson \(1979\)](#), in this paper we seek a solution to a beam problem based on average through-thickness displacement and rotation variables. In a departure from previous work, we introduce *strain moments*, which are analogous to the stress moments used in the equilibrium equations. These strain moments remove the restriction of working with an isotropic, homogeneous beam. This is an essential component of the present approach, as sandwich and layered orthotropic beams are often used for high-performance, aerospace applications ([Flower and Soutis, 2003](#)).

Another important feature of our theory is the use of certain statically determinate beam problems that we use to construct the relationship between stress and strain moments, and to reconstruct the stress and strain solution in a post-processing step. We call these solutions the *fundamental states* of the beam. The present theory was first pursued by [Hansen and Almeida \(2001\)](#) and [Hansen et al. \(2005\)](#), and an extension of this theory to the analysis of plates was presented by [Guiamatsia and Hansen \(2004\)](#), [Tafeuvoukeng \(2007\)](#) and [Guiamatsia \(2010\)](#).

This paper begins with a brief discussion of two classical methods used to calculate the shear correction factor in Section (2). Section (3) describes the proposed theory and Section (3.2) introduces the fundamental states. In Section (4), calculations are presented for a beam composed of multiple isotropic layers. Section (5) briefly presents the modified equations of motion for an isotropic beam. In Section (6), comparisons are made with finite-element calculations. Section (7) outlines conclusions based on the theory presented herein.

2. The shear correction factor

One of the main difficulties in using Timoshenko beam theory is the proper selection of the shear correction factor. Many authors have published definitions of the shear

correction factor and have proposed various methods to calculate it. Most of these approaches fall into one of two categories. The first approach is to use the shear correction factor to match the frequencies of vibration of various beam constructions with exact solutions to the theory of elasticity. The second approach is to use the shear correction factor to account for the difference between the average shear or shear strain and the actual shear or shear strain using exact solutions to the theory of elasticity.

[Timoshenko \(1922\)](#) developed the frequency-matching approach. He calculated the shear correction factor by equating the frequency of vibration determined using the plane stress equations of elasticity to those computed using his beam theory. Although not explicitly written in the paper, the shear correction factor obtained in this manner for a rectangular beam is

$$k_{xy} = \frac{5(1 + \nu)}{6 + 5\nu}. \quad (1)$$

[Cowper \(1966\)](#) calculated the shear correction factor using an approach from the second category described above. Using residual displacements designed to take into account the distortion of the cross sections under shear loads, Cowper was able to derive a formula for the shear correction factor based on solutions of a cantilever beam subjected to a tip load. For a rectangular isotropic homogeneous beam, Cowper found the following shear correction factor:

$$k_{xy} = \frac{10(1 + \nu)}{12 + 11\nu}. \quad (2)$$

Following Cowper's approach, [Stephen \(1980\)](#) computed the shear correction factor for beams of various cross sections by using the exact solutions for a beam subject to a uniform gravity load. He employed a modified form of the Kennard–Leibowitz method ([Leibowitz and Kennard, 1961](#)), to obtain the shear correction factor by equating the average centerline curvature of the exact result with the Timoshenko solution. He obtained a modified form of Timoshenko's shear correction factor for rectangular sections that approached Equation (1) for thin cross-sections.

Using the frequency matching approach, [Hutchinson \(1981\)](#) computed the shear correction factor by performing a comparison between Timoshenko beam theory and three solutions from the theory of elasticity, the Pochhammer–Chree solution in [Love \(1920\)](#), a Fourier solution due to [Pickett \(1944\)](#) and a series solution computed by [Hutchinson \(1980\)](#). Hutchinson found that the best shear correction factor was dependent on the frequency and Poisson's ratio of the beam, but that Timoshenko's value was better than Cowper's.

Later, [Hutchinson \(2001\)](#) introduced a new Timoshenko beam formulation and computed the shear correction factor for various cross sections based on a comparison with a tip-loaded cantilever beam. For a beam with a rectangular cross section, Hutchinson obtained a shear correction factor that depends on the Poisson ratio and the width to depth ratio. In a later discussion of the paper, [Stephen \(2001\)](#) showed that the shear correction factors he obtained in [Stephen \(1980\)](#) were equivalent.

More recently [Dong et al. \(2010\)](#), presented a semi-analytic finite-element technique for calculating the shear correction factor based either on the Saint–Venant warping function or the free vibration of a beam.

Some experimental studies have been performed to try and measure the shear correction factor based on the original equations proposed by Timoshenko. [Spence and Seldin](#)

(1970) obtained experimental values of the shear correction factor for a series of square and circular beams composed of both isotropic and anisotropic materials by determining their natural frequencies. Kaneko (1975) performed an extensive review of the shear correction factors for rectangular and circular cross sections obtained by various authors using either experimental techniques or analysis. Experimental studies have generally used a natural frequency approach to determine the shear correction factor and have generally found that Timoshenko's value is superior to Cowper's. This is perhaps not surprising, since Timoshenko's correction was obtained by matching frequencies in the same manner in which the experiments are performed. However, the frequency matching approach fails to provide a theoretical explanation as to why the value of a factor that modifies the relationship between the shear resultant and the average shear strain should be determined by the natural frequency of vibration. It is this deficiency that motivates the work presented here.

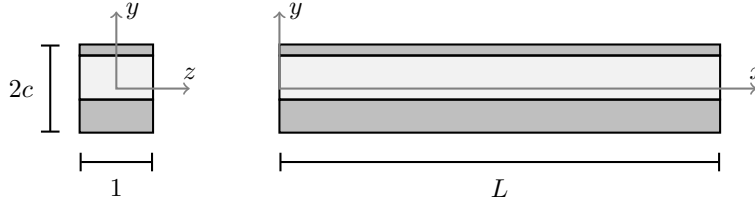


Figure 1: The geometry of the beam composed of layers of different materials.

3. The theory

The geometry of the beam under consideration is shown in Figure 1. The beam extends along the x -direction subject to forces on the top and bottom surfaces in the y -direction. The reference axis is placed at the centroid of the cross-section. The half-thickness in the y -direction is c , while the length of the beam in the x -direction is L . The beam is of uniform composition in both the x and z -directions and so consists of a series of layers with different material properties. We assume that each layer is composed of an orthotropic material, with material properties aligned with the coordinate axes. These assumptions eliminate the possibility of twisting and allow the beam to be modeled using a plane stress assumption in the z plane. In each layer k , numbered from the bottom to the top of the beam, the following constitutive law holds:

$$\boldsymbol{\sigma}_{(k)} = \mathbf{C}_{(k)} \boldsymbol{\epsilon}_{(k)},$$

where $\boldsymbol{\sigma}_{(k)} = [\sigma_x \ \sigma_y \ \sigma_{xy}]_{(k)}^T$ and $\boldsymbol{\epsilon}_{(k)} = [\epsilon_x \ \epsilon_y \ \gamma_{xy}]_{(k)}^T$. Since the beam is composed of an orthotropic material, there is no coupling between shear and normal stresses. Although variation in the Poisson's ratio between layers would lead to a violation of the plane stress assumption, we include this possibility and ignore the edge effects in the cross-section in such situations.

These assumptions are an extension of the conditions originally used by Timoshenko, who limited his analysis to plane stress beams composed of a single isotropic material (Timoshenko, 1922).

