Quantum Corrections to Spinning String in AdS$_5 \times$ S$^5$

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Summary

- Review of String / Gauge theory duality (AdS/CFT)
- Folded Spinning string solution (Gubser, Klebanov, Polyakov, 02)
- Short spinning string, Quantum corrections (A. Tseytlin, A. T, 08, M. Beccaria, A. T., to appear)
- Long spinning string, Quantum Corrections (M. Beccaria, V. Forini, A. Tseytlin, A. T., to appear)
- Conclusions
AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

**Gauge theory**

\[ \mathcal{N} = 4 \text{ SYM} \text{ SU}(N) \text{ on } \mathbb{R}^4 \]

\[ A_\mu, \Phi^i, \Psi^a \]

Operators w/ conf. dim. \( \Delta \)

**String theory**

IIB on AdS\(_5\)xS\(_5\)

radius \( R \)

String states w/ \( E = \frac{\Delta}{R} \)

\[ g_s = g_{YM}^2 \; ; \quad R / l_s = (g_{YM}^2 N)^{1/4} \]

\[ N \to \infty , \; \lambda = g_{YM}^2 N \; \text{fixed} \]

\( \lambda \) large \( \to \) string th.

\( \lambda \) small \( \to \) field th.
**Folded spinning string**
(Gubser, Klebanov, Polyakov)

\[ ds^2 = - \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2 \]

\[ t = \kappa \tau, \quad \phi = w \tau, \quad \rho = \rho(\sigma) \]

Equation for \( \rho \):

\[ \rho' - \kappa \sqrt{1 - \eta \sinh^2 \rho} \]

\[ \coth^2 \rho_0 = \frac{w^2}{\kappa^2} \equiv 1 + \eta \]

Complicated classical solution:

\[ \sinh \rho = \frac{1}{\sqrt{\eta}} \, \text{sn} \left[ \kappa \sqrt{\eta} \, \sigma, -\frac{1}{\eta} \right] \]

Dual to minimal twist gauge theory operator \( \text{tr}(\Phi D_+ S \Phi) \)
Periodicity condition implies

\[ \kappa = \frac{1}{\sqrt{\eta}} \ _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \]

Energy and spin

\[ E = \sqrt{\lambda} \mathcal{E} \quad S = \sqrt{\lambda} \mathcal{S} \]

\[ \mathcal{E} = \frac{1}{\sqrt{\eta}} \ _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}\right) \quad \mathcal{S} = \frac{\sqrt{1+\eta}}{2\eta \sqrt{\eta}} \ _2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta}\right) \]

Cannot obtain exactly \( \mathcal{E} = \mathcal{E}(\mathcal{S}) \)

Perturbatively in large \( \mathcal{S} \)

\[ E - S = \frac{\sqrt{\lambda}}{\pi} \ln \mathcal{S} + ... \]

\( \ln \mathcal{S} \) scaling obtained also on the gauge theory side
Difficult to quantize string on $AdS_5 \times S^5$

solution:

construct various classical solutions at quantize them semi-classically

starting action for string in $AdS_5 \times S^5$

(Metsaev, Tseytlin, 98)

\[ S = T \int d^2 \sigma \left[ G_{mn}(x) \partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta \partial x + \bar{\theta}\theta \partial x \partial x + ... \right] \]

-- complicated solution – hard to quantize semi-classically even at 1-loop

-- this is the case for folded string solution

-- possible to quantize in different limits.
Short spinning string -- Quantum corrections

Folded string solution in flat space

\[ ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 \]

Solution is

\[ t = c\tau, \quad \rho = c \sin \sigma, \quad \phi = \tau \]

string tension like in AdS

\[ T = \frac{1}{2\pi \alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi} \]

Classical Energy and Spin satisfy usual flat-space Regge relation

\[ E_0 = \epsilon \sqrt{\lambda} \quad S = \frac{\epsilon^2}{2} \sqrt{\lambda} \]
\[ E_0(S, \lambda) = \lambda^{1/4} \sqrt{2S} \]

