Thermodynamics of higher spin black holes

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G. Compère and W.S., arXiv: 1306.0014
G. Compère, J. Jottar and W.S., arXiv: 1308.2175

Black holes in string theory, University of Michigan
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Why higher spin?

In the tensionless limit of string theory, all higher spin($s > 2$) excitations become massless. So far it is not clear how to describe this limit. Vasiliev provides a framework to study higher spin theories. A direct connection between string (field) theory and Vasiliev theory is still not clear.

Holography, or gauge/gravity duality is an efficient tool to explore problems in both gravity and field theory. Higher spin gravity/CFT duality is one of the few examples where both sides of the duality can be explicitly constructed.

Higher spin gravity/CFT duality

In the bulk, the gravity theory is described by Vasiliev higher spin theory, *Vasiliev* which has been constructed in various dimensions.

In four dimensions, Vasiliev theory is described by some master equations, for which AdS_4 is a solution. Expanding the equation around the vacuum, we get an infinite tower of excitations with all spins. The holographic dual is the singlet sector of $O(N)$ vector model, or its generalizations. *Klebanov-Polyakov*

In three dimensions, the bulk theory contains all higher spin fields and some scalar fields with mass parameterized by a continuous parameter λ. The holographic dual is conjectured to be the large N limit of $\mathcal{W}_N$ minimal model. *Gaberdiel-Gopakumar*

Black holes in higher spin theories

Black holes play a significant role in understanding the quantum theory of gravity.

Despite lots of progress has been made recently in four dimensions, black holes remains puzzling. On the one hand, the CFT partition function calculation shows that there is no Hawking-Page transition at finite temperature *Shenker-Yin.*

On the other hand, there are some candidate black hole solutions in Vasiliev theory *Didenko-Vasiliev* and hyperbolic black holes. However, it is not clear how to define thermodynamic quantities, such as mass, charge, temperature, and entropy.

Higher spin black holes in 3d

In 3d, The pure higher spin sector can be formulated as Chern Simons gauge theory, which is a direct generalization of $SL(2, \mathbb{R})$ formulation of Einstein gravity. In particular, this truncation has an action.

CFT calculations show that there is again no Hawking-Page transition *Banerjee-Castro-Hellerman-Hijiano-Lepage-Jutier-Malony-Shenker, Gaberdiel-Gopakumar-Rangamani.*

In the bulk, BTZ black holes exist. Explicit higher spin black hole solutions have been constructed *Gutperle-Kraus.*

Thermodynamic properties have been studied, despite with some discrepancy *Ammon-Gutperle-Kraus-Perlmutter, Perez-Tempo-Troncoso, de Boer-Jottar, Compère-Song, Compère-Jottar-Song, Henneaux-Pérez-Tempo-Troncoso.*

$SL(N,\mathbb{R})$ Chern-Simons gauge theory

A truncation of Vasiliev theory is the $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons gauge theory, with the action

$$S[A,\bar{A}] = S_{CS}[A] - S_{CS}[\bar{A}]$$

where $S_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ The equations of motion are given by

$$F \equiv dA + A\wedge A = 0$$

where

$$A = \sum_s (\omega + \frac{e}{\ell})_\mu^{a_1\cdots a_{s-1}} T_{a_1\cdots a_{s-1}} dx^\mu$$

There are similar equations for $\bar{A}$. The metric and spin 3 fields are extracted from this by

$$g_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(e_\mu e_\nu), \quad \varphi_{\mu\nu\gamma} = \frac{1}{9\sqrt{-\sigma}} tr\left(e_{(\mu}e_\nu e_{\gamma)}\right)$$