3.1. The displacement representation

Following Prescott (1942) and Cowper (1966), the average through-thickness displacements and average rotation are defined as follows:

$$\begin{aligned} u_0(x, t) &= \frac{1}{2c} \int_{-c}^c u(x, y, t) dy, \\ u_1(x, t) &= \frac{3}{2c^3} \int_{-c}^c yu(x, y, t) dy, \\ v_0(x, t) &= \frac{1}{2c} \int_{-c}^c v(x, y, t) dy, \end{aligned} \tag{3}$$

where u and v are the displacements in the x and y directions, respectively. The average displacements and rotation are defined regardless of the through-thickness behavior of u and v , which are piecewise continuous through the thickness of the beam in this problem. The average displacements are an incomplete representation of the total displacement field in the beam, in the sense that the average quantities do not capture the point-wise behavior of the exact displacement. In order to capture this behavior, it is necessary to introduce residual displacements that account for the difference between the average and point-wise quantities in the following manner:

$$\begin{aligned} u(x, y, t) &= u_0(x, t) + yu_1(x, t) + \tilde{u}(x, y, t), \\ v(x, y, t) &= v_0(x, t) + \tilde{v}(x, y, t), \end{aligned} \tag{4}$$

where \tilde{u} and \tilde{v} are the residual displacements in the x and y directions, as introduced by Cowper (1966). Given the definitions of the average displacements (3), the zeroth and first moments of \tilde{u} , and the zeroth moment of \tilde{v} through the thickness, must be zero:

$$\begin{aligned} \int_{-c}^c \tilde{u}(x, y, t) dy &= 0, \\ \int_{-c}^c y\tilde{u}(x, y, t) dy &= 0, \\ \int_{-c}^c \tilde{v}(x, y, t) dy &= 0. \end{aligned}$$

The average displacements and displacement residuals may be used to determine the strain at any point in the beam. In this approach however, we are interested in the average through-thickness strain. To this end, we introduce the following strain moments:

$$\epsilon_0(x, t) = \int_{-c}^c \frac{\partial u}{\partial x} dy = 2c \frac{\partial u_0}{\partial x}, \tag{5a}$$

$$\kappa(x, t) = \int_{-c}^c y \frac{\partial u}{\partial x} dy = \frac{2c^3}{3} \frac{\partial u_1}{\partial x}, \tag{5b}$$

$$\gamma(x, t) = \int_{-c}^c \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] dy = 2c \left[u_1 + \frac{\partial v_0}{\partial x} \right] + \int_{-c}^c \frac{\partial \tilde{u}}{\partial y} dy, \tag{5c}$$

that represent the axial, bending and shear strain moments, respectively. These strain moments are analogous to the stress moments that are used to define the equilibrium equations for a beam. Note that these strain moments are not normalized and as a result have different dimensions than the point-wise strain. The main advantage of using the strain moments (5) over point-wise strain variables is that they are always defined, regardless of the through-thickness distribution of the strain. This is an important property, since the point-wise shear strain can be discontinuous at material interfaces.

Thus far, no assumptions beyond those of linear elasticity have been made. The combination of the average and residual displacements can be used to capture an arbitrary displacement field. Next, we examine the state of stress within the beam.

3.2. The fundamental states

The basic assumption made in the development of the present beam theory is that the stress and strain state in the beam can be approximated using a linear combination of axially-invariant solutions. We call these axially-invariant solutions the *fundamental states*. The fundamental states can be used to capture the complex interaction between the stresses in layered orthotropic beams, away from the ends of the beam. As is the case in many beam theories, end effects cannot be captured using this approach. In this section we address how to determine the fundamental states.

The fundamental states are determined from a hierarchy of statically determinate beam problems. These beam problems are formulated using a series of self-equilibrating loads applied to a beam with the same sectional properties as the beam under consideration. Rigid body translation and rotation modes are removed from the solution by imposing three displacement constraints so that no stress concentrations are present. The first four loading conditions leading to the first four fundamental states are shown in Figure 2. N , M and Q are the axial, bending, and shear resultants, defined as follows:

$$\begin{aligned} N(x, t) &= \int_{-c}^c \sigma_x(x, y, t) dy, \\ M(x, t) &= \int_{-c}^c y \sigma_x(x, y, t) dy, \\ Q(x, t) &= \int_{-c}^c \sigma_{xy}(x, y, t) dy. \end{aligned} \tag{6}$$

We also refer to these as the stress moments.

The beam in Figure 2 has the same cross-sectional properties as the beam under consideration, but is extended between the coordinates $x = -L_f$ to $x = L_f$. L_f is the half-length of the beam for the fundamental state analysis, and must be sufficiently large such that the end effects do not influence the state of stress or strain at the middle of the beam.

The fundamental states are obtained from the solution of the beam problems illustrated in Figure 2 by taking the through-thickness stress and strain distribution at $x = 0$. As a result, the fundamental states represent a distribution of stress and strain only in the y direction. The loading conditions are constructed such that only one stress resultant or load is non-zero at $x = 0$. For instance, in the third fundamental state, which corresponds to a shear load, the bending resultant is zero at the mid-section and the

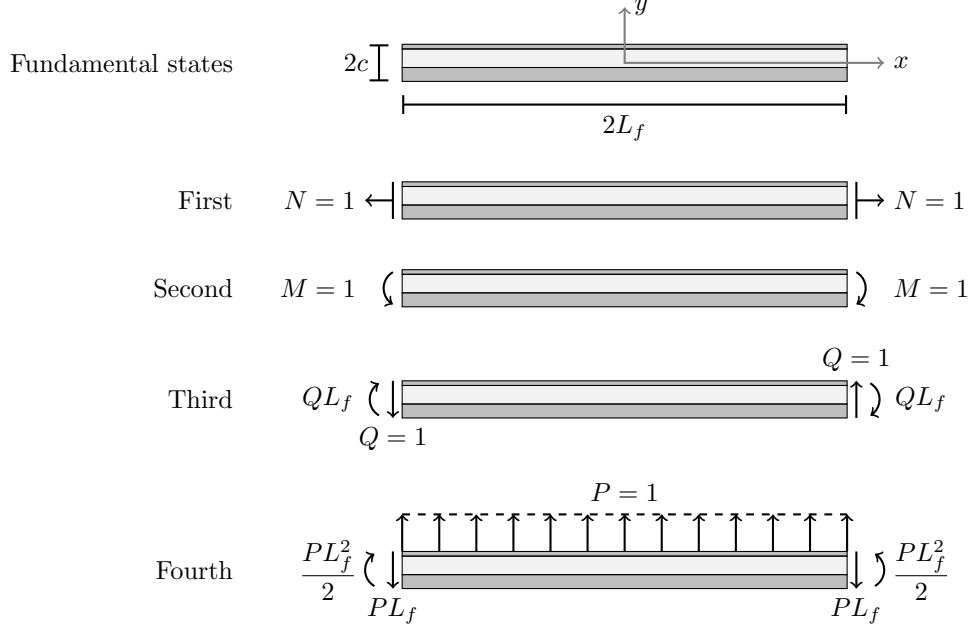


Figure 2: An illustration of the loading conditions used to obtain the first four fundamental states. The states are: axial loading, bending moment, shear and pressure load. The fundamental states are extracted from the solution at the $x = 0$ plane. L_f , the half-length of the beam used to calculate the fundamental states, must be large enough that the end effects do not influence stress distribution at $x = 0$.

shear resultant is unity, while in the fourth fundamental state, which corresponds to a pressure load, the shear resultant and bending moment are zero and the pressure load is unity.

