Quantized Black Hole Charges and Freudenthal Duality

M. J. Duff
Blackett Laboratory, Imperial College, London

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University of Michigan

Summary

Freudenthal duality

F-dual: A discrete symmetry transformation $x \to \tilde{x}$, of a Freudenthal triple system x for which $\tilde{\tilde{x}} = -x$, preserving the quartic invariant $\Delta(x)$.

Originally introduced as a symmetry of extremal black holes in supergravity

[L. Borsten, D. Dahanayake, M. J. Duff, and W. Rubens, arXiv:0903.5517]

Other applications

- Generalised to a full symmetry of black hole potential [Ferrara , Marrani, Yeranyan, arXiv:1102.4857]
- The attractor mechanism [Ortin, Shahbazi, arXiv:1206.3190]
- Freudenthal gauge theory [Marrani, Qiu, Shih, Tagliaferro, Zumino, arXiv:1208.0013]
- Symmetry of supergravity [Borsten, Duff, Ferrara, Marrani arXiv1212.3254]

Summary

The Framework

- In quantum theory, charges are quantized
- We exploit the mathematical framework of integral Jordan algebras, the integral Freudenthal triple system and, in particular, the work of Krutelevich.
- [Krutelevich, J. Algebra (2002); J. Algebra (2007), arXiv:math/0411104]

Extremal Black Holes

Einstein-Maxwell System

- Einstein-Maxwell action and e.o.m:

$$S[g_{\mu\nu}, A_\mu] = \frac{1}{16\pi} \int d^4x \sqrt{|g|} \left[R - \frac{1}{4} F^2 \right]$$

$$R_{\mu\nu} = \frac{1}{2} \left[F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} F^2 \right]$$

Reissner-Nordstöm Black Hole

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$Q = \frac{1}{16\pi} \int_{S^2_\infty} \star F$$

Quantum Black Holes

RN Black Hole Thermodynamics

Hawking temperature:

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})}$$

Bekenstein-Hawking entropy:

$$S_{BH} = \pi(M + \sqrt{M^2 - Q^2})^2$$

The Extremal Limit: $M \to Q$

- Horizons collapse $r_+ = r_-$.
- Stable against Hawking radiation $T_H \to 0$
- But, have non-vanishing entropy $S_{BH} = \pi Q^2$

These special properties allow one derive the entropy quantum mechanically using string theory.

Supergravity

$\mathcal{N} = 8$

- $\mathcal{N} = 8$ single supermultiplet has bosonic sector

$$g_{\mu\nu}, \quad 70\phi, \quad 28C_{\mu}$$

- Scalars parametrise a homogeneous space

$$\frac{G_4}{H_4} = \frac{E_{7(7)}}{SU(8)}$$

- EOM invariant under G_4, the U-duality group
- The fields strengths plus their duals transform linearly as the **56** of E_7

Generic supergravity

- Scalars may or may not parametrise a homogeneous space $\frac{G_4}{H_4}$
- EOM invariant under G_4, the U-duality group
- The fields strengths plus their duals transform linearly under G_4

Extremal Black Holes in supergravity

- Solutions similar to Reissner–Nordström, but
- More electromagnetic charges in the game (56 in the $\mathcal{N} = 8$ case)

$$p^a = \frac{1}{4\pi}\int_{S^2} F^a, \qquad q_a = \frac{1}{4\pi}\int_{S^2} G_a, \qquad Q = \begin{pmatrix} p^a \\ q_a \end{pmatrix}$$

- There are also scalar equations of motion
- Extremal solutions "attractor mechanism" kicks in: the scalars at the horizon must be fixed in terms of the charges independent of their asymptotic values. [Ferrara:1995, Strominger:1996, Ferrara:1996, Ferrara:1996, Ferrara:1997]; the horizon area loses all memory of the scalars and is a (non-polynomial) quadratic function of only the charges, just as in the Reissner–Nordström case.
- The entropy is given by a quartic U-duality invariant $\Delta(p, q)$,

$$S_{\mathrm{BH}} = \pi\sqrt{|\Delta(p, q)|}.$$

Cubic Jordan algebras

Jordan Algebras

A Jordan algebra $\mathfrak{J}$ is vector space defined over a ground field $\mathbb{F}$ equipped with a bilinear product satisfying [Jordan:1933]

$$X \circ Y = Y \circ X, \quad X^2 \circ (X \circ Y) = X \circ (X^2 \circ Y), \quad \forall\, X, Y \in \mathfrak{J}.$$