Asymptotic AdS_3 solutions

- The solution with zero energy is

$$A = e^{\rho} L_1 dx^+ + L_0 d\rho$$

The metric is AdS_3 with radius ℓ,

$$ds^2 = \ell^2 (d\rho^2 + e^{2\rho} dx^+ dx^-)$$

- In the highest weight gauge, asymptotically AdS_3 can be written as

$$\begin{aligned} A &= b^{-1} a b + b^{-1} db \\ b &= e^{\rho L_0}, \quad a = (L_1 - \frac{\mathcal{L}}{k} L_{-1}) dx^+ \end{aligned}$$

EOM:

$$\partial_- \mathcal{L} = 0$$

The asymptotic symmetry group under Dirichlet boundary conditions is generated by Virasora algebra, with $\mathcal{L}$ being the stress tensor

- A particular class of solutions are the BTZ(BBZ) black holes, with $\mathcal{L}$, $\bar{\mathcal{L}}$ being constant. BTZ black holes are locally quotients of AdS_3. The mass and angular momentum are given by

$$M = \mathcal{L} + \bar{\mathcal{L}}, \quad J = \bar{\mathcal{L}} - \mathcal{L}$$

Thermodynamics of BTZ black holes

- Definition of charges: Note that $a_\phi = a_+ = L_1 - \frac{\mathcal{L}}{k} L_{-1}$ in the highest weight gauge. The spin 2 charge Q_2 can be read from the coefficient of lowest weight generator L_{-1} in a_ϕ. In a gauge invariant way, $Q_2 \equiv \frac{k}{2} Tr[a_\phi^2] = \mathcal{L}$
- In Euclidean space, $z \sim z + 2\pi \sim z + 2\pi\tau$. One can check that the following equation holds $\tau a_z + \bar{\tau} a_{\bar{z}} = \tau a_\phi$
- Smoothness condition at the horizon: the holonomy along thermal circle is trivial $Tr[(\tau a_z + \bar{\tau} a_{\bar{z}})^2] = -\frac{1}{2} \Longrightarrow \mathcal{L} = -\frac{k}{4\tau^2}$
- The Bekenstein-Hawking entropy

$$S_{BH} = S_A + S_{\bar{A}},\ S_A = -2\pi i k Tr[a_\phi(\tau a_z + \bar{\tau} a_{\bar{z}})]$$

Non-trivial checks

- The first law of thermodynamics $\tau = \frac{i}{2\pi}\frac{\delta S}{\delta \mathcal{L}}$
- Matching with the dual CFT

$$S_{BH} = \frac{A_H}{4G} = 2\pi\left(\sqrt{\frac{c\mathcal{L}}{6}} + \sqrt{\frac{\bar{c}\bar{\mathcal{L}}}{6}}\right), \quad c = \bar{c} = 6k$$

The charge $\mathcal{L}$ is the zero mode of the Virasoro generator.

Generalization to $SL(N,R)$

Definitions

- Definition of $2(N-1)$ charges, Q_i, $\bar{Q}_i$
- Definition of $2(N-1)$ chemical potentials α_i, $\bar{\alpha}_i$
- Smoothness condition at the horizon—holonomy conditions, setting α_i, $\bar{\alpha}_i$ as functions of Q_i, $\bar{Q}_i$.
- The Bekenstein-Hawking entropy

Checks

- The first law of thermodynamics $\alpha_i = \frac{i}{2\pi}\frac{\delta S}{\delta Q_i}$
- Matching with the dual CFT

Key features of higher spin black holes

- Spin 3 black hole is characterized by deformation parameters μ, $\bar{\mu}$, as well as spin 3 field $\mathcal{W}$, $\bar{\mathcal{W}}$. When $\mu = \bar{\mu} = \mathcal{W} = \bar{\mathcal{W}} = 0$, the solutions go back to BTZ black holes.
- For $\mu\bar{\mu} \neq 0$, the metric grows like $e^{4\rho}$, violating the Brown-Henneaux boundary conditions.