This is exact in flat space

**Folded string solution in AdS**

\[ 0 < \rho < \rho_{\text{max}} \quad \text{coth} \, \rho_{\text{max}} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}} \]

\[ \sinh \rho = \epsilon \, \text{sn}(\kappa \epsilon^{-1} \sigma, -\epsilon^2) \]

\( c \) measures the length of the string

We expand in small \( \epsilon \)

\[ \rho_{\text{max}} = c - \frac{1}{6} c^3 + O(c^5) \]
\[ \epsilon = \sqrt{2S} - \frac{1}{4\sqrt{2}} S^{3/2} + \ldots \]

Short string limit corresponds to small semi-classical spin \( S \ll 1 \)

Classical energy

\[ E_0(S, \lambda) = \lambda^{1/4} \sqrt{2S} + \frac{3}{4\sqrt{2}} \lambda^{-1/4} S^{3/2} + O(S^{5/2}) \]

This small spin expansion is an example of a near flat space expansion: the leading-order in solution \( \epsilon \) can be identified with the folded spinning string solution in the flat space.
Quantum corrections

\[ \left( \frac{1}{\sqrt{\lambda}} \right) \text{ Corrections respect the structure at classical level} \]

Semiclassical quantization

\[ \lambda \gg 1, \quad \frac{S}{\sqrt{\lambda}} = \text{fixed} \ll 1 \]

Energy has the following structure

\[ E(S, \lambda) = \lambda^{1/4} \sqrt{2S} \left[ h_0(\lambda) + h_1(\lambda)S + h_2(\lambda)S^2 + \ldots \right] \]

\[ h_n = \frac{1}{(\sqrt{\lambda})^n} \left( a_{n0} + \frac{a_{n1}}{\sqrt{\lambda}} + \frac{a_{n2}}{(\sqrt{\lambda})^2} + \ldots \right) \]
Classical string

\[
a_{00} = 1, \quad a_{10} = \frac{3}{8}, \quad a_{20} = -\frac{21}{128}, \ldots
\]

1-loop string computation gives

\[
a_{01} = 1, \quad a_{11} = \frac{41}{64} - \frac{1}{2} \zeta(3) \approx 0.039
\]

UV finiteness of superstring implies

\[
h_0(\lambda) = 1
\]

**Gauge theory**

Corresponding operator in SL(2) sector

low twist operator \( \text{tr}(\Phi D^S_+ \Phi) \) with \( S \sim 1 \)
anomalous dimension scale as
(A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, V.N. Velizhanin)

\[ \Delta(S, \lambda) = q_1(\lambda)S + q_2(\lambda)S^2 + O(S^3) \]

\[ q_1(\lambda) = 1 + d_{01}\lambda + d_{02}\lambda^2 + \ldots \]

\[ q_2(\lambda) = d_{21}\lambda + d_{22}\lambda^2 + \ldots \]

\[ \lambda \ll 1, \quad S = \text{fixed} \]

formally expanded in small $S$ limit
cannot directly continue string expansion to small $S$ and small $\lambda$
To relate the "small spin" string theory and gauge theory expansions one would need to re-sum the series in both arguments \( (\lambda, S) \) and then re-expand the result first in large \( \lambda \) for fixed \( S \) and then in small \( S \).

\[
S = \frac{S}{\sqrt{\lambda}}
\]

1-loop correction at strong coupling – some details

Work in conformal gauge with flat 2d metric expand the \( AdS_5 \times S^5 \) superstring action near solution at quadratic order in fluctuations for bosons and fermions
\[ \tilde{L}_B = -\partial_a \tilde{t} \partial^a \tilde{t} - \mu_t^2 \tilde{t}^2 + \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu_\phi^2 \tilde{\phi}^2 \]
\[ + 4\tilde{\rho}(\kappa \sinh \rho \partial_0 \tilde{t} - w \cosh \rho \partial_0 \tilde{\phi}) + \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \mu_\rho^2 \tilde{\rho}^2 \]
\[ + \partial_a \beta_u \partial^a \beta_u + \mu_\beta \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s , \]
\[ \mu_\phi^2 = 2\rho^2 - w^2 , \quad \mu_\rho^2 = 2\rho^2 - w^2 - \kappa^2 , \quad \mu_\beta = 2\rho^2 \]