We label the fundamental states with a superscript for the corresponding condition: N , M and Q for the axial resultant, bending moment, and shear resultant, and P for any externally applied load. For instance, $\sigma^M(y)$ and $\epsilon^M(y)$ is the fundamental state corresponding to bending with strain moments ϵ_0^M , κ^M and γ^M . Note that the strain moments of the fundamental states are scalar values independent of any coordinate.

In the present work, we obtain the fundamental states through a series of analytic calculations presented below. In these calculations the beam used to calculate the fundamental states is essentially of infinite length since the stress resultants satisfy the loading conditions illustrated in Figure 2 in an average sense for any length L_f . The fundamental states could also be obtained approximately using a finite-element approach.

The stress and strain state in the beam can be written as a linear combination of the fundamental states and stress and strain residuals, $\bar{\sigma}$ and $\bar{\epsilon}$, respectively. Using this

linear superposition, the stress and strain state in the beam is given by:

$$\boldsymbol{\sigma}(x, y, t) = N\boldsymbol{\sigma}^N + M\boldsymbol{\sigma}^M + Q\boldsymbol{\sigma}^Q + P\boldsymbol{\sigma}^P + \tilde{\boldsymbol{\sigma}}(x, y, t), \quad (7a)$$

$$\boldsymbol{\epsilon}(x, y, t) = N\boldsymbol{\epsilon}^N + M\boldsymbol{\epsilon}^M + Q\boldsymbol{\epsilon}^Q + P\boldsymbol{\epsilon}^P + \tilde{\boldsymbol{\epsilon}}(x, y, t), \quad (7b)$$

where the magnitudes of the fundamental states — the axial, bending, and shear resultants and the pressure load — are functions of x and t while the fundamental states are functions only of the through-thickness coordinate y . The stress and strain residuals $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\epsilon}}$ represent deviations due to end effects and higher-order fundamental states. For instance, a linear or quadratic pressure load would induce stresses and strains not captured by the first four states discussed here. It is important to recognize that as a result of Equation (6), the stress residuals $\tilde{\boldsymbol{\sigma}}$ do not contribute to the axial, bending, or shear resultants.

The assumption that the stress and strain state in the beam can be approximated by a linear combination of the fundamental states is equivalent to assuming that the terms $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\epsilon}}$ may be omitted in the analysis. As a result, end effects are not captured within the theory. Furthermore, rapidly varying loads produce similar terms from linear, quadratic, and higher-order polynomial loading fundamental states. If these higher-order fundamental states are not included in the analysis, they will essentially produce additional $\tilde{\boldsymbol{\epsilon}}$ terms.

The fundamental states also provide a self-consistent method for reconstructing the stress and strain distribution within the beam in a post-processing step using Equation (7). This reconstruction includes stress and strain components that are not normally considered in classical approaches without recourse to a post-analysis integration of the equilibrium equations through the thickness. However, as is well known, this integration procedure can introduce compatibility problems, whereas Equation (7) does not suffer from this issue.

3.3. The constitutive relation and pressure correction

We now develop a constitutive relationship between moments of stress and moments of strain. A pressure correction is also introduced to account for the influence of externally applied loads. To develop these relationships it is necessary to examine the stress and strain moments in the context of the stress and strain decomposition in Equation (7). By construction, the stress resultants found in Equation (6) are always equal to the magnitudes of the fundamental states. On the other hand, the strain moments may have contributions from all fundamental states and the strain residuals. Using Equation (7b), the required moments of strain are,

$$\begin{bmatrix} \epsilon_0 \\ \kappa \\ \gamma \end{bmatrix} = \begin{bmatrix} \epsilon_0^N & \epsilon_0^M & \epsilon_0^Q \\ \kappa^N & \kappa^M & \kappa^Q \\ \gamma^N & \gamma^M & \gamma^Q \end{bmatrix} \begin{bmatrix} N \\ M \\ Q \end{bmatrix} + P \begin{bmatrix} \epsilon_0^P \\ \kappa^P \\ \gamma^P \end{bmatrix} + \begin{bmatrix} \tilde{\epsilon}_0 \\ \tilde{\kappa} \\ \tilde{\gamma} \end{bmatrix}, \quad (8)$$

where $\tilde{\epsilon}_0$, $\tilde{\kappa}$ and $\tilde{\gamma}$ are the moments of the strain residuals. Note that the left-hand side of Equation (8) is equal to the moments from Equation (5). Recall also that the moments of the fundamental states are constant.

It is important to distinguish between the three different terms in the expression for the strain moments (8). The first term is due to the stress resultants, the second term

is due to the applied loads, and the remaining term is due to the strain residuals. The final term is neglected based on the assumption that its contribution will be small.

Setting $\tilde{\epsilon}_0$, $\tilde{\kappa}$ and $\tilde{\gamma}$ to zero, and re-arranging Equation (8) results in the following constitutive relation:

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \mathbf{D} \left\{ \begin{bmatrix} \epsilon_0 \\ \kappa \\ \gamma \end{bmatrix} - P \begin{bmatrix} \epsilon_0^P \\ \kappa^P \\ \gamma^P \end{bmatrix} \right\}, \quad (9)$$

where the components of the constitutive matrix \mathbf{D} can be found as follows:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_0^N & \epsilon_0^M & \epsilon_0^Q \\ \kappa^N & \kappa^M & \kappa^Q \\ \gamma^N & \gamma^M & \gamma^Q \end{bmatrix}^{-1}. \quad (10)$$

Note that this matrix is not necessarily symmetric. Due to the orthotropic construction of the beam, γ^N , γ^M , ϵ_0^Q , and κ^Q are zero. As a result D_{13} , D_{23} , D_{31} , and D_{32} are also zero.

If only axial, bending and shear loads are applied to the beam, then there is no load-dependent strain moment contribution. However, when external loads are applied to the beam, the relationship between the strain moments and stress moments is modified as follows:

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \mathbf{D} \begin{bmatrix} \epsilon_0 \\ \kappa \\ \gamma \end{bmatrix} - P \begin{bmatrix} N^P \\ M^P \\ Q^P \end{bmatrix}, \quad (11)$$

where N^P , M^P and Q^P are the product of the strain moments, ϵ_0^P , κ^P and γ^P , and the constitutive matrix \mathbf{D} . N^P , M^P and Q^P represent a load-dependent pressure correction to the constitutive equations.

Note that the constitutive matrix \mathbf{D} is derived using the strain moments from the first three fundamental states. The only assumption used to derive this relationship is that the moments of the strain residuals are small. The influence of externally applied loads can be accounted for by including strain moment terms from the fundamental state corresponding to pressure loading. Higher-order loading effects could be included by taking into account the strain moments due to linear, quadratic, and polynomial pressure distributions in general. Neglecting these effects is equivalent to introducing a non-zero strain residual moment.