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Cubic Jordan Algebras

Let V be a vector space equipped with a cubic norm and a base point:

- A *cubic* form:

$$N_3 : V \to \mathbb{R} \quad \text{s.t.} \quad N_3(\alpha X) = \alpha^3 N_2(X) \quad \forall \alpha \in \mathbb{R}, X \in V,$$

$$\begin{aligned} N_3(X, Y, Z) =& N_3(X + Y + Z) - N_3(X + Y) - N_3(Y + Z) - N_3(X + Z) \\ &+ N_3(X) + N_3(Y) + N_3(Z) \end{aligned}$$

  is trilinear.

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  is trilinear.
- A *base point*: element $c \in V$ satisfying $N_3(c) = 1$.

The Freudenthal triple system

Freudenthal triple system over a Jordan algebra $\mathfrak{J}$ [Freudenthal:1954,Brown:1969]:

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J} \oplus \mathfrak{J}.$$

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{R} \quad \text{and} \quad A, B \in \mathfrak{J}.$$

Defining relations

1. Quadratic form $\{\bullet, \bullet\}$: $\mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \to \mathbb{R}$

$$\{x, y\} = \alpha\delta - \beta\gamma + \mathrm{Tr}(A, C) - \mathrm{Tr}(B, D), \quad \text{where} \quad x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \; y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}.$$

2. Quartic form $\Delta : \mathfrak{F}(\mathfrak{J}) \to \mathbb{R}$

$$\Delta(x) = -(\alpha\beta - \mathrm{Tr}(A, B))^2 - 4[\alpha N(A) + \beta N(B) - \mathrm{Tr}(A^\sharp, B^\sharp)].$$

3. Triple product $T : \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \to \mathfrak{F}(\mathfrak{J})$ which is uniquely defined by

$$\{T(x, y, w), z\} = 2\Delta(x, y, w, z).$$

FTS symmetries

Automorphism group

$$\mathrm{Aut}(\mathfrak{F}) := \{\sigma | \{\sigma x, \sigma y\} = \{x, y\}, \quad \Delta(\sigma x, \sigma y, \sigma z, \sigma w) = \Delta(x, y, z, w)\}.$$
$$\Rightarrow T(\sigma x, \sigma y, \sigma z) = \sigma T(x, y, z)$$

$\mathfrak{J}_3$	$\mathrm{Str}_0(\mathfrak{J}_3)$	$\mathrm{Aut}(\mathfrak{F}(\mathfrak{J}_3))$	$\dim \mathfrak{F}(\mathfrak{J}_3)$
$\mathbb{R}$	$-$	$SL(2,\mathbb{R})$	4
$\mathbb{R}\oplus\mathbb{R}$	$\mathrm{SO}(1,1;\mathbb{R})$	$[\mathrm{SL}(2,\mathbb{R})]^2$	6
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	$[\mathrm{SO}(1,1;\mathbb{R})]^2$	$[\mathrm{SL}(2,\mathbb{R})]^3$	8
$\mathbb{R}\oplus\Gamma_{1,n-1}$	$\mathrm{SO}(1,1;\mathbb{R})\times\mathrm{SO}(1,n-1;\mathbb{R})$	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n;\mathbb{R})$	$2n+4$
$\mathbb{R}\oplus\Gamma_{5,n-5}$	$\mathrm{SO}(1,1;\mathbb{R})\times\mathrm{SO}(5,n-5;\mathbb{R})$	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n;\mathbb{R})$	$2n+12$
$J_3^{\mathbb{R}}$	$\mathrm{SL}(3,\mathbb{R})$	$\mathrm{Sp}(6,\mathbb{R})$	14
$J_3^{\mathbb{C}}$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SU}(3,3;\mathbb{R})$	20
$J_3^{\mathbb{H}}$	$\mathrm{SU}^\star(6,\mathbb{R})$	$\mathrm{SO}^\star(12,\mathbb{R})$	32
$J_3^{\mathbb{O}}$	$E_{6(-26)}(\mathbb{R})$	$E_{7(-25)}(\mathbb{R})$	56
$J_3^{\mathbb{O}^s}$	$E_{6(6)}(\mathbb{R})$	$E_{7(7)}(\mathbb{R})$	56