Model in $SL(2,R)$ theory

The analog of a *Gutperle and Kraus* higher spin black hole in $SL(2,R)$ theory is not a BTZ black hole in the highest weight gauge, but in a gauge with explicit deformation parameter turned on

$$a = (L_1 - \frac{\mathcal{L}}{k} L_{-1}) dx^+ + \mu_2 (L_1 - \frac{\mathcal{L}}{k} L_{-1}) dx^-$$

The EOM becomes

$$\partial_- \mathcal{L} = \mu_2 \partial_+ \mathcal{L} + 2\mathcal{L} \partial_+ \mu_2 - \frac{k}{2} \partial_+^3 \mu_2$$

When $\mu_2 = 0$, the solution is a BTZ black hole in the highest weight gauge. When $\mu_2 \neq 0$, the solution goes as $e^{2\rho}$, also violating the Brown-Henneaux boundary condition.

For constant μ_2 and $\mathcal{L}$, the metric becomes standard BTZ with the new variables

$$\tilde{x}^+ = \frac{x^+ + \mu_2 x^-}{1-\mu_2}, \quad \tilde{\mathcal{L}} = (1-\mu_2)^2 \mathcal{L}$$

In Euclidean space,

$$\tilde{z} \sim \tilde{z} + 2\pi \sim \tilde{z} + 2\pi\tilde{\tau}$$

where

$$\tilde{\tau} = \frac{\tau - \bar{\tau}\mu_2}{1-\mu_2}$$

Holomorphic formalism *Gutperle and Kraus*

- Definition of charges, $\mathcal{L}$, the same as $\mu_2 = 0$
- In Euclidean space, $z \sim z + 2\pi \sim z + 2\pi\tau$.
- Smoothness condition at the horizon $Tr[(\tau a_z + \bar{\tau} a_{\bar{z}})^2] = -\frac{1}{2}$
- Getting the entropy by integrating the first law of thermodynamics $\tau = \frac{i}{2\pi}\frac{\delta S_{hol}}{\delta \mathcal{L}}$

- Matching with the dual CFT

$$S_{hol} = 2\pi\left(\sqrt{\frac{c\mathcal{L}}{6}} + \sqrt{\frac{\bar{c}\bar{\mathcal{L}}}{6}}\right), \quad c = \bar{c} = 6k$$

- However, The Bekenstein-Hawking entropy

$$S_{BH} = \frac{A_H}{4G} = 2\pi\left(\sqrt{k\tilde{\mathcal{L}}} + \sqrt{k\tilde{\bar{\mathcal{L}}}}\right) \neq S_{hol}$$

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- The energy is defined as the conjugate variable of τ
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Modified canonical formalism *Compère-Jottar-Song*

- Definition of charges, $\tilde{\mathcal{L}} \equiv \frac{k}{2} Tr[a_\phi^2]$—zero mode of the Virasoro generator
-
$$\tau a_z + \bar{\tau} a_{\bar{z}} \equiv \tilde{\tau} a_\phi, \quad \tilde{\tau} = \frac{\tau - \bar{\tau}\mu_2}{1-\mu_2}$$
  Then it follows that $\tilde{z} \sim \tilde{z} + 2\pi \sim \tilde{z} + 2\pi\tilde{\tau}$
- Smoothness condition at the horizon $Tr[(\tau a_z + \bar{\tau} a_{\bar{z}})^2] = -\frac{1}{2}$
- The Bekenstein-Hawking entropy
$$S_{BH} = \frac{A_H}{4G} = 2\pi(\sqrt{k\tilde{\mathcal{L}}} + \sqrt{k\tilde{\bar{\mathcal{L}}}})$$
- The first law of thermodynamics $\tilde{\tau} = \frac{i}{2\pi}\frac{\delta S}{\delta \tilde{\mathcal{L}}}$
- Matching with the dual CFT
$$S_{BH} = \frac{A_H}{4G} = 2\pi(\sqrt{\frac{c\tilde{\mathcal{L}}}{6}} + \sqrt{\frac{\bar{c}\tilde{\bar{\mathcal{L}}}}{6}}), \quad c = \bar{c} = 6k$$

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In general, black solutions are given by a_ϕ and $\tau a_z + \bar{\tau} a_{\bar{z}}$, with $[a_\phi, \tau a_z + \bar{\tau} a_{\bar{z}}] = 0$. All thermal dynamical quantities should be defined in terms of these two.

