Cosmological Singularities from Matrices

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Space-time from Matrices

- In string theory space and time are not fundamental, but derived concepts which emerge out of more fundamental structures.
- In a few cases we have some hint of what this structure could be – these are situations where the space-time physics has a holographic description – usually in terms of a field theory of matrices.
- These are in fact descriptions of closed string dynamics in terms of open strings.
## Examples

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We will describe some recent attempts to construct toy models of cosmological singularities in each of these
2d Closed String from Double scaled Matrix Quantum Mechanics

- \( M_{ij}(t) \) - \( N \times N \) hermitian matrix. This is the degree of freedom of open strings joining D0 branes

\[
S = \int dt \frac{1}{2} Tr[(D_t M)^2 + M^2]
\]

- Gauging – states are singlet under SU(N)
- Eigenvalues are fermions. Single particle hamiltonian

\[
H = \frac{1}{2} (p^2 - x^2)
\]

- Density of fermions

\[
\partial_x \phi(x, t) = \frac{1}{N} \text{Tr} \delta(M(t) - x \cdot I)
\]
To leading order in $1/N$, the dynamics of the scalar field is given by the action

$$S = N^2 \int dx dt \left[ \frac{1}{2} \left( \frac{\partial_t \phi}{\partial_x \phi} \right)^2 - \frac{\pi^2}{6} \left( \partial_x \phi \right)^3 - (\mu - \frac{1}{2} x^2) \partial_x \phi \right]$$

- This **collective field theory** would be in fact the **field theory of closed strings in two dimensions** – the **space dimension has emerged out of the matrix**
- The fundamental quantum description is in terms of **fermions**
- Collective field theory used to find the **emergent space-time** as seen by closed strings – at the **semiclassical level**
Physics of the ground state

- The ground state is a filled fermi sea – for which the collective field is static
  \[ \partial_x \phi_0 = \frac{1}{\pi} \sqrt{x^2 - 2\mu} \]

- Fluctuations \( \phi(x,t) = \phi_0(x) + \eta(x,t) \) are described at the semiclassical level by two scalar fields living in the two regions \( |x| \geq \sqrt{2\mu} \) with \( \eta(\pm\sqrt{2\mu},t) = 0 \)
In fact, at the semiclassical level these fluctuations may be thought to live in a relativistic space-time.

In terms of coordinates

\[ t = \tau \quad x = \sqrt{2 \mu} \cosh \sigma \]

\[
H = \frac{1}{2} \int d\sigma \left\{ [\Pi^2 + (\partial_\sigma \eta)^2] + \frac{1}{\mu \sinh^2 \sigma} [\Pi^2 \partial_\sigma \eta + \frac{\pi^2}{3} (\partial_\sigma \eta)^3] \right\}
\]

These two massless scalars are related to the only two dynamical fields of 2d string theory by a transform which is non-local at the string scale.

Both these scalars live in the same space-time.

\[ I^\pm \quad \text{Weakly coupled} \]
Space-like boundaries


- The infinite $W_\infty$ symmetry of the theory may be used to find time-dependent classical solutions –
  (Karczmarek and Strominger; S.R.D., J. Davis, F. Larsen and P. Mukhopadhyay)
- Fluctuations around such solutions are once again massless scalars, but the global nature of the space-time can be rather non-trivial.
- One of these examples

\[
\partial_x \phi_0 = \frac{1}{\pi(1 + e^{2t})} \sqrt{x^2 - (1 + e^{2t})}
\]

\[
\partial_t \phi_0 = -\frac{xe^{2t}}{1 + e^{2t}} \partial_x \phi_0
\]
- The **semiclassical space-time** perceived by these fluctuations are again best described in terms of Minkowskian coordinates $\tau, \sigma$

\[
x = \cosh \sigma \sqrt{1 + e^{2t}} \quad e^{\tau} = \frac{e^t}{\sqrt{1 + e^{2t}}}
\]

- As the fundamental time of the problem $t$ runs over its full range, the time $\tau$ stops

\[
-\infty \leq t \leq \infty \quad \Leftrightarrow \quad -\infty \leq \tau \leq 0
\]

It appears that there is a space-like boundary

Note: $x \to \infty$ over the entire $I^+$
This is geodesically incomplete.

Normally one would simply extend the space-time to complete it.

However in this case there is a fundamental definition of time provided by the matrix model – the time $t$ - It does not make sense to extend the space-time beyond this boundary.