3.4. The shear strain correction

The additional integral in the expression for the shear strain moment in Equation (5c), involves a correction from the residual displacements. The value of this integral depends on the distribution of the shear strain through the thickness. Several authors have suggested that this shear strain correction should be computed under different loading conditions. For example, Cowper (1966) computes his value of the shear correction factor for a beam subject to a constant shear load, while Stephen (1980) and Hutchinson (2001) compute the correction for a beam subject to a gravity load.

In a similar approach to Cowper, we set the shear strain correction equal to the ratio of the shear strain moment over the average shear strain computed using the fundamental

state corresponding to shear:

$$k_{xy} = \frac{\gamma^Q}{2c \left[u_1 + \frac{\partial v_0}{\partial x} \right]_Q} = 1 + \frac{\int_{-c}^c \frac{\partial \tilde{u}}{\partial y} dy \Big|_Q}{2c \left[u_1 + \frac{\partial v_0}{\partial x} \right]_Q}. \quad (12)$$

The subscript Q is used to denote that the expression is evaluated using the fundamental state corresponding to shear.

The corrected shear strain moment is therefore:

$$\gamma = 2ck_{xy} \left[u_1 + \frac{\partial v_0}{\partial x} \right].$$

It is important to realize that this is not a correction for the shear stiffness of the beam, but rather a correction of the discrepancy between the average shear strain and the displacement representation. It is therefore more correct to refer to it as a *shear strain correction*.

3.5. Equilibrium equations

The equilibrium equations for the stress resultants are obtained by the standard approach of integrating the two-dimensional momentum equations. When the density of the material ρ is constant, these equations are:

$$\frac{\partial N}{\partial x} = 2c\rho \frac{\partial^2 u_0}{\partial t^2}, \quad (13a)$$

$$\frac{\partial M}{\partial x} - Q = \frac{2c^3}{3} \rho \frac{\partial^2 u_1}{\partial t^2}, \quad (13b)$$

$$\frac{\partial Q}{\partial x} + P = 2c\rho \frac{\partial^2 v_0}{\partial t^2}. \quad (13c)$$

If the density of the material varies in the through-thickness direction, these equations would involve integrals of the residual displacements.

3.6. Discussion

Our proposed theory fits almost entirely within Timoshenko's original beam theory (Timoshenko, 1921, 1922). While the displacement variables involved have a different interpretation, the equations themselves take essentially the same form, except for the pressure correction. The pressure correction can be treated as an additional force arising from the application of a pressure load. As a result, beyond the calculation of the fundamental states, the theory does not require much more computational effort than classical Timoshenko beam theory. In addition, the proposed theory can handle any combination of boundary conditions typically imposed for classical Timoshenko beam problems. Within the context of our theory, different boundary conditions result in additional strain residual moment terms in Equation (8).

Not only does our proposed theory take a similar form to Timoshenko's beam theory, but the additional modifications proposed above have several important benefits. As with Cowper's theory, the proposed approach has a completely general displacement representation (4). We have introduced a stress and strain decomposition (7), based on

the fundamental states, that also provides a self-consistent method for the reconstruction of the through-thickness stress and strain distributions. Finally, the theory contains a consistent method for predicting the shear strain correction (12), the pressure correction (11), and the stiffness (10) and using the fundamental states. These additions enhance the capabilities of classical Timoshenko beam theory.

4. Isotropic layered beam

In this section we derive the fundamental states, the stress-strain moment constitutive equation, the shear correction factor, and the pressure strain moment corrections for a beam composed of K isotropic layers. Each layer has Young's modulus E_k , Poisson's ratio ν_k , and is situated between $y = h_k$ and $y = h_{k+1}$, where h_k is defined relative to the centroid of the cross section. It is often convenient to use the ratio of the Young's moduli α_k , defined such that $E_k = E\alpha_k$, where E may be chosen as the Young's modulus in any convenient layer. Furthermore, we use the non-dimensional ratio of the stations, $\xi_k = h_k/c$. For convenience in presenting various formula, we define $\Delta_k^n = h_{k+1}^n - h_k^n$ and $\delta_k^n = \xi_{k+1}^n - \xi_k^n$. The weighted area A , the weighted second moment of area I , and a stretching-bending parameter t_b , are defined as follows:

$$A \equiv \sum_{i=1}^K \alpha_i \Delta_k, \quad I \equiv \sum_{i=1}^K \frac{\alpha_i}{3} \Delta_k^3, \quad t_b \equiv \frac{1}{A} \sum_{i=1}^K \frac{\alpha_i}{2} \Delta_k^2.$$

In the following formula, a subscript k is used to represent the stress or strain distribution in the k -th layer, lying between $h_k \leq y \leq h_{k+1}$.

4.1. Axial and bending states

The first fundamental state solution corresponds to a beam subject to a unit axial load that results in the following stress:

$$\sigma_{x(k)}^N = \frac{I}{A} \frac{1}{I - At_b^2} \alpha_k (1 - ry),$$

where $r = t_b A / I$. The strain moments in this fundamental state are:

$$\epsilon_0^N = \frac{2cI}{A} \frac{1}{E(I - At_b^2)}, \quad \kappa^N = -\frac{2c^3 t_b}{3} \frac{1}{E(I - At_b^2)},$$

and $\gamma^N = 0$.

The second fundamental state solution corresponds to a unit bending moment that results in the following stress:

$$\sigma_{x(k)}^M = \frac{1}{I - At_b^2} \alpha_k (y - t_b).$$

The strain moments in this fundamental state are:

$$\epsilon_0^M = -2ct_b \frac{1}{E(I - At_b^2)}, \quad \kappa^M = \frac{2c^3}{3} \frac{1}{E(I - At_b^2)},$$

and $\gamma^M = 0$. Using Equation (10), the relationship between the strain moments and the stress moments, can be determined as follows:

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = E(I - At_b^2) \begin{bmatrix} 2cI/A & -2ct_b \\ -2c^3t_b/3 & 2c^3/3 \end{bmatrix}^{-1} = \frac{3EA}{4c^4} \begin{bmatrix} 2c^3/3 & 2ct_b \\ 2c^3t_b/3 & 2cI/A \end{bmatrix}.$$

This equation defines the constitutive relationship for the first two fundamental states.

4.2. Shear state and shear strain correction

The third fundamental state corresponds to a constant unit shear load. The stresses in the beam corresponding to this case are:

$$\boldsymbol{\sigma}_{(k)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_{(k)} = \frac{1}{2(I - At_b^2)} \begin{bmatrix} 2\alpha_k x(y - t_b) \\ 0 \\ \alpha_k (c_k + 2t_b y - y^2) \end{bmatrix}, \quad (14)$$

where the c_k terms are determined to ensure a continuous variation of the shear stress through the thickness. The c_k coefficient in the first layer is $c_1 = h_1^2 - 2t_b h_1$, and can be obtained for subsequent layers using the following formula:

$$c_k = ((\alpha_{k-1} - \alpha_k)(2t_b h_k - h_k^2) + \alpha_{k-1} c_{k-1}) / \alpha_k.$$

The fundamental state consists of only the stresses corresponding to the axial-invariant components of the solution. These are obtained from Equation (14) by setting $\sigma^Q(y) = \sigma(x = 0, y)$.