- To define charges we first take solutions with no deformations in the highest weight gauge. The gauge invariant expressions of these charges generalize naturally to solutions with deformations. The charges are in terms of traces of polynomials of a_ϕ. For example $Q_2 \equiv \tilde{\mathcal{L}} \sim Tr[a_\phi^2]$, $Q_3 \equiv \tilde{\mathcal{W}} \sim Tr[a_\phi^3], \cdots$
  From $SL(3, R)$ black holes, it is explicitly shown that these charges can be obtained from the ASG.
- The chemical potentials can be read from expressing $\tau a_z + \bar{\tau} a_{\bar{z}}$ in terms of polynomials of a_ϕ. $\tau a_z + \bar{\tau} a_{\bar{z}} = \sum_{j \geq 2} \alpha_j b_{j-1}(a_\phi)$.

The canonical entropy is obtained from the Legendre transformation of the on-shell action, after adding proper boundary terms to make the variational principle well-defined. Or alternatively, go to the metric formalism, and use the Wald formula.

$$S_{BH} = S_L + S_R,\; S_L = -2\pi i k_{cs} Tr[a_\phi(\tau a_z + \bar{\tau} a_{\bar{z}})]$$

this entropy is also confirmed by recent entanglement entropy analysis.

Advantages of the modified canonical formalism

- All thermodynamical quantities are expressed in a gauge invariant way
- The charges agree with the ASG analysis
- The entropy has an microscopic interpretation

An alternative way to introduce the chemical potential

Henneaux-Pérez-Tempo-Troncoso

$$a = (L_1 - \frac{\mathcal{L}}{k} L_{-1}) dx^+ + \frac{\nu}{\ell} (L_1 - \frac{\mathcal{L}}{k} L_{-1}) dt$$

The chemical potential appears in the connection, instead of on the identifications .

Advantages

- The charges are unchanged under the deformation, as a_ϕ does not depend on the deformation parameter.
- The asymptotic symmetry group is also unchanged.

Disadvantages:

The original higher spin black holes are not written in this gauge.

$SL(3,\mathbb{R})$ algebra

$$
\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j} \\
[L_i, W_m] &= (2i-m)W_{i+m} \\
[W_m, W_n] &= -\frac{1}{3}(2m^2+2n^2-mn-8)L_{m+n}
\end{aligned}
$$

where $i = 0, \pm 1,\ m, n = 0, \pm 1, \pm 2$

- Principal embedding: L_i as $SL(2,\mathbb{R})$ generators, W generators transform under L_i as spin 2 multiplet.
- Diagonal embedding: $\hat{L}_0 = \frac{1}{2}L_0,\ \hat{L}_{\pm 1} = \pm\frac{1}{4}W_{\pm 2}$ as the $SL(2,\mathbb{R})$ generators. $\hat{J}_0 = \frac{1}{2}W_0$ transform as spin 0 multiplet, and $\hat{G}^{\pm}_{\frac{1}{2}} = \frac{1}{\sqrt{8}}(W_1 \mp L_1)$ and $\hat{G}^{\pm}_{-\frac{1}{2}} = \frac{1}{\sqrt{8}}(L_{-1} \pm W_{-1})$ both transform as spin $\frac{1}{2}$ muitiplet.