\( \beta_u \ (u = 1, 2) \) AdS\(_5\) fluctuations transverse to AdS\(_3\)
\( \varphi, \chi_s \ (s = 1, 2, 3, 4) \) fluctuations in S\(_5\)

The fermionic part of the quadratic fluctuation Lagrangian -- 4+4 2d Majorana fermions with \( \sigma \) -dependent mass

\[ \tilde{L}_F = 2i(\bar{\Psi} \gamma^a \partial_a \Psi - \mu_F \bar{\Psi} \Gamma_{234} \Psi) , \quad \mu_F^2 = \rho^2 \]
Expanding coefficients in small $\epsilon$

$$\mu_r^2 = \epsilon^2 \cos 2\sigma + \ldots, \quad \mu_\phi^2 = -1 + (\cos 2\sigma + \frac{1}{2})\epsilon^2 + \ldots,$$

$$\mu_\rho^2 = -1 + (\cos 2\sigma - \frac{1}{2})\epsilon^2 + \ldots, \quad \mu_\beta^2 = 2\mu_F^2 = 2\epsilon^2 \cos^2 \sigma + \ldots$$

Fluctuation Lagrangian is $\sigma$ dependent, not easy to compute spectrum

1-loop correction to string energy

$$E_1 = \frac{\Gamma_1}{\kappa T} \quad T \equiv \int d\tau \to \infty$$

Fluctuation Lagrangian does not depend on time

$$\det [-\partial_1^2 - \partial_0^2 + 2\epsilon^2 \cos^2 \sigma] = T \int \frac{d\omega}{2\pi} \det [-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma]$$
We can now use perturbation theory in $\epsilon^2$

\[
\ln \frac{\det[A + \epsilon^2 B]}{\det A} = \epsilon^2 Tr[A^{-1} B] + O(\epsilon^4)
\]

\[
Z = \frac{\det^{\frac{8}{2}}[-\partial^2_0 - \partial^2_1 + \epsilon^2 \cos^2 \sigma] \det^{\frac{2}{2}}[-\partial^2_0 - \partial^2_1]}{\det^{\frac{2}{2}}[-\partial^2_0 - \partial^2_1 + 2\epsilon^2 \cos^2 \sigma] \det^{\frac{5}{2}}[-\partial^2_0 - \partial^2_1] \det^{\frac{1}{2}} Q}
\]

Example: for decoupled bosons

\[
\ln \frac{\det[-\partial^2_1 + \omega^2 + 2\epsilon^2 \cos^2 \sigma]}{\det[-\partial^2_1 + \omega^2]} \approx \epsilon^2 \sum_n \frac{2}{n^2 + \omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \sigma
\]

\[
= \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2}
\]

Q is 3 x 3 matrix coupled fluctuation operator
Leading 1-loop $\epsilon^2$ correction to energy vanishes
Expected energy is like in flat space. It should be true to all loops.

Higher order in expansion to get first non-zero coeff. $\epsilon^4$.

\[
\ln \frac{\det[A + \epsilon^2 B + \epsilon^4 C]}{\det A}
\]

\[
= \epsilon^2 \text{Tr}[A^{-1} B] - \frac{\epsilon^4}{2} \text{Tr}[A^{-1} B A^{-1} B] + \epsilon^4 \text{Tr}[A^{-1} C]
\]