FTS symmetries

The automorphism group is generated by [Brown: 1969]:

$$\phi(W): \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + (Y, W) + (X, W^\sharp) + \beta N(W) & X + \beta W \\ Y + X \times W + \beta W^\sharp & \beta \end{pmatrix},$$

$$\psi(Z): \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & X + Y \times Z + \alpha Z^\sharp \\ Y + \alpha Z & \beta + (X, Z) + (Y, Z^\sharp) + \alpha N(Z) \end{pmatrix},$$

$$T(s): \begin{pmatrix} \alpha & X \\ Y & \beta \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}\alpha & s(X) \\ s'^{-1}(Y) & \lambda\beta \end{pmatrix}.$$

Ranks

Natural, Aut invariant rank conditions:

$$\begin{aligned}
\mathrm{Rank} x = 1 &\Leftrightarrow 3T(x,x,y) + \{x,y\}x = 0 \qquad \forall y \in \mathfrak{F}; \\
\mathrm{Rank} x = 2 &\Leftrightarrow T(x,x,x) = 0, \quad \exists y \quad \text{s.t.} \quad 3T(x,x,y) + \{x,y\}x = 0 \neq 0; \\
\mathrm{Rank} x = 3 &\Leftrightarrow \Delta(x) = 0, \quad T(x,x,x) \neq 0; \\
\mathrm{Rank} x = 4 &\Leftrightarrow \Delta(x) \neq 0.
\end{aligned}$$

$\mathfrak{F}^{\mathbb{O}^s}$: black holes

- In the classical supergravity limit the 28+28 electric/magnetic black hole charges x_I ($I = 1, \ldots, 56$) transform as the fundamental **56** of the continuous U-duality group $E_{7(7)}(\mathbb{R})$. Under $SO(1,1;\mathbb{R}) \times E_{6(6)}(\mathbb{R})$ the **56** breaks as

$$\mathbf{56} \to \mathbf{1}_3 + \mathbf{1}_{-3} + \mathbf{27}_1 + \mathbf{27}'_{-1}.$$

  The charges may be represented as $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}^s}) = \mathfrak{F}^{\mathbb{O}^s}$,

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{\mathbb{O}^s}.$$

  Here, p^0, q_0 are the graviphotons and P, Q are the magnetic/electric $\mathbf{27}'$ and $\mathbf{27}$.

- Leading order black hole entropy:

$$S_{D=4,\mathrm{BH}} = \pi\sqrt{|\Delta(x)|} \qquad (\Delta = \mathbf{1} \in [\mathbf{56} \times \mathbf{56} \times \mathbf{56} \times \mathbf{56}]_s)$$