The shear strain moment is determined by integrating the shear strain through the thickness:

$$\gamma^Q = \sum_{k=1}^K \frac{(1 + \nu_k)}{E(I - At_b^2)} \left(c_k \Delta_k + t_b \Delta_k^2 - \frac{1}{3} \Delta_k^3 \right).$$

The relationship between the shear stress resultant and the shear strain resultant is, $Q = D_{33}\gamma$, where

$$D_{33} = 1/\gamma^Q. \quad (15)$$

This is not a simple average of the shear-modulus through the thickness, which is often used in beam theories. Equation (15) is a weighted average dependent on the relative distribution of shear through the thickness.

The shear correction factor for the multi-layer beam k_{xy} , is determined from Equation (12). It is a dimensionless quantity that depends only on the Poisson ratio, the relative position of the layers, and the relative magnitudes of the stiffnesses of each layer. As such, it is expressed using dimensionless quantities.

The dimensionless bending-stretching coupling constant τ is given by

$$\tau = \frac{1}{2} \sum_{k=1}^K \alpha_k (\xi_{k+1}^2 - \xi_k^2) \bigg/ \sum_{k=1}^K \alpha_k (\xi_{k+1} - \xi_k).$$

We next introduce the constants C_k , B_k , and A_k , which are defined sequentially for each layer. For $k = 1$, $C_1 = \xi_1^2 - 2\tau\xi_1$, $B_1 = -2(1 + \nu_1)C_1$ and $A_1 = 0$. For each subsequent

layer,

$$\begin{aligned}
C_k &= ((\alpha_{k-1} - \alpha_k)(2\tau\xi_k - \xi_k^2) + \alpha_{k-1}C_{k-1}) / \alpha_k, \\
B_k &= (\nu_{k-1} - \nu_k)(\xi_k^2 - 2\tau\xi_k) + B_{k-1}, \\
A_k &= 2\xi_k \left((1 + \nu_{k-1})C_{k-1} - (1 + \nu_k)C_k + \frac{1}{2}(B_{k-1} - B_k) \right) \\
&\quad + (\nu_{k-1} - \nu_k)(\tau\xi_k^2 - \xi_k^3/3) + A_{k-1}.
\end{aligned}$$

The shear correction factor for the layered, isotropic beam is

$$k_{xy} = D/F, \quad (16)$$

where

$$\begin{aligned}
D &= \sum_{k=1}^K (1 + \nu_k) \left\{ C_k \delta_k + \tau \delta_k^2 - \frac{1}{3} \delta_k^3 \right\}, \\
F &= \sum_{k=1}^K \left\{ \frac{1}{2} \delta_k^3 (2(1 + \nu_k)C_k + B_k) + \frac{1}{40} (2 + \nu_k) (15\tau \delta_k^4 - 4\delta_k^5) \right. \\
&\quad \left. + \frac{3}{4} A_k \delta_k^2 - \frac{1}{2} B_k \delta_k + \frac{\nu_k}{2} (\tau \delta_k^2 - \frac{1}{3} \delta_k^3) \right\}.
\end{aligned}$$

For a single-layer beam, this expression simplifies to Cowper's shear correction factor (2).

4.3. Pressure state and pressure strain correction

The fourth fundamental state corresponds to a pressure load applied to the beam. The total force in the y -direction per unit length of the beam is distributed between a traction on the top surface P_t , and a traction on the bottom surface P_b . Both tractions act in the positive y direction. The total force is such that the contributions sum to unity $P_t + P_b = 1$.

The pressure load causes a linearly varying shear and quadratically varying moment in the beam, resulting in the following state of stress:

$$\boldsymbol{\sigma}^{(k)} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_{(k)} = \frac{1}{2(I - At^2)} \begin{bmatrix} -\alpha_k (x^2 y - t_b x^2 - 2y^3/3 + 2t_b y^2 + e_k y + f_k) \\ \alpha_k (d_k + c_k y + t_b y^2 - y^3/3) \\ -\alpha_k x (c_k + 2t_b y - y^2) \end{bmatrix}. \quad (17)$$

The fundamental state is determined by taking only the axially-invariant components of the stress state given in Equation (17): $\boldsymbol{\sigma}^P(y) = \boldsymbol{\sigma}(x = 0, y)$.

The coefficients d_k are determined from the inter-layer continuity of σ_y , while the coefficients e_k and f_k are used to satisfy two equilibrium equations: $\int_{-c}^c y \sigma_x dy = -x^2/2$ and $\int_{-c}^c \sigma_x dy = 0$, as well as $K - 2$ inter-layer displacement continuity constraints. The d_k coefficients can be determined using the following relationship, $d_1 = -2(I - At_b^2)P_b/\alpha_1 - (c_1 h_1 + t h_1^2 - h_1^3/3)$ for the first layer, and in all subsequent layers using

$$d_k = 1/\alpha_k [(\alpha_{k-1} - \alpha_k)(t_b h_k^2 - h_k^3/3) + h_k(\alpha_{k-1} c_{k-1} - \alpha_k c_k) + \alpha_{k-1} d_{k-1}].$$

The additional equations for the inter-layer continuity of the displacements are

$$(e_k - e_{k-1})h_k - (f_k - f_{k-1}) = (\nu_{k-1} - \nu_k) (t_b h_k^2 - h_k^3/3) - \nu_k (d_k + c_k h_k) + \nu_{k-1} (d_{k-1} + c_{k-1} h_k),$$

$$e_k - e_{k-1} = c_k(2 + \nu_k) - c_{k-1}(2 + \nu_{k-1}) + (\nu_{k-1} - \nu_k)(h_k^2 - 2t_b h_k),$$

for $k = 2, \dots, K$. The two additional equilibrium equations are

$$\sum_{i=1}^K \alpha_k \left\{ \frac{e_k}{3} \Delta_k^3 + \frac{f_k}{2} \Delta_k^2 \right\} = \sum_{i=1}^K \alpha_k \left\{ \frac{2}{15} \Delta_k^5 - \frac{t_b}{2} \Delta_k^4 \right\},$$

$$\sum_{i=1}^K \alpha_k \left\{ \frac{e_k}{2} \Delta_k^2 + f_k \Delta_k \right\} = \sum_{i=1}^K \alpha_k \left\{ \frac{1}{6} \Delta_k^4 - \frac{2t_b}{3} \Delta_k^3 \right\}.$$

Using the values obtained by solving these for e_k and f_k with the above $2K$ equations, the strain moments for this fundamental state can be written as

$$\epsilon_0^P = \frac{1}{2E(I - At_b^2)} \left\{ \frac{4}{3} t_b c^3 + \sum_{k=1}^K \left(\frac{e_k}{2} \Delta_k^2 + f_k \Delta_k + \nu_k \left(d_k \Delta_k + \frac{c_k}{2} \Delta_k^2 + \frac{t_b}{3} \Delta_k^3 - \frac{1}{12} \Delta_k^4 \right) \right) \right\}, \quad (18a)$$

$$\kappa^P = \frac{1}{2E(I - At_b^2)} \left\{ -\frac{4}{15} c^5 + \sum_{k=1}^K \left(\frac{e_k}{3} \Delta_k^3 + \frac{f_k}{2} \Delta_k^2 + \nu_k \left(\frac{d_k}{2} \Delta_k^2 + \frac{c_k}{3} \Delta_k^3 + \frac{t_b}{4} \Delta_k^4 - \frac{1}{15} \Delta_k^5 \right) \right) \right\}. \quad (18b)$$

The shear strain moment for this fundamental state is zero, $\gamma^P = 0$.