Asymptotic AdS_3 solutions in principal embedding

Upon gauge choice, asymptotically AdS_3 under Dirichlet boundary conditions can be written as

$$\begin{aligned} A &= b^{-1}a(x^+)b + b^{-1}db \\ b &= e^{\rho L_0} \\ a(x^+) &= (L_1 - \frac{\mathcal{L}(x^+)}{k}L_{-1} - \frac{\mathcal{W}(x^+)}{4k}W_{-2})dx^+ \end{aligned}$$

The asymptotic symmetry group under Dirichlet boundary conditions is generated by $\mathcal{W}_3$ algebra, with $\mathcal{L}$ being the stress tensor, while $\mathcal{W}$ transform as a spin three field.

- A particular class of solutions are the BTZ(BBZ) black holes, with $\mathcal{W}=0$ and $\mathcal{L}$, $\bar{\mathcal{L}}$ being constant. BTZ black holes are locally quotients of AdS_3. The mass and angular momentum are given by
$$M=\mathcal{L}+\bar{\mathcal{L}}, \quad J=\bar{\mathcal{L}}-\mathcal{L}$$
- Generic solutions obeying the boundary conditions are BTZ black holes with $\mathcal{W}_3$ hair.

Higher spin black holes *Gutperle and Kraus*

Having a solution with spin 3 charges, corresponds to adding a source term linear in μ in the dual CFT.Take the anstz

$$a = (L_1 - \frac{\mathcal{L}}{k} L_{-1} + \frac{\mathcal{W}}{4k\sigma} W_{-2}) dx^+ + (\mu W_2 + \cdots) dx^-$$

with $\mathcal{L}$ and $\mathcal{W}$ arbitrary function of (x^+, x^-). $\cdots$ means linear combinations of $SL(3,\mathbb{R})$ generators, with the coefficients determined by the bulk equation of motion. The higher spin black hole solutions corresponds to constant $\mathcal{L}$ and $\mathcal{W}$, The metric and spin 3 fields are extracted from this by

$$g_{\mu\nu} = \frac{1}{2} \mathrm{Tr}(e_\mu e_\nu), \quad \varphi_{\mu\nu\gamma} = \frac{1}{9\sqrt{-\sigma}} tr\big(e_{(\mu} e_\nu e_{\gamma)}\big)$$

The spacetime geometry

- In the presence of higher spin gauge transformations, the existence of event horizons and singularities in the metric become gauge dependent. The gauge used so far describes a traversable wormhole connecting two asymptotic regions. But an explicit higher spin gauge transformation can make it describe a black hole with smooth horizons.

  Ammon-Gutperle-Kraus-Perlmutter

- When $\mu = \bar{\mu} = \mathcal{W} = \bar{\mathcal{W}} = 0$, the solution is just the BTZ black hole solution. Turning on μ is to turn on a chemical potential for the spin three field $\mathcal{W}$.
- For $\mu\bar{\mu} \neq 0$, the metric grows like $e^{4\rho}$, violating the generalized Brown-Henneaux condition found by *Henneaux-Rey, Campoleoni-Fredenhagen-Pfenninger-Theisen*

Algorithm for Asymptotic Symmetry Group

- A choice of boundary condition (falloff conditions on the initial data at $t = 0$)
- Allowed symmetry transformations: generate 'gauge' transformations that preserve the b.c.
- Asymptotic Symmetry Group(ASG) $= \frac{\text{Allowed symmetry transformations}}{\text{Trivial symmetry transformations}}$.
- The boundary condition is consistent if all charges are finite, conserved, and integrable.

Dirichlet boundary conditions at fixed μ

We use the ansatz

$$a=(L_1-\frac{\mathcal{L}}{k}L_{-1}+\frac{\mathcal{W}}{4k\sigma}W_{-2})dx^+ +(\mu W_2+\cdots)dx^-$$

with $\mathcal{L}$ and $\mathcal{W}$ arbitrary function of (t,ϕ), or equivalently functions of (x^+,x^-), where $x^\pm=t\pm\phi$.
Bulk field equations determine $\cdots$ in terms of $\mu,\mathcal{L}$ and $\mathcal{W}$, as well as the relations between $\mathcal{L}$ and $\mathcal{W}$.

$$\begin{aligned}\partial_-\mathcal{L} &= -2\mu\partial_+\mathcal{W}\\ \partial_-\mathcal{W} &= \mu\Big(\frac{2}{3}\partial_+^3\mathcal{L}-\frac{32}{3k}\mathcal{L}\partial_+\mathcal{L}\Big)\end{aligned}$$

are the so called "good" Boussinesq equation, also known as the time independent Kadomtsev-Petviashvili(KP) II equation.