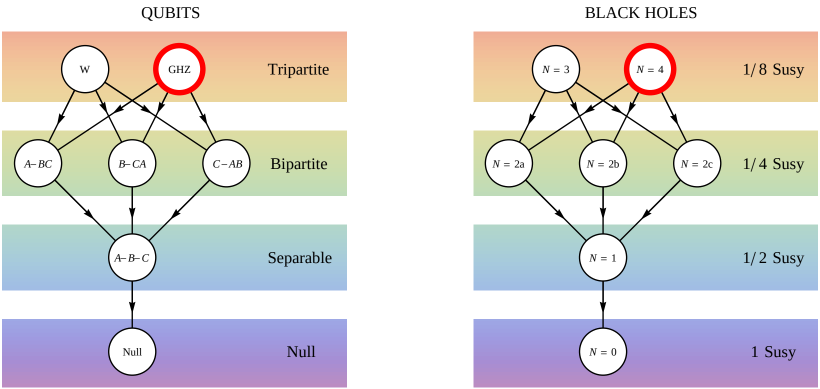
Black hole charge orbits

Rank	Rank cond. non-vanishing	Rep state	Orbit	dim	SUSY
1	x	$\begin{pmatrix} 1 & (0,0,0) \\ 0 & 0 \end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(6)}(\mathbb{R}) \ltimes \mathbb{R}^{27}}$	28	1/2
2	$3T(x,x,y) + x\{x,y\}$	$\begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{O(6,5;\mathbb{R}) \ltimes \mathbb{R}^{32} \times \mathbb{R}}$	45	1/4
3	$T(x,x,x)$	$\begin{pmatrix} 0 & (1,1,1) \\ 0 & 0 \end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{F_{4(4)}(\mathbb{R}) \ltimes \mathbb{R}^{26}}$	55	1/8
4	$\Delta(x) > 0$	$\begin{pmatrix} 1 & (1,1,k) \\ 0 & 0 \end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(2)}(\mathbb{R})}$	55	1/8
4	$\Delta(x) < 0$	$\begin{pmatrix} 1 & (1,1,-k) \\ 0 & 0 \end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(6)}(\mathbb{R})}$	55	0

[Ferrara, Gunaydin: hep-th/9708025; Lu, Pope, Stelle: hep-th/9708109]
[Sukuzawa: 2006; LB, Duff, Ferrara, Marrani, Rubens: arXiv:1108.0424, arXiv:1108.0908]

Generalisations

- This Jordan algebra and FTS framework can be use to classify the black holes of a large number of theories
  [Ferrara, Günaydin:hep-th/0606108; Bellucci, Ferrara, Günaydin, Marrani:hep-th/0606209]
- In particular $D = 4, \mathcal{N} = 2$ theories which arise from compactifying type II string theory on a Calabi-Yau
- See "Small Orbits" and "Explicit Orbit Classification of Reducible Jordan Algebras and Freudenthal Triple Systems"
  [LB, Duff, Ferrara, Marrani, Rubens: arXiv:1108.0424, arXiv:1108.0908]



- N= number of charges / number of kets

Freudenthal duality of black holes in supergravity with U-duality of type E_7

Definition

$$\tilde{}: \mathfrak{F} \to \mathfrak{F} \qquad x \mapsto \tilde{x} = \frac{T(x,x,x)}{\sqrt{|\Delta(x)|}}$$

Key properties

$$\{\tilde{x}, x\} = 2\,\mathrm{sgn}(\Delta)\sqrt{|\Delta(x)|}$$

$$\tilde{\tilde{x}} = -x$$

$$\Delta(\tilde{x}) = \Delta(x)$$

$\Rightarrow$ The F-dual is a nonlinear transformation acting on the black hole charges which leaves the leading order Bekenstein-Hawking entropy invariant.

Quantization of charges

Dirac-Schwinger-Zwanziger quantization

Dirac-Schwinger-Zwanziger quantization condition for two dyons $x = (q_I, p^I)$ and $x' = (\tilde{q}_I, \tilde{p}^I)$:

$$\frac{1}{2}[p^I \tilde{q}_I - \tilde{p}^I q_I] \in \mathbb{Z}.$$

The charges live on an lattice and the U-duality is broken to a discrete subgroup

$$G_4(\mathbb{R}) \to G_4(\mathbb{Z})$$

Example: M-theory on a 7-torus

$D = 4, \mathcal{N} = 8$ sugra has U-duality group $E_{7(7)}(\mathbb{R})$: gets broken by stringy corrections

$$E_{7(7)}(\mathbb{Z})$$

Fundamental symmetry of M-theory [Hull, Townsend: hep-th/9410167]

DSZ in FTS language

$$\{x, x'\} \in \mathbb{Z}$$

[LB, Dahanayake, Duff, W. Rubens: arXiv:0903.5517]

Quantization of charges: Implications

Integral FTS

- The integral charges implies we must use an "integeral FTS"

$$\mathfrak{F}(\mathbb{R}) \to \mathfrak{F}(\mathbb{Z})$$

  based on integral Jordan algebra $\mathfrak{J}(\mathbb{Z})$

- Notion made precise by S. Krutelevich

- We focus on the $\mathcal{N} = 8$ with $E_{7(7)}(\mathbb{Z})$ symmetry theory hereafter

[Krutelevich, J. Algebra (2002); J. Algebra (2007), arXiv:math/0411104]
[LB, Dahanayake, Duff, W. Rubens: arXiv:0903.5517]
[LB et al: arXiv:1002.4223]

Integral Freudenthal triple system $\mathfrak{F}^{\mathbb{O}^s_{\mathbb{Z}}}$

The integral Freudenthal triple system $\mathfrak{F}^{\mathbb{O}^s_{\mathbb{Z}}}$ provides a natural model for $E_{7(7)}(\mathbb{Z})$ acting on lattice of charges [Krutelevich: 2004].