A is massless propagator; B,C are $\sigma$-dependent insertions. Technically more involved.
The result is:
\[
\Gamma_1(\epsilon^4)
= -\frac{T \epsilon^4}{4\pi} \int_{-\infty}^{\infty} d\omega \left\{ \sum_n \left[-\frac{7}{8 n^2 + \omega^2} - \frac{1}{32 n^2 + (\omega + i)^2} - \frac{1}{32 n^2 + (\omega - i)^2}\right] \right. \\
+ \left. \frac{1}{2} \sum_n \left[-\frac{\omega^2}{[n^2 + (\omega + i)^2]^2} - \frac{\omega^2}{[n^2 + (\omega - i)^2]^2}\right] \right. \\
+ \left. \frac{1}{4 n^2 + \omega^2} \left(\frac{1}{(n-2)^2 + \omega^2} + \frac{1}{(n+2)^2 + \omega^2}\right) + \frac{1}{2} \frac{1}{[n^2 + (\omega + i)^2][n^2 + (\omega - i)^2]} \right. \\
+ \left. \omega^2 \left(\frac{1}{(n+1)^2 + \omega^2} + \frac{1}{(n-1)^2 + \omega^2}\right) \left(\frac{1}{n^2 + (\omega + i)^2} + \frac{1}{n^2 + (\omega - i)^2}\right) \right. \\
+ \left. \frac{(1 + i\omega)^2}{4} \frac{1}{n^2 + (\omega - i)^2} \left(\frac{1}{(n-2)^2 + (\omega - i)^2} + \frac{1}{(n+2)^2 + (\omega - i)^2}\right) \right. \\
+ \left. \frac{(1 - i\omega)^2}{4} \frac{1}{n^2 + (\omega + i)^2} \left(\frac{1}{(n-2)^2 + (\omega + i)^2} + \frac{1}{(n+2)^2 + (\omega + i)^2}\right) \right\} 
\]
Remarkable both sum and then the integral can be computed exactly.

The summation gives

\[
\sum_{n=3}^{\infty} S_n = \frac{\pi^2(\omega^2 + 1)\text{csch}^2 \pi \omega}{2\omega^2} + \frac{\pi(5\omega^2 + 4) \coth \pi \omega}{8\omega^3(\omega^2 + 1)} - \frac{53}{48(\omega^2 + 1)} - \frac{27}{32(\omega^2 + 4)}
\]

\[
+ \frac{3}{16(\omega^2 + 9)} + \frac{19}{96(\omega^2 + 16)} - \frac{5}{8\omega^2} - \frac{1}{4(\omega^2 + 1)^2} + \frac{6}{(\omega^2 + 4)^2} - \frac{1}{\omega^4}
\]

1-loop correction to energy

\[
E_1 = \frac{1}{\sqrt{2}} \left[ \frac{41}{32} - \zeta(3) \right] S^{3/2} + O(S^{5/2})
\]

\(\zeta(3)\) also in dimensions of short operators at weak coupling
Generalization to non-zero $J$ in $S^5$

String spinning in AdS, and around a big circle in $S^5$

Important for relation to SL(2) sector operators

$$\text{tr}(D^S_+ \Phi^J)$$

$J$ interpreted as the length of the corresponding spin chain

Expanding in short string limit $\epsilon \ll 1$ two possible cases

- if $\nu = J \gg 1$ fast short string, BMN like limit

$$E_0 = \nu + S + \frac{S}{2\nu^2} + \ldots , \quad \nu \gg 1, \quad \frac{S}{\nu} \ll 1$$
• if $\nu \ll \sqrt{S} \ll 1$ slow short string limit

Classical energy has near flat-space expansion

$$E_0 = \sqrt{2S} \left(1 + \frac{\nu^2}{4S} + \ldots\right) + \frac{3}{4\sqrt{2}}S^{3/2} \left(1 + \frac{5\nu^2}{12S} + \ldots\right) + \ldots$$