5. Equations of motion for an isotropic beam

Before examining several static cases using the shear and pressure corrections derived above, we will briefly examine the natural frequency of vibration of an isotropic beam with a body-force correction. For this isotropic case, $I = 2c^3/3$, $A = 2c$ and $t_b = 0$.

Under a constant body load with a value of $1/2c$, the stress state in an isotropic beam is,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{1}{2I} \begin{bmatrix} -x^2 y + 2y^3/3 - 2c^2 y/5 \\ (c^2 y - y^3)/3 \\ xy^2 - xc^2 \end{bmatrix}. \quad (19)$$

The stresses have a linear varying shear and a quadratically varying bending moment, as in the pressure state described above. From Equation (19), the fundamental state corresponding to a body load is $\boldsymbol{\sigma}^B(y) = \boldsymbol{\sigma}(x = 0, y)$. The strain moments corresponding to this fundamental state are $\epsilon_0^B = 0$, $\gamma^B = 0$, and

$$\kappa^B = -\frac{1}{2EI} \frac{\nu}{3} \left(\frac{2c^5}{3} - \frac{2c^5}{5} \right) = -\frac{\nu c^2}{15E}. \quad (20)$$

The bending moment correction is $M^B = -\nu c^2/15$. Under conditions of free-vibration, the magnitude of this body-force fundamental state is equal to the inertial force per unit span. As a result, Equation (11) becomes

$$M = EI \frac{\partial u_1}{\partial x} + \rho A M^B \frac{\partial^2 v_0}{\partial t^2}. \quad (21)$$

Using this relationship, the equation of motion for a freely vibrating beam is

$$EI \frac{\partial^4 v_0}{\partial x^4} + \rho A \frac{\partial^2 v_0}{\partial t^2} - \rho I \left[1 + \frac{E}{k_{xy} G} + \frac{AM^B}{I} \right] \frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{k_{xy} G} \frac{\partial^4 v_0}{\partial t^4} = 0. \quad (22)$$

The classical equation of motion may be obtained by setting $M^B = 0$. The equation of motion for an isotropic beam, using the body-force correction (20) and Cowper's shear correction factor (2), is

$$EI \frac{\partial^4 v_0}{\partial x^4} + \rho A \frac{\partial^2 v_0}{\partial t^2} - \frac{17 + 10\nu}{5} \rho I \frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{12 + 11\nu}{5} \left(\frac{\rho I}{A} \right)^2 \frac{\partial^4 v_0}{\partial t^4} = 0. \quad (23)$$

While for the classical equation, with Timoshenko's shear correction factor (1), the equation of motion is

$$EI \frac{\partial^4 v_0}{\partial x^4} + \rho A \frac{\partial^2 v_0}{\partial t^2} - \frac{17 + 10\nu}{5} \rho I \frac{\partial^4 v_0}{\partial t^2 \partial x^2} + \frac{12 + 10\nu}{5} \left(\frac{\rho I}{A} \right)^2 \frac{\partial^4 v_0}{\partial t^4} = 0. \quad (24)$$

Equations (23) and (24) differ only in the coefficient of the fourth term by $1/5\nu(\rho I/A)^2$. The relative difference between these terms is 2% for $\nu = 0.3$. This suggests that for vibration problems, using the proposed theory with Cowper's shear correction factor and a body-force correction is essentially equivalent to using Timoshenko's shear correction factor and the equations of motion he originally derived. This agreement should be expected, as experiments based on the natural frequencies of vibration have typically demonstrated that Timoshenko's shear correction factor is superior (Spence and Seldin, 1970; Kaneko, 1975).

6. Results

In this section, we examine the shear strain and pressure corrections, and the constitutive relationship obtained above, for two cases: a three-layer symmetric beam, and a multi-layer beam composed of alternating materials. Results from a finite-element analysis are used to compare with the formulas derived above.

The first beam considered is composed of three layers, where the middle layer is made of a material that has a lower Young's modulus than the outer layers. This problem is designed to model a sandwich structure in which the inner core material is less stiff than the outer material. The outer two layers have Young's modulus E and Poisson's ratio ν , while the inner core has Young's modulus αE and Poisson ratio ν . The depth of the beam is $2c$ and the inner core extends from $y = -rc$ to $y = rc$, where r is the fraction of the beam that is composed of the core material. For this beam, simplifications from the general formulas above are possible. The average shear stiffness (15) simplifies to

$$D_{33} = \frac{3EI}{2(1 + \nu)(2c^3 - 3c^3 r(1 - s))}, \quad (25)$$

where $s = (1 - (1 - \alpha)r^2)/\alpha$ and the shear correction factor (16) becomes

$$k_{xy} = \frac{(1 + \nu)(30r(s - 1) + 20)}{30(1 + \nu)s - (6 + 8\nu) + 15(1 + \nu)(1 - s)(2 + r^3 - 3r)}. \quad (26)$$

As the ratio of the Young's modulus of the core decreases, it is interesting to note that a limiting case is reached that is independent of the Poisson's ratio. This limit as $\alpha \rightarrow 0$ is,

$$k_{xy} = \frac{2}{3 - \nu^2}. \quad (27)$$

The second beam considered is composed of alternating isotropic layers that have relative Young's modulus $E_1/E_2 = 10$ and Poisson's ratios of $\nu_1 = 0.2$ and $\nu_2 = 0.4$. For this case, we vary the number of layers, keeping the depth of the beam constant, $c = 1/2$ while altering the thickness of the layers to match. As a result $h_k = -c + 2c(k - 1)/K$. The plies are composed of alternating material starting from the bottom layer. The beam is symmetric for odd K .

For finite-element calculations, we use bi-cubic Lagrange plane stress elements with a standard formulation. We choose these high-order elements because they capture the piecewise parabolic shear stress accurately through the thickness of the beam.

The finite-element model is constructed with $L/2c = 10$ with 50 elements along the length of the beam. For the three-layer beam, we take 20 elements through the thickness resulting in 18422 degrees of freedom. For the multi-layer beam the number of through-thickness elements varies so that the number of elements in each layer is the same, while the total number of elements through the thickness does not fall below 20. The number of elements through the thickness is $K \lceil 20/K \rceil$.

In order to compare the value of the shear correction factor derived above with finite-element results, we use results from a beam subject to a shear load at the tip, with the root fully fixed. An approximate shear correction factor is computed from the finite-element solution based on Equation (12). This approximate shear correction factor, k_{xy}^{FE} , is computed as follows

$$k_{xy}^{FE}(x) = \frac{\int_{-c}^c \gamma_{xy} dy}{2c \left[u_1 + \frac{\partial v_0}{\partial x} \right]}, \quad (28)$$

where numerical integration is used to evaluate u_1 and v_0 from the finite-element results based on Equation (3), and the derivative is performed using a central-difference calculation with $\Delta x = 10^{-5}$. k_{xy}^{FE} is calculated at every Gauss point along the x -direction.