- The quantized black hole charge vector is given by,

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{Z} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{\mathbb{O}^s_{\mathbb{Z}}}.$$

- Here $\mathfrak{J}$ is given by

$$Q = \begin{pmatrix} q_1 & Q_s & \overline{Q_c} \\ \overline{Q_s} & q_2 & Q_v \\ Q_c & \overline{Q_v} & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{Z} \quad \text{and} \quad Q_{v,s,c} \in \mathbb{O}^s_{\mathbb{Z}}.$$

Integral black hole canonical form

Canonical form

Every element $x \in \mathfrak{F}^{\mathbb{O}^s_{\mathbb{Z}}}$ is $E_{7(7)}(\mathbb{Z})$ equivalent to a diagonally reduced canonical form,

$$x_{\text{can}} = \alpha \begin{pmatrix} 1 & k\,\text{diag}(1, l, lm) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad \alpha > 0.$$

Canonical quartic norm

$$\Delta(x) = -(j^2 + 4k^3 l^2 m)\alpha^4 \quad \Rightarrow \quad \Delta(x) \in \{0, 1\} \mod 4$$

Square of the horizon area is quantized!

- Proof [Krutelevich: 2004]

Integral black hole canonical form

- Given a discrete subgroup $G(\mathbb{Z}) \subset G$ we can define new invariants:

$$\gcd(\mathbf{rep}).$$

[Krutelevich:2004]:

$$\begin{aligned} d_1(x) &:= \gcd(x), \\ d_2(x) &:= \gcd(3T(x,x,y) + \{x,y\}x), \ \forall y \\ d_3(x) &:= \gcd(T(x,x,x)), \\ d_4(x) &:= \Delta(x). \end{aligned}$$

plus one more

$$d_4'(x) := \gcd(x \wedge T(x)).$$

- Insufficient to fix the can form!

$$\alpha \begin{pmatrix} 1 & (0,0,0) \\ (0,0,0) & j \end{pmatrix}, \qquad \alpha \begin{pmatrix} 1 & (j,0,0) \\ (0,0,0) & j \end{pmatrix},$$

- However, in particular sub-cases they do the job.

>1/8-BPS and *projective* black holes.

>1/8-BPS black hole orbits

Assuming >1/8-BPS (Rank < 3) improved canonical form:
Every element $x \in \mathfrak{F}^{\mathbb{O}^s_\mathbb{Z}}$ is $E_{7(7)}(\mathbb{Z})$ equivalent to a diagonally reduced canonical form,

$$x_{\text{can}} = \alpha \begin{pmatrix} 1 & k\,\text{diag}(1,0,0) \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha > 0.$$

- Uniquely fixed by: $d_1(x) := \gcd(x), d_2(x) := \gcd(3T(x,x,y) + \{x,y\}x), \ \forall y$ since $d_1(x_{\text{can}}) = \alpha$ and $d_2(x_{\text{can}}) = 2\alpha^2 k$.
-

>1/8-BPS (Rank < 3) black hole orbit classification:

1. The complete set of distinct 1/2-BPS charge vector orbits is given by,

$$\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha > 0\}.$$

2. The complete set of distinct 1/4-BPS charge vector orbits is given by,

$$\{\alpha \begin{pmatrix} 1 & k(1,0,0) \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha, k > 0\}.$$

Projective black holes

The concept of a projective element was originally introduced for the case $\mathfrak{J}_3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by Manjul Bhargava (2004) in the context of generalising Gauss's composition law for quadratic forms.