The allowed symmetry transformations

The allowed symmetry transformations are parameterized by two parameters $\epsilon(x^+,x^-)$ and $\chi(x^+,x^-)$ with the condition

$$\begin{aligned} \partial_-\chi &= 2\mu\partial_+\epsilon \\ \partial_-\epsilon &= -\frac{2\mu}{3}\partial_+^3\chi + \frac{32\mu}{3k}\mathcal{L}\partial_+\chi \end{aligned}$$

$\mathcal{L}$ and $\mathcal{W}$ transform as

$$\begin{aligned} \delta\mathcal{L} &= -\partial_+\mathcal{L}\epsilon - 2\mathcal{L}\partial_+\epsilon + \frac{k}{2}\partial_+^3\epsilon + 2\chi\partial_+\mathcal{W} + 3\partial_+\chi\mathcal{W} \\ \delta\mathcal{W} &= -\epsilon\partial_+\mathcal{W} - 3\partial_+\epsilon\mathcal{W} \\ &\quad + \frac{\sigma}{3}\Big(2\chi\partial_+^3\mathcal{L} + 9\partial_+\chi\partial_+^2\mathcal{L} + 15\partial_+^2\chi\partial_+\mathcal{L} + 10\partial_+^3\chi\mathcal{L} - \frac{k}{2}\partial_+^5\chi \\ &\quad - \frac{32}{k}(\chi\mathcal{L}\partial_+\mathcal{L} + \partial_+\chi\mathcal{L}^2)\Big) \end{aligned}$$

The asymptotic symmetry group at $\mu = 0$

- To find the ASG, the first step is to solve the bulk equations of motion and the equation for the gauge parameters, at $\mu = 0$, $\partial_- \mathcal{L} = \partial_- \mathcal{W} = \partial_- \epsilon = \partial_- \chi = 0$, with the associated charges at $t = 0$,

$$\mathcal{Q} = \frac{1}{2\pi} \int d\phi \Big(\epsilon(0,\phi)\mathcal{L}(0,\phi) - \chi(0,\phi)\mathcal{W}(0,\phi) \Big)$$

- The second step is to read the algebra defined from the Poisson algebra

$$\{\mathcal{Q}_{\Lambda^{(1)}}, \mathcal{Q}_{\Lambda^{(2)}}\} \equiv \delta_{\Lambda^{(1)}} \mathcal{Q}_{\Lambda^{(2)}}$$

  where $\Lambda = (\epsilon, \chi)$. Fourier expanding in ϕ, we learn that the algebra is the classical $\mathcal{W}_3$, with $\mathcal{L}_m$ forming the Virasoro algebra, and $\mathcal{W}_n$ forming the spin 3 part of the algebra.

Therefore canonical analysis shows that the ASG is holomorphically factorized $W_3 \times \bar{W}_3$

- When $\mu \neq 0$, holomorphic factorization breaks down, as can be seem from the equations of motion.

$$
\begin{aligned}
\partial_- \mathcal{L} &= -2\mu\partial_+\mathcal{W} \\
\partial_- \mathcal{W} &= \mu\Big(\frac{2}{3}\partial_+^3\mathcal{L} - \frac{32}{3k}\mathcal{L}\partial_+\mathcal{L}\Big) \\
\partial_- \chi &= 2\mu\partial_+\epsilon \\
\partial_- \epsilon &= -\frac{2\mu}{3}\partial_+^3\chi + \frac{32\mu}{3k}\mathcal{L}\partial_+\chi
\end{aligned}
$$