For comparison with the pressure corrections, we calculate a solution of a cantilevered beam subject to a pressure load distributed on the top and bottom with $P_b = 1/5$ and $P_t = 4/5$. The pressure correction is evaluated using a combination of finite-element and beam theory values where the total strain and stress moments are computed from the finite-element method, while the constitutive relation is used from Equation (9). This gives the following equation for the pressure correction to the axial resultant:

$$N_{FE}^P = 2cD_{11} \left. \frac{\partial u_0}{\partial x} \right|_{FE} + \frac{2c^3}{3} D_{12} \left. \frac{\partial u_1}{\partial x} \right|_{FE} - N_{FE}. \quad (29)$$

Similar expressions are used for the bending and shear corrections.

Typical results for the variation of the approximate shear correction factor, shear stiffness and approximate pressure corrections with axial direction are plotted in Figure 3 for the multi-layer beam with $K = 5$. These show that there is a strong variation of these approximations close to the ends of the beam but that these variations quickly settle to a constant value over most of the length of the beam. For all comparisons that follow,

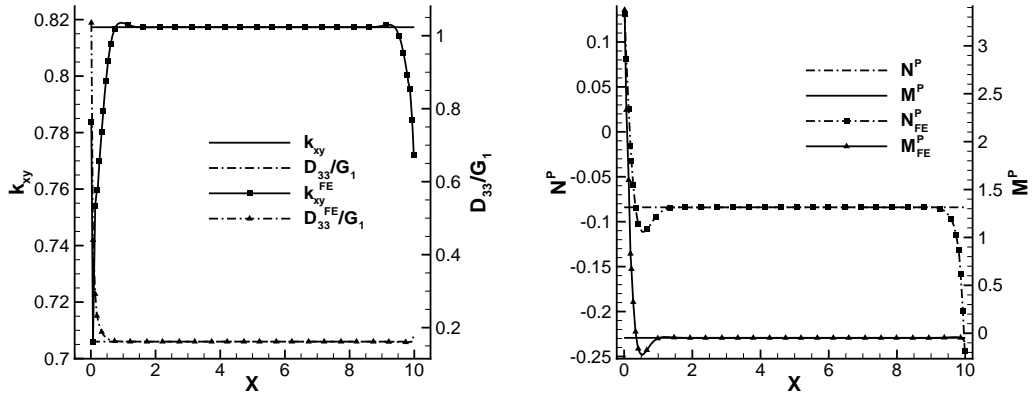


Figure 3: The variation of the approximate shear correction factor, \tilde{k}_{xy} , homogenized shear stiffness D_{33} and the pressure corrections N^P and M^P per unit length of the multi-layer beam with $K = 5$. These results clearly show the end effects.

we average the approximate shear correction factor (28), the shear stiffness and the approximate pressure corrections (29) obtained from the finite-element method over the span $x = 4$ to $x = 6$.

Figure 4 shows the variation of the shear correction factor and the average shear stiffness computed using Equations (26) and (25) respectively. The finite-element calculations were performed at a core ratios of $r = 0.2, 0.5, 0.8, 0.9, 0.95, 0.98$ and at relative stiffness ratios of $\alpha = 1, 0.5, 0.1, 0.01$. Good agreement is obtained at all values. Figure 4 shows the limiting case from Equation (27) for zero core stiffness.

Figure 5 shows the variation of the shear correction factor computed using the general form from Equation (16) and the homogenized shear stiffness computed using Equation (15) for the multi-layer beam. Finite-element calculations were performed for the first 10 beams with $K = 1, \dots, 10$, while the analytic formulas are used up to $K = 50$ to show the trend. As previously mentioned, for odd K the beams are symmetric, and for even K the beams exhibit bending-stretching coupling. As K becomes larger, the coefficients tend towards a limiting case. Excellent agreement is obtained. The average relative error for $K = 1, \dots, 10$ is 3.3×10^{-6} and 1.7×10^{-5} for the shear strain correction and shear stiffness respectively, while the maximum errors are 1.0×10^{-5} and 9.3×10^{-5} , respectively.

Figure 6 shows the pressure corrections for the axial resultant and bending moment for the multi-layer beam. The theoretical results were computed by first finding the strain moment corrections from Equation (18a) and Equation (18b) and multiplying by the average constitutive relation from Equation (9). The average relative error for the pressure corrections are 3.6×10^{-5} and 4.6×10^{-5} for N^P and M^P , respectively, while the maximum errors are, 1.8×10^{-4} and 1.2×10^{-4} . These results demonstrate that the constitutive equation is modified by the presence of an externally applied pressure load, otherwise the predicted correction would be zero. In addition, these results show that

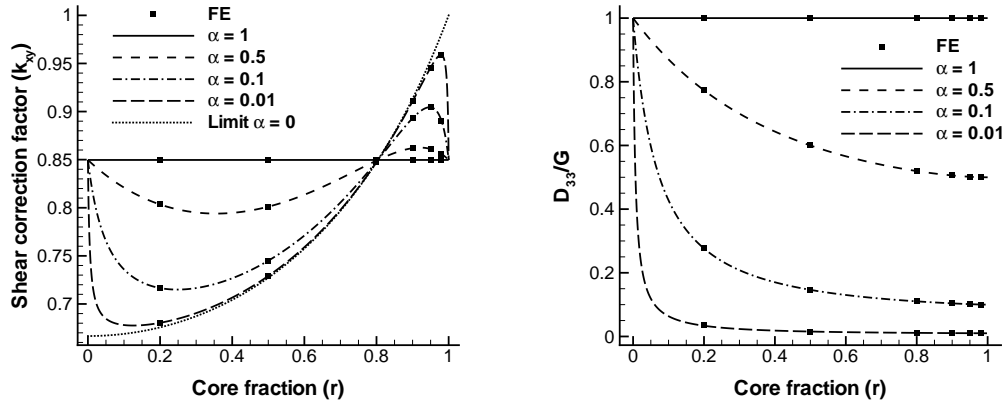


Figure 4: A comparison between the shear correction factor k_{xy} and homogenized shear stiffness D_{33} computed using the theory and the finite-element method for the three-layer beam.

these corrections are correctly predicted by Equation (18).

6.1. Impact of the corrections

We have demonstrated good agreement between the shear strain correction factor and the pressure correction when compared with finite-element computations. To put these results in perspective, it is necessary to assess the relative importance of these values in predicting the stress or strain distribution and the deflection of a beam. This is a complex task that is highly problem-dependent. To make a concrete comparison, we examine two cases: the deflection of a tip-loaded cantilever beam and the stress distribution in a clamped-clamped pressure loaded beam.