An element x is said to be *projective* if its U-duality orbit contains a diagonal reduced element

$$x = \begin{pmatrix} \alpha & (X_1, X_2, X_3) \\ 0 & \beta \end{pmatrix},$$

satisfying

$$\begin{aligned} \gcd(\alpha X_1, \alpha\beta, X_2 X_3) &= 1; \\ \gcd(\alpha X_2, \alpha\beta, X_1 X_3) &= 1; \\ \gcd(\alpha X_3, \alpha\beta, X_1 X_2) &= 1. \end{aligned}$$

Projective black hole canonical form

Any projective element x is U-duality equivalent to an element (Krutelevich:2004):

$$\begin{pmatrix} 1 & (1,1,m) \\ (0,0,0) & j \end{pmatrix},$$
$$j \in \{0,1\}, \;\; m \in \mathbb{Z},$$

where the values of m and j are uniquely determined by $\Delta(x) = -(j + 4m)$.

- When Δ is odd, $d_3(x) = 1$ iff x is projective

In the projective case all black holes with the same quartic norm and hence lowest order entropy are U-duality related.

Implications for F-duality

Reminder: F-duality

$$\tilde{}: \mathfrak{F} \to \mathfrak{F} \qquad x \mapsto \tilde{x} = \frac{T(x,x,x)}{\sqrt{|\Delta(x)|}}$$

Not every black hole admits a well defined F-dual

Necessary and sufficient conditions

- Requiring that $\tilde{x}$ is integer therefore restricts us to that subset of black holes for which $|\Delta(x)|$ is a perfect square and for which $|\Delta(x)|^{1/2}$ divides $T(x)$:

$$d_4(x) = \left[\frac{d_3(x)}{d_1(\tilde{x})}\right]^2 = \left[\frac{d_3(\tilde{x})}{d_1(x)}\right]^2 = d_4(\tilde{x}),$$

where $d_1(x) = \gcd(x)$, $d_3(x) = \gcd(T(x))$ and $d_4(x) = |\Delta(x)|$.

Implications for F-duality

Discrete invariants under F-duality

- Not all discrete U-duality invariants are F-dual invariant
- For example, the product $d_1(x)d_3(x)$ is invariant but $d_1(x)$ and $d_3(x)$ separately need not be.
- However not only $d_4(x)$ but also $d_2(x)$, $d_2'(x)$ and $d_4'(x)$ are F-dual invariant.
- The invariance of $d_4'(x)$ follows from

$$\begin{aligned} \tilde{x} \wedge T(\tilde{x}) &= T(x)|\Delta|^{-1/2} \wedge T(T(x)|\Delta|^{-1/2}) \\ &= -|\Delta|^{-2} T(x) \wedge \Delta^2 x \\ &= \mathrm{sgn}(\Delta) x \wedge T(x) \end{aligned}$$

  and, hence, $d_4'(x) = d_4'(\tilde{x})$.
- Since corrections to the black hole can depend on the discrete invariants whether F-duality preserves the entropy to all orders is an open question.

Implications for F-duality

Comments

- Large class of F-dual black holes
- We don't know how they are characterised
- Projective black holes:

  In the projective case all black holes with the same quartic norm and hence lowest order entropy are U-duality related.

- Non-projective black holes:

  Non-projective black holes related by an F-duality not conserving d_1 *provide examples of configurations with the same quartic norm and hence lowest order entropy that are definitely not U-duality related,*

- But more surprisingly

  Non-projective black holes related by an F-duality conserving d_1 *provide examples of configurations with the same quartic norm, and same discrete invariants, that are apparently not U-duality related*

Open questions

- Is it possible that the full space of 4-dimensional orbits could be resolved if the complete list of independent arithmetic invariants was known?
- To proceed further, it would serve us well to have a full classification of the independent $E_{7(7)}(\mathbb{Z})$ arithmetic invariants.
- When is Δ a perfect square?
- Can the class of black holes admitting an F-dual be classified in a useful way?
- If so what is its physical significance?
- Does the F-dual leave the entropy invariant to all orders? Requires a U-duality invariant black hole entropy formula for arbitrary charges
  For $d_2(x) = 1$ (almost, but not quite, projectivity) entropy is determined by $d'_4(x)$ and $\Delta(x)$ [Sen: 0804.0651; Sen: 0908.0039] [Bianchi, Ferrara, Kallosh; 0912.0057, 0910.3674]
- Promote to symmetry of supergravity Lagrangians?
  YES! [Borsten, Duff, Ferrara, Marrani arXiv1212.3254]