The Theory of Flux Tubes

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Introduction

- Theories of Goldstone bosons are often non-renormalizable; they have a natural cutoff being the “decay constant.”
- Yet these theories are useful and highly predictive (because of the nonlinearly realized symmetry). The reason is that in such theories we can count operators not according to their naive dimension, but according to some other “scaling” dimension.
- If operators are arranged in this way, according to their scaling, there are just finitely many coefficients for any scaling, and the number of operators with given scaling is often infinite (or much larger than the number of independent coefficients).
- This is what makes such theories predictive.

Introduction

Let us consider some examples. The most canonical one is non-Abelian pion physics. For example, the leading Lagrangian for $SU(2)\times SU(2)\to SU(2)$ is

$$\mathcal{L}=f_\pi^2 Tr(g^{-1}dgg^{-1}dg)\sim\frac{(\partial\vec{\pi})^2}{1+\vec{\pi}^2/f_\pi^2}$$

We see that infinitely many terms are fixed by just one coefficient, f_π. The appropriate notion of scaling is such that $S[\pi]=0, S[\partial]=1$. If we classify operators by the scaling S there are finitely many coefficients at every S and infinitely many operators with this S.

The operators with $S=4$ were first classified by Gasser and Leutwyler. They are nontrivial and the corresponding few "new" coefficients in QCD are fairly well known.

Introduction

When one classifies terms at higher order, such as terms with $S = 4$, one needs to count them modulo the equations of motion of the $S = 2$ theory. Indeed, terms at $S = 4$ that are proportional to the equations of motion of the $S = 2$ are called "redundant" because they can be removed by a change of variables. (In other words, they do not contribute to the S-matrix.)

Introduction

One very interesting fact about this theory (well, the case of $SU(3) \times SU(3) \to SU(3)$) is that there is a special operator with $S = 4$. It looks like $\pi \partial^4 \pi^4$ with an epsilon tensor. This operator explains the process $KK \to \pi\pi\pi$ that would otherwise be impossible to account for.

The coefficient of this $S = 4$ term is fixed by an anomaly argument (Wess&Zumino, Witten). One symptom of it being special is that it is impossible to write this term locally in 4d using the matrix of pions g. However it can be written using an extension of the matrix g into some auxiliary 5d space, where we write a closed form of the type

$$\int_{\mathcal{M}_5} Tr \left(g^{-1} dg\right)^3$$

Introduction

It is definitely a legal term, when we expand it in components in 4d it is perfectly invariant under the non-linearly realized chiral symmetry. It is just that one cannot write down this term "covariantly" directly in 4d.

Introduction

Another example is the spontaneous breaking of $\mathcal{N}=1$ supersymmetry. The Goldstone field is the Goldstino G_α and the natural scaling S in this problem is $S[G_\alpha]=-1/2$ and $S[\partial]=1$. Because of the Fermi statistics we learn that

- There are only terms with $S\geq 0$
- For any S there are only finitely many terms one can write.

However the theory is still predictive. For example, at $S=0$ we encounter the familiar Akulov-Volkov action

$$\mathcal{L}=F^2+i\bar{G}\partial G+\frac{1}{F^2}\bar{G}^2\partial^2G^2+\frac{1}{F^4}\bar{G}^4\partial^4G^4\ .$$

One can also classify the subleading terms. There are always finitely many operators at any given S, but less independent coefficients. Thus, the theory is predictive.

Introduction

One may ask whether there is something like the Wess-Zumino term in supersymmetric theories. As far as I know there is none.

The folklore is that superspace has no topology and so no WZ term.

Spontaneously Broken Conformal Symmetry

Our final example is the spontaneous breaking of conformal symmetry. We will discuss for concreteness $d = 4$, other interesting cases such as $d = 6$ were discussed in the paper of Elvang, Freedman, Hung, Kiermaier, Myers, and Theisen. The discussion below is based on ZK&Schwimmer (see also Schwimmer&Theisen).

- Some conformal field theories have a moduli space of vacua.
- We can then travel on this moduli space of vacua. This breaks the conformal symmetry spontaneously. The NG boson is the dilaton τ. (Associated to $SO(d, 2) \to SO(d - 1, 1)$.)
- The appropriate scaling in this case is like in pion physics, $S[\tau] = 0$, $S[\partial] = 1$.

Spontaneously Conformal Symmetry

One finds a universal two derivative term

$$\int e^{-2\tau}(\partial\tau)^2 \ .$$

Next one attempts to classify the terms at $S = 4$ that can be written covariantly in $d = 4$. One remembers to divide by the equations of motion of the $S = 2$ theory. The answer is that there are no such terms.

This is unlike pion physics, where there are plenty of terms at $S = 4$. Hence the case of the dilaton seems "more predictive."