1-loop computation in the second case

Result:

$$E = \lambda^{1/4} \sqrt{2S} \left[1 + \frac{J^2}{4\sqrt{\lambda}S} (1 + 0 + \ldots) - \frac{J^4}{32\lambda S^2} (1 + 0 + \ldots) + O(J^6)\right]$$

$$+ \frac{3}{4\sqrt{2}} \lambda^{-1/4} S^{3/2} \left[\left(1 + \frac{4}{3\sqrt{\lambda}} \left(\frac{41}{32} - \zeta(3)\right) + \ldots\right) + \frac{J^2}{\sqrt{\lambda} S} \left(\frac{5}{12} + \frac{1}{3\sqrt{\lambda}} + \ldots\right) \right.$$

$$\left.- \frac{J^4}{\lambda S^2} \left(\frac{7}{96} + \frac{1}{12\sqrt{\lambda}} + \ldots\right) + O(J^6)\right] + O(S^{5/2})$$
Computed 1-loop correction to order $S^{5/2}$

The result contains rational numbers, $\zeta(3)$ and $\zeta(5)$

Higher order in $S$, more zeta functions appear at $J=0$

Interesting to compute two-loop string corrections but hard, and, of course, to sum up the series

Understand strong coupling limit of anomalous dimension $\Delta$ for short operators -- finite $S$

Beyond asymptotic BA
Long spinning string -- Quantum Corrections

Start with spinning string solution

\[ \sinh \rho = \frac{1}{\sqrt{\eta}} \ \text{sn} \left[ \kappa \sqrt{\eta} \ \sigma, -\frac{1}{\eta} \right], \quad 0 \leq \sigma \leq \frac{\pi}{2} \]

Maximum length \( \rho_0 \)

\[ \cot h^2 \rho_0 = \frac{w^2}{\kappa^2} \equiv 1 + \eta \]

Small \( \eta \) expansion

\[ \eta \rightarrow 0 \quad \text{solution is} \quad \rho = \kappa_0 \sigma \quad \kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta} \]

String touches the boundary of AdS \( \rho_0 = \infty \)
At leading order this leads to the energy $E - S \sim \log S$
Here we want to go to next orders in large $S$
Solution can be expanded as

$$\sinh \rho = \sinh(\kappa_0 \sigma) - \frac{\eta}{8} \left[ \sinh(2\kappa_0 \sigma) - \frac{4}{\pi} \sigma \right] \cosh(\kappa_0 \sigma) + \mathcal{O}(\eta^2)$$

Energy and spin expansion

$$E = \frac{2}{\pi \eta} + \frac{\pi \kappa_0 + 1}{2\pi} - \frac{\eta}{32\pi} (2\pi \kappa_0 - 3) + \mathcal{O}(\eta^2)$$

$$S = \frac{2}{\pi \eta} - \frac{\pi \kappa_0 - 3}{2\pi} - \frac{\eta}{32\pi} (2\pi \kappa_0 + 13) + \mathcal{O}(\eta^2)$$

Next to leading order string does not touch the boundary

Classical energy is given by
\[ E = \sqrt{\lambda} \mathcal{E}(S), \quad S = \frac{S}{\sqrt{\lambda}}, \]

\[ \mathcal{E}(S)_{S \gg 1} = S + a_0 \ln S + a_c + \frac{1}{S} (a_{11} \ln S + a_{10}) \]

\[ + \frac{1}{S^2} (a_{22} \ln^2 S + a_{21} \ln S + a_{20}) + \mathcal{O}\left(\frac{\ln^3 S}{S^3}\right) \]

\[ a_0 = \frac{1}{\pi}, \quad a_c = \frac{1}{\pi} (\ln 8\pi - 1) \]