- Solution of ϵ and χ depend on $\mathcal{L}$ and $\mathcal{W}$. One consequence is that we can not write down directly the finite version of charges. We have to first define infinitesimal charges, and then integrate over the phase space.
- Note that it seems that the transformation laws of $\mathcal{L}$ and $\mathcal{W}$ does not depend on μ. But canonical analysis requires writing everything at $t = 0$. Since $\mathcal{L}$ and $\mathcal{W}$ are not purely holomorphic, there are additional μ dependence due to time dependence.

Observations at $\mu = 0$

- At $\mu = 0$, note that the boundary conditions only require two initial data $\mathcal{L}(0,\phi)$ and $\mathcal{W}(0,\phi)$. All time derivatives are determined by the equations of motion. Similarly, the allowed symmetry transformations are also parameterized by two initial data $(\epsilon(0,\phi), \chi(0,\phi))$. Therefore all the conserved charges can be written in terms of the initial data $\mathcal{L}(0,\phi)$, $\mathcal{W}(0,\phi)$ and $\Lambda \equiv (\epsilon(0,\phi), \chi(0,\phi))$
- To get the algebra, we need to get the infinitesimal conserved charges associated with the symmetry in a good basis. At $\mu = 0$, $\Lambda_{L_m} \equiv (e^{im\phi}, 0)$ generates the Virasora algebra, while $\Lambda_{W_n} \equiv (0, e^{in\phi})$ generates the spin three algebra.

The stragegy at $\mu \neq 0$

- After turning on μ, the equations of motion can be expanded to any given order in μ, which express $\dot{\mathcal{L}}(0,\phi)$ and $\dot{\mathcal{W}}(0,\phi)$ in terms of $\mathcal{L}(0,\phi)$ and $\mathcal{W}(0,\phi)$ and their spatial derivatives. Therefore, the initial data problem with $\mathcal{L}(0,\phi)$ and $\mathcal{W}(0,\phi)$ as initial data is always well defined at any order in μ.
- We will determine the basis for (ϵ,χ) by two criteria: first, the associated infinitesimal charges should be integrable and second, the resulting algebra should be closed and well defined. After try and error, it turns out that we can choose a basis such that the final algebra is still $\mathcal{W}_3$. We did the explicit calculation up to $\mathcal{O}(\mu^4)$ but we expect that this result will extend to all orders in perturbation theory.

The basis up to $\mathcal{O}(\mu^4)$

The spin 2 generator

$$\begin{aligned}\epsilon &= \left(1+\frac{3}{2}\mu^2\partial_\phi^2\right)\epsilon_0 + O(\mu^4),\\ \chi &= -\left(\mu+\frac{1}{2}\mu^3\partial_\phi^2\right)\epsilon_0 + O(\mu^4)\end{aligned}$$

the spin 3 generator

$$\begin{aligned}\epsilon &= \Big(\mu\big(\partial_\phi^2-\frac{32\mathcal{L}}{3k}\big)-\frac{16\mu^2}{k}\mathcal{W}\\ &\quad -\frac{\mu^3}{6}\partial_\phi^4+\frac{32\mu^3}{3k}\big(\mathcal{L}\partial_\phi^2-3\mathcal{L}'\partial_\phi-\frac{8}{3}\mathcal{L}''\big)\Big)\chi_0+O(\mu^4),\\ \chi &= \Big(1-\frac{\mu^2}{2}\big(\partial_\phi^2+\frac{32\mathcal{L}}{3k}\big)\Big)\chi_0+O(\mu^4).\end{aligned}$$