Several other examples of this type

World-sheet formulation not settled. We have a proposal – space-like tachyon condensation
Details of tachyon condensation

- Use Macroscopic loops to guess the perturbation to the world-sheet action which represent such classical solutions

\[ T(\phi, t) = 1 - \sqrt{2\mu (1 + e^{2t})^{-1} e^{-\phi} K_1 \left( \sqrt{2\mu (1 + e^{2t}) e^{-\phi}} \right)} \]

- At early times this is the usual Liouville wall

\[ T(\phi \to +\infty, t \to -\infty) = \mu e^{-2\phi} (\phi + \text{const}) \]

- Generally this represents a space-like tachyon condensation

\[ T(\phi >> 0, t < \phi) = \frac{e^{2t}}{1 + e^{2t}} + \mu e^{-2\phi} (\phi + \text{const} - \ln \sqrt{1 + e^{2t}}) \]
Beyond semiclassical approximation

(S.R.D. and Luiz dos Santos)

What is really happening is that unlike the ground state, the future boundary is not a weakly coupled region. In fact the hamiltonian \( \partial_\tau \) is again

\[
H = \frac{1}{2} \int d\sigma \left\{ [\Pi^2 + (\partial_\sigma \eta)^2] + \frac{1}{\sinh^2 \sigma} [\Pi^2 \partial_\sigma \eta + \frac{\pi^2}{3} (\partial_\sigma \eta)^3] \right\}
\]

Except at the very edge there is no true space-time interpretation in this region

However the fermion theory is perfectly well defined
The time-dependent background is in fact a non-normalizable state of the fermion theory

\[ |\alpha >= e^{i\alpha W_02} |\mu > \]

Various expectation values in this state may be calculated in terms of corresponding quantities in the ground state. For example the fermion density

\[ <\rho>_{\alpha} = \frac{\text{Re}}{\sqrt{1 + \alpha e^{2t}}} \int_0^\infty ds e^{i\mu s} \left( (-4\pi i \sinh s)^{1/2} \exp \left[ i \frac{x^2}{2(1 + \alpha e^{2t})} \tanh^2 \frac{s}{2} \right] \right) \]

Expressions like this show that the exact answer differs significantly from the semiclassical expression over almost the entire \( I^+ \)

There is no S-Matrix. However the time evolution of the wave function seems to make sense
Lesson

- The open string time – in this case the time of the matrix model - can go over the full range
- The closed string time – the time which is perceived by fluctuations in a semiclassical interpretation - can be terminated
- At the end of this semiclassical time, there is no valid relativistic interpretation of the model – though the model itself seems to make sense
Something similar happens in Matrix Big Bangs of Craps, Sethi and Verlinde

We will discuss this for IIB pp-wave backgrounds with two compact directions $x^{-}$ and a space $x^{8}$

\[
\begin{align*}
 ds^2 & = 2dx^+ dx^- - 4\mu^2 [(x^1)^2 + \cdots (x^6)^2] (dx^+)^2 \\
 & \quad - 8\mu x^7 dx^8 dx^+ + [(dx^1)^2 + \cdots (dx^8)^2] \\
 F_{+1234} & = F_{+5678} = \mu \ e^{Qx^+} \\
 \Phi & = -Qx^+
\end{align*}
\]

\[\text{IIB: } g_s, l_s \]
\[x^- \approx x^- + 2\pi \ R \quad x^8 \approx x^8 + 2\pi \ R_B\]
The holographic theory is a 2+1 dimensional SU(J) Yang Mills theory on a torus \((\rho, \sigma)\)

\[
\mathcal{L} = \text{Tr} \left\{ \frac{1}{2} \left[ (D_\tau X^a)^2 - (D_\sigma X^a)^2 - e^{2Q\tau}(D_\rho X^a)^2 \right] + \frac{1}{2(G_{YM}e^{Q\tau})^2} \left[ F_{\sigma\tau}^2 + e^{2Q\tau}(F_{\rho\tau}^2 - F_{\rho\sigma}^2) \right] \right. \\
- \frac{\mu^2}{2} \left[ (X^1)^2 + \cdots (X^6)^2 + 4(X^7)^2 \right] + \frac{(G_{YM}e^{Q\tau})^2}{4} [X^a, X^b]^2 \\
- \frac{4\mu}{(G_{YM}e^{Q\tau})} e^{Q\tau} X^7 F_{\rho\sigma} - 4\mu i(G_{YM}e^{Q\tau}) X^7 [X^5, X^6] \right. \\
\left. \right\}
\]