For the case of the tip-loaded beam, we assess the importance of the shear correction factor and homogenized shear stiffness. With no stretching-bending coupling, D_{12} and D_{21} are zero and the tip deflection is

$$v_0(L) = Q \left[\left(\frac{L}{2c} \right)^3 \frac{4}{D_{22}} + \left(\frac{L}{2c} \right) \frac{1}{D_{33}k_{xy}} \right].$$

The two terms in this expression represent a contribution to the deflection from the bending stiffness and a contribution from the shear stiffness. The ratio of these two terms is,

$$r_{sb} = \left(\frac{2c}{L} \right)^2 \frac{D_{22}}{4D_{33}k_{xy}},$$

where r_{sb} is the shear to bending displacement ratio. Clearly the slenderness ratio, $S_r = L/2c$, is the most important single factor. For an isotropic beam, $D_{22} = E$ and $D_{33} = G$, with k_{xy} equal to Cowper's shear correction factor (2). For a Poisson ratio of $\nu = 0.3$, this results in $r_{sb} = 0.765S_r^{-2}$. For a reasonable slenderness ratio of $S_r > 10$, the shear contributes very little to the deflection. On the other hand, for the three-layer

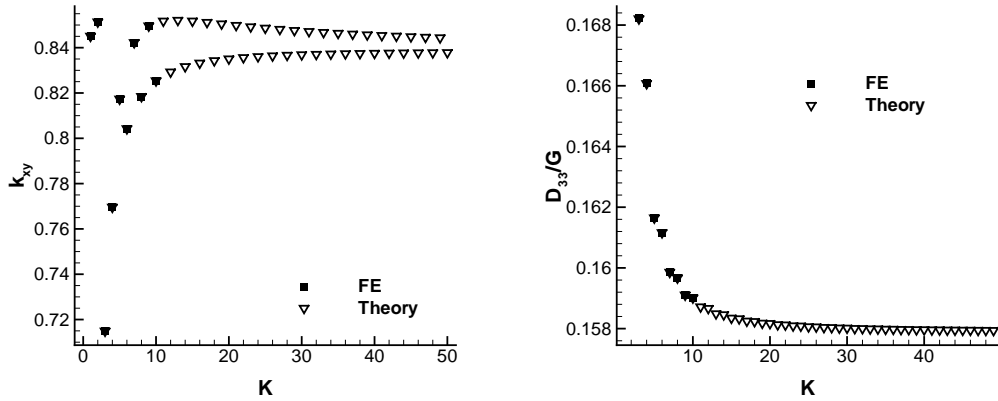


Figure 5: A comparison between the shear correction factor k_{xy} and homogenized shear stiffness D_{33} computed by theory and the finite-element method for the multi-layer beam.

symmetric beam discussed above, with $\nu = 0.3$, $\alpha = 0.01$, and a core ratio $r = 0.95$, the shear to bending displacement ratio is $r_{sb} = 10.14S_r^{-2}$. This suggests that the shear stiffness plays a much more important role in beams of this construction. Correct determination of the shear strain correction factor and homogenized shear stiffness is much more important for beams that have low shear stiffness such as sandwich beams.

We now examine a clamped-clamped beam subject to a distributed pressure load on the top and the bottom surfaces, $P_t = 4/5$ and $P_b = 1/5$. The beam is composed of alternating layers as described above for the $K = 5$ case. The dimensions of the beam are $L = 10$ and $c = 1/2$.

The pressure correction causes two effects: a modification of the constitutive relation, and additional contributions to the stress reconstruction (7). Using the constitutive Equation (9) and the force method, the stress resultants can be determined:

$$\begin{aligned}
 N(x) &= -PN^P, \\
 M(x) &= P \left(\frac{x}{2}(L-x) - \frac{L^2}{12} - M^P \right), \\
 Q(x) &= P \left(\frac{L}{2} - x \right).
 \end{aligned}$$

Note that even though the beam is symmetric, there is a non-zero axial compressive force and moment offset. This is due to the strain moments caused by the pressure on the top and bottom surface of the beam.

Figure 7 shows a comparison of σ_y and σ_{xy} predicted by the stress reconstruction and the finite-element method over the length of beam at a location $y = 0.6c$. These results show very good agreement between the stress reconstruction and the finite-element results. Neglecting the fundamental state corresponding to pressure would result in $\sigma_y = 0$.

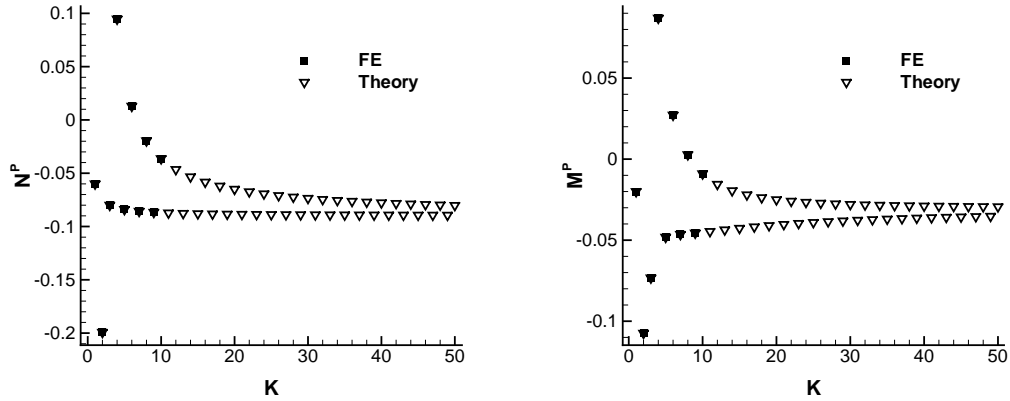


Figure 6: A comparison between the pressure corrections N^P and M^P computed using the theory $K = 1 \dots 50$ and the finite-element method $K = 1 \dots 10$.

7. Conclusions

A Timoshenko beam theory for layered orthotropic beams has been presented in this paper. Following the work of [Prescott \(1942\)](#) and [Cowper \(1966\)](#), the beam kinematics are developed in terms of average through-thickness displacement and rotation variables. The proposed theory includes a consistent method for calculating the stiffness of the beam, the shear strain correction factor, and the strain-moment corrections for externally applied loads. These values are based on the axially-invariant fundamental state solutions. We have demonstrated that the present approach easily handles layered beam constructions through the use of both stress and strain moments that admit solutions where components of stress and strain may be discontinuous across interfaces. The external load corrections proposed in the theory modify the constitutive relationship and the equations of motion. The analysis presented suggests that for vibration problems, using the proposed theory with Cowper's shear correction factor and a body-force correction is essentially equivalent to using Timoshenko's shear correction factor with the original equations of motion he derived. On the other hand, numerical comparisons using static analysis demonstrated the accuracy and the consistency of the definitions of the shear strain correction factor and the external load corrections. Both static and dynamic situations are treated by the theory without inconsistency, as a result of the external load correction terms.

Acknowledgments

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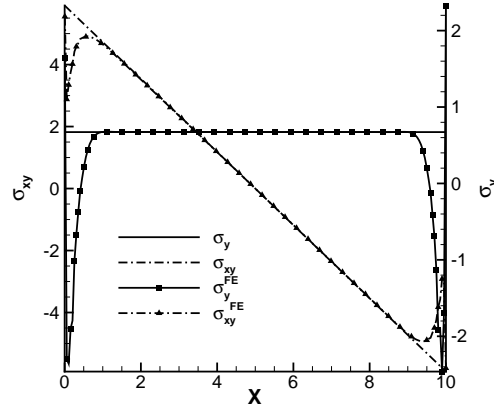


Figure 7: A slice of the stress distribution at $y = 0.6c$ for a clamped-clamped beam with distributed pressure load on the top and bottom surface, $P_t = 4/5$ and $P_b = 1/5$. This location corresponds to an interface between the top and next lowest layers.

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