Spontaneously Conformal Symmetry

Amazingly, at $S = 4$ one finds a term that is perfectly invariant under the nonlinearly realized conformal symmetry but cannot be written covariantly, and does not vanish by the equations of motion. It looks roughly like

$$\int d^4x \, (\partial\tau)^4$$

As our experience from pion physics suggests, it may be associated to an anomaly. Indeed, one can show that its coefficient is completely fixed by the a-anomaly. This story is very closely related to the a-theorem.

The Flux Tube

The dilaton effective action is associate to the breaking $SO(d,2) \to SO(d-1,1)$. We will now investigate another case where there is some space-time symmetry broken, of the type

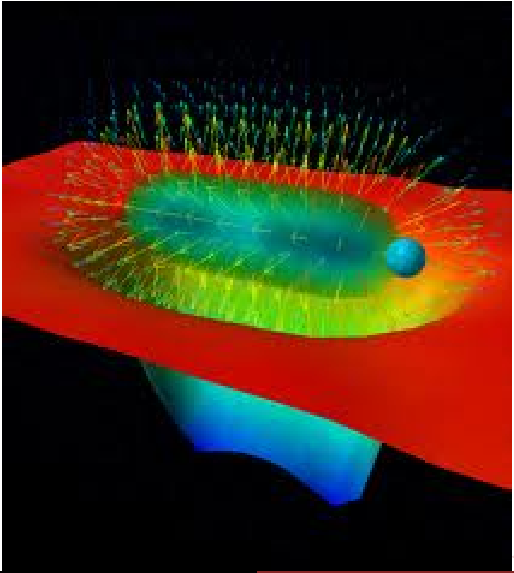
$$SO(d) \to SO(d-2) \times SO(2)$$

.

This happens when the d-dimensional theory admits a long stable string.

- The simplest example is the Abelian Higgs model. There one can see the long string at weak coupling.
- Such long strings can also result from a complicated RG flow, such as in pure Yang-Mills theory. There the long string is formed by the confined flux lines.

The Flux Tube



The Flux Tube

This is an example of an RG flow "across dimensions." (Using Bobev's terminology.)

The theory in the bulk can be completely massive but there are some degrees of freedom that live on the defect (vortex).

Surprisingly, in some supersymmetry theories, this RG flow across dimensions does retain some highly nontrivial information from the high energy theory, for example, BPS states, structure of vacua etc. For example, 4D/2D dualities that were discussed by Dorey, Hollowood, Tong, Shifman, Yung, Hanany, and others.

The Flux Tube

Our goal here is to understand the basics of this RG flow from 4d to 2d (or general D to 2d). Our discussion would apply even in strong coupled theories such as pure Yang-Mills theory.

The Flux Tube

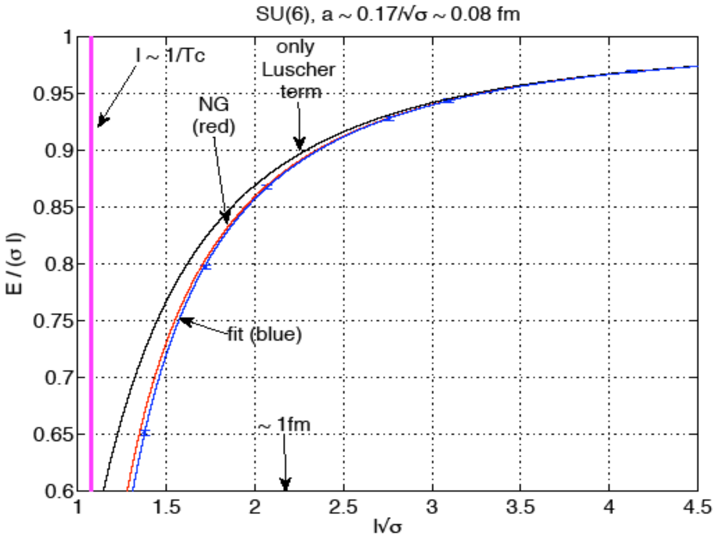
There has been remarkable progress on precision measurements of their properties on the lattice (for various gauge groups) [Teper et al.]. The measurements are of the energy levels, $E_n(L)$, where L is the length of the flux tube and n is the excitation level.
In general we have

$$E_n = TL + \frac{a_n^{(1)}}{L} + \frac{a_n^{(2)}}{TL^3} + \frac{a_n^{(3)}}{T^2L^5} \cdots$$

where T is the tension and the rest are some unknown coefficients. However, the lattice measurements done to date fit (within the available precision) extremely well with the formula (sometimes called the "NG formula" – it was first proposed by [Arwis])

$$E_n = TL\sqrt{1 + \frac{8\pi}{TL^2}\left(n - \frac{D-2}{24}\right)}$$

The Flux Tube



The Flux Tube

We will now explain the reasons for this phenomenon. We will see that some ingredients are very reminiscent of the dilaton effective action. The conclusions are

- $a_n^{(1)}, a_n^{(2)}, a_0^{(3)}$ MUST agree with the square root ansatz in any field theory. Basically, the current precision allows to go up to $1/L^3$. Hence we have a prediction about the ground state energy at $1/L^5$.
- There MUST be deviations from the square root formula already for $a_{n>0}^{(3)}$. These deviations are completely model independent and calculable.
- These predictions should hold even in strongly coupled field theories, and should be verifiable in the future when the lattice techniques develop further.
- We expect the coefficients $a_n^{(k>3)}$ to be model dependent.