\[
\sigma \approx \sigma + 2\pi \frac{l_s^2}{R} \\
\rho \approx \rho + 2\pi \frac{l_s^2}{R} g_s \\
G_{YM}^2 = \frac{RR_B^2}{g_s l_s^4}
\]

\[
G_{YM} \rightarrow G_{YM}e^{Q\tau} \\
\partial_\rho \rightarrow \partial_\rho e^{Q\tau}
\]

New feature
In IIB
At sufficiently early times

(i) only diagonal X’s survive

(ii) The gauge field $F_{\mu\nu}$ gets dualized into a scalar field – so we have 8 scalars now

(iii) The effective size of the $\rho$ direction is small – becomes a 1+1 dimensional theory

(iv) This 1+1 dimensional theory becomes the world-sheet theory of the original IIB string moving in this background in the $x^+ = \tau$ gauge

(v) The rank of the gauge group J becomes identified with the momentum in $x^-$ direction

$$p_- = J / R$$
Details of Dualization

**J. Michelson, (unpublished)**

- In the regime where the fields become abelian, introduce an auxiliary field – add \( \frac{1}{2} \epsilon_{\mu\nu\lambda} \partial_\mu \phi F_{\nu\lambda} \)

- Integrate out the gauge field

\[
\mathcal{L}' = -\frac{1}{2} \left[ \sum_{a=1}^{7} (\partial_\mu X^a)^2 + G_{YM}^2 (\partial_\mu \phi)^2 \right] - 2\mu^2 \left[ \sum_{i=1}^{6} (X^i)^2 + 4(X^7)^2 \right] + 4G_{YM}\mu X^7 \partial_\tau \phi
\]

- Perform a field redefinition

\[
X^i = Y^i, \quad i = 1, \cdots, 6, \\
X^7 = Y^7 \cos(2\mu\tau) + Y^8 \sin(2\mu\tau), \\
G_{YM}\phi = -Y^7 \sin(2\mu\tau) + Y^8 \cos(2\mu\tau)
\]

- Final form

\[
\mathcal{L}_{\text{diag}} = -\frac{1}{2} \sum_{I=1}^{8} (\partial_\mu Y^I)^2 - 2\mu^2 \sum_{I=1}^{8} (Y^I)^2
\]
Generically such a space-time interpretation is not valid. This is specifically true near $\tau \to -\infty$ - here the coupling of the YM theory is weak and nonabelian configurations are important.

From the point of view of the YM theory this is the far past –

From the point of view of the space-time string theory *forcibly extrapolated* to $x^+ \to -\infty$ this appears as a “beginning of time”

Once again the open string (YM) time runs over the full range – while the closed string time appears to begin.

However at this beginning the space-time interpretation is itself breaking down.
Matrix Membranes

- The quantity $G_{YM}^2 / \mu$ acts as a **semiclassical parameter** in the theory $\mu / G_{YM}^2 >> 1$ : classical solutions representing **fuzzy ellipsoids**

\[
\begin{align*}
X^5(\tau, \sigma) &= S(\tau)J^1 \\
X^6(\tau, \sigma) &= S(\tau)J^2 \\
X^7(\tau, \sigma) &= R(\tau)J^3
\end{align*}
\]

\[
[J^a, J^b] = i\varepsilon_{ac}^{\ ab} J^c
\]

Even though $R$ oscillates, the size of the fuzzy ellipsoid **always goes to zero at late times**
Brane Production

- The effect of factors of $e^{Q\tau}$ in front of $\partial_\rho$ may be thought of as a time-dependent size of the circle.

- States of the YM theory are labelled by $(m,n)$
  
  $m = \text{momentum along } \sigma$
  
  $n = \text{momentum along } \rho$

  $(m,0): \text{states of F-strings}$
  
  $(0,n): \text{states of D-strings}$

In the 1+1 theory in $(\tau,\sigma)$, states with $n \neq 0$ are KK modes with a time dependent mass

$$m_n^2 = 4\mu^2 + \left(\frac{nR}{g_s l_s^2}\right)^2 e^{2Q\tau}$$
This implies particle \((p,q)\) string production.