Expect the same structure when including string loop corrections – check at 1-loop. Structure is:

\[ E = S + f \ln S + f_c + \frac{1}{S} [f_{11} \ln S + f_{10}] \]

\[ + \frac{1}{S^2} [f_{22} \ln^2 S + f_{21} \ln S + f_{20}] + \mathcal{O}\left(\frac{\ln^3 S}{S^3}\right) \]
coefficients \( f, f_c, f_{11}, \ldots \) receive \( \frac{1}{(\sqrt{\lambda})^n} \) corrections:

\[
f = \frac{\sqrt{\lambda}}{\pi} \left( 1 - \frac{3 \ln 2}{\sqrt{\lambda}} + \ldots \right) \quad f_{11} = \frac{\lambda}{2\pi^2} \left( 1 - \frac{6 \ln 2}{\sqrt{\lambda}} + \ldots \right)
\]

\[
f_c = \frac{\sqrt{\lambda}}{\pi} \left( \ln \frac{8\pi}{\sqrt{\lambda}} - 1 - \frac{3 \ln 2}{\sqrt{\lambda}} \ln \frac{8\pi}{\sqrt{\lambda}} + \ldots \right)
\]

\[
f_{10} = \frac{\lambda}{2\pi^2} \left[ \ln \frac{8\pi}{\sqrt{\lambda}} - 1 - \frac{3 \ln 2}{\sqrt{\lambda}} (2 \ln \frac{8\pi}{\sqrt{\lambda}} - 1) + \ldots \right]
\]

String side: \( \sqrt{\lambda} \gg 1 \quad \frac{S}{\sqrt{\lambda}} \) =fixed and then \( \frac{S}{\sqrt{\lambda}} \gg 1 \)

Gauge theory side: \( \lambda \ll 1, \quad S\) =fixed and then \( S \gg 1 \)

Remarkable one obtains the same structure
Anomalous dimension for twist two scalar operators at four loops obtained from asymptotic BA.

\[ \gamma(S)_{S \gg 1} = f \ln \bar{S} + \bar{f}_c + \frac{f_{11} \ln \bar{S} + \bar{f}_{10}}{S} + \frac{f_{22} \ln^2 \bar{S} + \bar{f}_{21} \ln bS + \bar{f}_{20}}{S^2} + \frac{f_{33} \ln^3 \bar{S} + \bar{f}_{32} \ln^2 \bar{S} + \bar{f}_{31} \ln \bar{S} + \bar{f}_{30}}{S^3} + \mathcal{O}\left(\frac{\ln^4 bS}{S^4}\right). \]

\[ \bar{S} = e^{\gamma_E} S \]

Coefficients are power series in \[ \hat{\lambda} = \frac{\lambda}{16\pi^2} \]

Functions \( f, f_c, f_{11}, \ldots \) are interpolating functions.

Anomalous dimension for twist two scalar operators \( \text{Tr}(\Phi D_+^S \Phi) \) at four loops obtained from asymptotic BA. (Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07)

\[ f = 8\hat{\lambda} - \frac{8 \pi^2}{3} \hat{\lambda}^2 + \frac{88 \pi^4}{45} \hat{\lambda}^3 - \left( \frac{584 \pi^6}{315} + 64 \zeta_3^2 \right) \hat{\lambda}^4 \]

\[ \bar{f}_c = -24 \zeta_3 \hat{\lambda}^2 + \left( \frac{16}{3} \pi^2 \zeta_3 + 160 \zeta_5 \right) \hat{\lambda}^3 + \left( -\frac{56}{15} \pi^4 \zeta_3 - \frac{80}{3} \pi^2 \zeta_5 - 1400 \zeta_7 \right) \hat{\lambda}^4 \]

\[ f_{11} = 32 \hat{\lambda}^2 - \frac{64 \pi^2}{3} \hat{\lambda}^3 + \frac{96 \pi^4}{5} \hat{\lambda}^4 \]
f is universal function related to cusp anomaly of light-like Wilson loops