The conserved charges up to $\mathcal{O}(\mu^4)$

$$\not\delta \mathcal{Q}_{\Lambda_L} = \frac{1}{2\pi}\int_{t=0} d\phi\, \epsilon_0 \delta\tilde{\mathcal{L}}, \qquad \not\delta \mathcal{Q}_{\Lambda_W} = -\frac{1}{2\pi}\int_{t=0} d\phi\, \chi_0 \delta\tilde{\mathcal{W}}$$

where

$$\begin{aligned}\tilde{\mathcal{L}} &= \mathcal{L} + 3\mu\mathcal{W} + \mu^2\left(\frac{7}{2}\mathcal{L}'' + \frac{16}{3k}\mathcal{L}^2\right) + \frac{29}{6}\mu^3\mathcal{W}'' + O(\mu^4),\\ \tilde{\mathcal{W}} &= \mathcal{W} - \mu(3\mathcal{L}'' - \frac{32}{3k}\mathcal{L}^2) + \frac{\mu^2}{2}(3\mathcal{W}'' + \frac{32}{k}\mathcal{L}\mathcal{W}) - \frac{512}{27k^2}\mu^3\mathcal{L}^3\\ &\quad + \frac{16\mu^3}{9k}\left(9\mathcal{W}^2 - 66(\mathcal{L}')^2 - 43\mathcal{L}\mathcal{L}''\right) + \frac{77}{18}\mu^3\mathcal{L}'''' + O(\mu^4).\end{aligned}$$

Now the infinitesimal charges only depend on the initial data at $t=0$. In particular, the gauge parameters $\epsilon(0,\phi)$, $\chi(0,\phi)$ do not dependent on the fields $\mathcal{L}(0,\phi)$, $\mathcal{W}(0,\phi)$.

The asymptotic symmetry group

One important consequence is that the charges are integrable at $t=0$. We obtain the spin 2 and spin 3 charges

$$\mathcal{Q}^{t=0}_{\Lambda_L} = \frac{1}{2\pi}\int_{t=0} d\phi\, \epsilon_0 \tilde{\mathcal{L}}, \qquad \mathcal{Q}^{t=0}_{\Lambda_W} = -\frac{1}{2\pi}\int_{t=0} d\phi\, \chi_0 \tilde{\mathcal{W}}.$$

The Poisson bracket between the conserved charges is defined by

$$\{\mathcal{Q}_{\Lambda^{(1)}}, \mathcal{Q}_{\Lambda^{(2)}}\} \equiv \delta_{\Lambda^{(1)}}\mathcal{Q}_{\Lambda^{(2)}} = \oint \mathcal{Q}_{\Lambda^{(2)}}[\delta_{\Lambda^{(1)}}\mathcal{L}, \delta_{\Lambda^{(1)}}\mathcal{W}; \mathcal{L}(\phi), \mathcal{W}(\phi)]$$

where $\Lambda^{(1)} = (\epsilon^{(1)}, \chi^{(1)})$, $\Lambda^{(2)} = (\epsilon^{(2)}, \chi^{(2)})$ are two choices of generators (either Λ_L or Λ_W with a corresponding choice of ϵ_0 or χ_0). Here, the Poisson bracket can be computed using the infinitesimal charges even though we do not have at hand the conserved charge $\mathcal{Q}_\Lambda$ at all times. After some algebra, we recognize that the Poisson bracket is just the classical algebra of $\mathcal{W}_3$, with central charge $c = 6k$.

The asymptotic symmetry group

- Despite the fact that higher spin black holes violate the asymptotic AdS_3 boundary conditions, we can still find consistent boundary conditions, under which the asymptotic symmetry group is not modified by changing μ. Namely, if we start with principal embedding at $\mu = 0$, the asymptotic symmetry group remains $\mathcal{W}_3$, and if we start with diagonal embedding, the ASG remains $\mathcal{W}_3^{(2)}$.
- Turning on chemical potential leads to an integrable deformation.
- The charges we defined before are indeed generators of the ASG.

Thank You