The “out” vacuum at late times is a squeezed state of “in” particles.

\[
|0\rangle_{\text{out}} = \prod_{n.m} \left\{ (1 - |\gamma_m|^2)^{1/4} \exp\left[\frac{1}{2} \gamma_m^* a_m^{(\text{in})} a_m^{(\text{in})} \right]\right\} |0\rangle_{\text{in}}
\]

\[
\text{out} \langle 0 | a_{m,n}^{(\text{in})} a_{m,n}^{(\text{in})} |0\rangle_{\text{out}} = \frac{1}{e^{\omega_m/Q} - 1}
\]

\[
\gamma_m = \frac{\beta_m^*}{\alpha_m} = -ie^{-\frac{\pi \omega_m}{Q}}
\]

\[
\omega_m^2 = 4\mu^2 + \frac{m^2R^2}{l_B^4}
\]

In other words, if we require the state at late times to contain only fundamental strings, the state near the big bang must be a squeezed state of \((p,q)\) strings.

Does this say anything about the issue of initial conditions?
The IIB pp-wave has another dual – a large R-charge sector of a 3+1 dim YM theory – or rather some quiver version of the theory.

Can we address the issue of singularities in this AdS/CFT language?

This seems to require construction of the supergravity background before performing a Penrose limit – we have not yet succeeded in doing that.

But this led us to find an infinite class of time-dependent backgrounds which have natural CFT duals.
The supergravity solutions are

\[
    ds^2 = \left( \frac{r^2}{R^2} \right) \tilde{g}_{\mu\nu} dx^\mu dx^\nu + \left( \frac{R^2}{r^2} \right) dr^2 + R^2 d\Omega_5^2
\]

\[
    F_{(5)} = R^4 (\omega_5 + \ast_{10} \omega_5) \quad \phi(x^\mu)
\]

This is a solution if \( g_{\mu\nu}(x^\mu) \) and \( \phi(x^\mu) \) obey

\[
    \tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \quad \partial_\mu (\sqrt{-\det(\tilde{g})} \tilde{g}^{\mu\nu} \partial_\nu \phi) = 0
\]

These are deformations of \( AdS_5 \times S^5 \). In fact they are near-horizon limits of deformations of the full 3-brane geometry.

There are similar geometries which are deformations of \( AdS_m \times S^n \).
Examples

- It is easy to find lots of solutions of this form – e.g. Kasner-like geometries with space-like singularities.

- A particularly interesting set of solutions are those with potential null singularities

\[ d\tilde{s}^2 = e^{f(X^+)}(-2dX^+dX^- + dx_2^2 + dx_3^2) \]
\[ \phi = \phi(X^+) \]

In this case we must have

\[ \frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\partial_+\phi)^2 \]

Solutions retain half of the supersymmetries

Pick a \( f(X^+) \), find \( \phi(X^+) \)

Looks like Liouville + c=1 matter

This class of solutions also discussed in

Chu and Ho, hep-th/0602054
Details of supersymmetry

- The null solutions retain the following susy's

\[ \Gamma^4 \epsilon = \epsilon, \quad \gamma^+ \epsilon = 0, \quad \epsilon = Z^{-1/8} e^{f/4} \eta \]

- where

\[ \Gamma^4 = i \Gamma^{0123} \quad Z = Z(x^m) = \frac{R^4}{r^4} \]
There are examples where the string coupling is always bounded.

$$e^f(x^+) = \tanh^2 x^+$$
$$e^\phi = g_s \left( \tanh \frac{x^+}{2} \right)^{\sqrt{8}}$$

Even though curvature invariants vanish, there is a singularity at $x^+ = 0$. This is reached by geodesics at finite proper time.

Here the string coupling vanishes at $x^+ = \pm\infty$ the space-time is pure $AdS_5 \times S^5$. 
In such backgrounds there is a natural CFT dual

- Note that we have turned on a non-normalizable mode. This means we have sources in the gauge theory.

- The natural dual is in fact the gauge theory which lives in the metric $\tilde{g}_{\mu\nu}$ and has a coupling $e^{\phi/2}$ - may be seen e.g. from DBI action of a 3-brane in this background.

- The supersymmetries of the bulk translate to those in this candidate CFT.

- Correlation functions of suitably dressed operators are non-singular at $X^+ = 0$.

- Interesting question: How does the gauge theory encode the space-time singularity?