Interesting property: coefficients of \( \frac{\ln^k}{S^k} \) seem to be universal in twist and flavor. all these coefficients can be determined from f:

\[
\gamma(S)_{S \gg 1} = \int \ln S + f_c + \frac{f_{11} \ln S + f_{10}}{S} + \frac{f_{22} \ln^2 S + f_{21} \ln S + f_{20}}{S^2} + \frac{f_{33} \ln^3 S + f_{32} \ln^2 S + f_{31} \ln S + f_{30}}{S^3} + \mathcal{O}\left(\frac{\ln^4 S}{S^4}\right)
\]

\[f_{11} = \frac{1}{2} f^2, \quad f_{22} = -\frac{1}{8} f^3, \quad f_{33} = \frac{1}{24} f^4, \quad \ldots\]
Why these functional relations happen?  
(B. Basso, G.P. Korchemsky, 07)

-- operators \( \text{tr}(D^S_+ \Phi J) \) classified according to representations of \( \text{SL}(2,\mathbb{R}) \) subgroup of \( \text{SO}(2,4) \)

-- representations labeled by conformal spin \( s = \frac{1}{2}(S + \Delta) \)

-- argue that anomalous dimension is a function of \( S \) only through conformal spin

\[
\Delta = S + J + \gamma(S, J)
\]

-- implies the existence of a simpler function \( f \)

\[
\gamma(S) = f\left(S + \frac{1}{2} \gamma(S)\right) \quad \text{``functional relation''}
\]
Function $f$ simpler and more fundamental: should not contain $\frac{\ln^k S}{S^k}$ in large $S$

gauge theory large $S$ expansion consistent with functional relation:

$$\gamma(S) = f \ln \left( S + \frac{1}{2} f \ln S + \ldots \right) + \ldots$$

$$= f \ln S + \frac{f^2}{2} \frac{\ln S}{S} - \frac{f^3}{8} \frac{\ln^2 S}{S^2} + \frac{f^4}{24} \frac{\ln^3 S}{S^3} + \ldots$$

This gives $f_{11}$, $f_{22}$, $f_{33}$, ..... in terms of $f$

Indeed consistent with gauge theory perturbative expansions
Another interesting observed fact: reciprocity property

Function $f$ in functional relation at large $S$ runs in inverse even powers of quadratic Casimir of SL(2,R)

$$f(S) = \sum_{n=0}^{\infty} \frac{a_n \ln C}{C^{2n}}$$

$C$ is bare quadratic operator defined in terms of conformal spin $C^2 \equiv s_0(s_0 - 1)$ or in terms of spins

$$C^2 = (S + \frac{1}{2}J)(S + \frac{1}{2}J - 1)$$

Reciprocity condition implies relations among some of the coefficients of $\ln^k \frac{S}{S_m}$, $k < m$
For twist $J = 2$

\[ f_{10} = \frac{1}{2} f (f_c + 1) \]

\[ f_{32} = \frac{1}{16} f \left[ f^3 - 2f^2 (f_c + 1) - 16f_{21} \right] \]

Functional relation and reciprocity hold at strong coupling? Yes, check to 1-loop in string theory

Functions $f, f_c, f_{10}, f_{11}$ extended at strong coupling
1-loop correction at strong coupling -- some details

Expand in large semi-classical parameter \( S = \frac{S}{\sqrt{\lambda}} \)

Expanding in small \( \eta \) quadratic fluctuation Lagrangian

\[
\tilde{L}_B = \tilde{L}_0 + \eta \tilde{L}_1 + ...
\]

\[
\tilde{L}_0 = \partial_a \chi \partial^a \chi + \partial_a \xi \partial^a \xi + 2\kappa_0 \chi \xi' - 2\kappa_0 \chi' \xi - 4\kappa_0 \tilde{\rho} \dot{\xi}
\]

\[
+ \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \partial_a \beta_u \partial^a \beta_u + 2\kappa_0^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s
\]

\[
\tilde{L}_1 = -\kappa_0^2 \cosh(2\kappa_0 \sigma) \xi^2 - \kappa_0^2 \cosh(2\kappa_0 \sigma) \tilde{\rho}^2 - \kappa_0^2 \sinh(2\kappa_0 \sigma) \xi \chi
\]

\[
- \frac{\kappa_0}{\pi} \left[ \kappa_0 \pi \cosh(2\kappa_0 \sigma) - 2 \right] \beta_u^2 + (\chi \xi' - \xi \chi') \left[ \frac{1}{\pi} - \frac{\kappa_0}{2} \cosh(2\kappa_0 \sigma) \right]
\]

\[
\tilde{\rho} \chi \kappa_0 \sinh(2\kappa_0 \sigma) \quad \tilde{\rho} \dot{\xi} \left[ \frac{2}{\pi} \right] \quad \kappa_0 \cosh(2\kappa_0 \sigma)
\]
1-loop effective action $\Gamma_1$

$$\Gamma_1 = -\frac{T}{4\pi} \int_{-\infty}^{\infty} d\omega \left[ 8 \ln \frac{\det \left[ -\partial_1^2 + \omega^2 + \rho'^2 \right]}{\det \left[ -\partial_1^2 + \omega^2 + \kappa_0^2 \right]} - 2 \ln \frac{\det \left[ -\partial_1^2 + \omega^2 + 2\rho'^2 \right]}{\det \left[ -\partial_1^2 + \omega^2 + 2\kappa_0^2 \right]} \right]$$

$$+ \ln \frac{\det^8 \left[ -\partial_1^2 + \omega^2 + \kappa_0^2 \right]}{\det^2 \left[ -\partial_1^2 + \omega^2 + 2\kappa_0^2 \right]\det^6 \left[ -\partial_1^2 + \omega^2 \right]} - \ln \frac{\det Q_\omega}{\det Q_\omega^{(0)}} + \ln \frac{\det P_\omega}{\det Q_\omega^{(0)}}$$

Expand ratio of determinants with

$$\ln \frac{\det [A + \eta B]}{\det A} = \eta \ Tr [A^{-1} B] + O(\eta^2)$$

Obtain a contribution

$$\Gamma_1^{(1)} = -\frac{T\eta}{4\pi} \sum_{n=-\infty}^{\infty} A_n$$
\[ A_n = \frac{8\kappa_0}{\sqrt{n^2 + \kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2 + 2\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2 + 4\kappa_0^2}} \]

another contribution

\[ E_1^{(0)} = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ 2\sqrt{n^2 + 2\kappa_0^2} + \sqrt{n^2 + 4\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa_0^2} \right] \]

Extract leading order at large \( \kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta} \)

1-loop the same structure as at classical level

\[ E_1 = b_0 \ln S + b_c + \frac{b_{11} \ln S + b_{10}}{S} + O\left(\frac{\ln^2 S}{S^2}\right) \]
\[ b_0 = -\frac{3 \ln 2}{\pi} \quad b_c = -\frac{3 \ln 2}{\pi} \ln 8\pi \]
\[ b_{11} = -\frac{3 \ln 2}{\pi^2} \quad b_{10} = -\frac{3 \ln 2}{\pi^2} \left( \ln 8\pi - \frac{1}{2} \right) \]

Functional and reciprocity relations at strong coupling imply:

\[ b_{11} = a_0 b_0 \quad b_{10} = \frac{1}{2}(a_0 b_c + b_0 a_c) \]

Recalling classical values

\[ a_0 = \frac{1}{\pi} \quad a_c = \frac{1}{\pi}(\ln 8\pi - 1) \]

Satisfied by the above coefficients!
Conclusions

● developed method to compute 1-loop corrections to spinning folded string in particular limits: long and short spinning string

● Long string: relations among coefficients of energy expansion in large $S$ shown to hold at strong coupling to a few orders $\log S, S^0, 1/S, \log S/S$

interesting: check this at higher orders in large $S$ expansion. Also, extend to $(S,J)$ solution.

interesting: understand better functional and reciprocity relations on both gauge and string theory

● Short string: structure of energy expansion obtained to 1-loop at strong coupling

interesting: understand BA for short operators $S \sim 1$