Effective Action on the Flux Tube

Assumptions:

- The theory admits long stable strings.
- There is a gap in the bulk and the only massless modes on the string are the $D-2$ NG bosons X^i that describe the bending of the string into the orthogonal directions.

Both assumptions hold true in the Abelian Higgs model and in Yang-Mills theory. However, the assumptions about the gap can be violated in very interesting ways in the presence of supersymmetry. (Hanany&Tong, Shifman&Yung, Konishi et al.) It would be interesting to extend our analysis to these cases.

Effective Action on the Flux Tube

As long as our assumptions hold true, the Renormalization Group flow would wipe out all the degrees of freedom except for the NG bosons describing the fluctuations of the flux tube.

A convenient choice is thus to write an action for $X^\mu(\sigma_1, \sigma_2)$, and we impose Lorentz invariance in space-time and diff-invariance on the worldhseet of the flux tube.

$$S = \int d^2\sigma \mathcal{L}(X^\mu(\sigma_1, \sigma_2))$$

After fixing the Lagrangian $\mathcal{L}$ we can fix a gauge where the string just sits in the 01 plane $X^0 = \sigma_1$, $X^1 = \sigma_2$ and perform computations in this unitary gauge.

Effective Action on the Flux Tube

The problem is thus reduced to classifying actions

$$S = \int d^2\sigma \mathcal{L}(X^\mu(\sigma_1, \sigma_2))$$

which are Lorentz invariant in space-time and diff-invariant on the world-sheet. This is a well-defined mathematical problem.

A 2d surface embedded in some ambient $\mathbb{R}^d$ is characterized by the first fundamental form

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu$$

and the second fundamental form

$$\Omega^\mu_{ab} = \nabla_a \partial_b X^\mu$$

Effective Action on the Flux Tube

One can work a little to convince oneself that we don't need any other object other than Ω and h. The natural scaling here is $S[X] = -1$, $S[\partial] = 1$ and so $S[\Omega] = 1$ and $S[h] = 0$.

So now we can classify Lagrangians. We begin with $S = 0$. We cannot use the second fundamental form, and the only option is thus

$$-T \int d^2\sigma \sqrt{h}$$

In static a gauge this becomes an action for X^i, $i = 2, ..., d-1$

$$-T \int d^2\sigma \sqrt{-\det\left(\eta_{ab} + \partial_a X^i \partial_b X^i\right)}$$

$$= T \int d^2\sigma \left(-1 + \frac{1}{2}\partial_a X^i \partial_b X^i + ...\right)$$

Effective Action on the Flux Tube

This is the familiar NG action from string theory. Here it is just the $S = 0$ approximation. Let us now comment that there are no terms with $S = 1$ unless we are willing to use some epsilon tensors, which we are not going to do (since they cannot appear in Yang-Mills theory).

At the order $S = 2$ we find two terms

$$\alpha \int d^2\sigma \sqrt{-h}h^{ab}(\Omega_{ab;A})^2 + \beta \int d^2\sigma \sqrt{-h}h^{ac}h^{bd}\Omega_{ab;A}\Omega_{cd;A}$$

Using the Gauss-Codazzi equation one can show that these terms are topological+redundant.

Effective Action on the Flux Tube

This phenomenon is very reminiscent of the dilaton effective action.

The next step would be to search for an anomaly term. Maybe there is some anomaly at $S = 2$. We have not managed to find one. Therefore, the claim is

The effective theory on the flux tube thus has no operators at $S = 2$. This means that the predictions of the $S = 0$ theory cannot be corrected before the order $S = 4$.

Effective Action on the Flux Tube

We learn that the first irrelevant operator correcting

$$-T\int d^2\sigma\sqrt{-\det\left(\eta_{ab}+\partial_a X^i\partial_b X^i\right)}$$

has $S=4$ and thus looks roughly like

$$\partial_-^3 X\partial_+^3 X\partial_- X\partial_+ X$$

and simple dimensional analysis shows that it can affect the series

$$E_n = TL+\frac{a_n^{(1)}}{L}+\frac{a_n^{(2)}}{TL^3}+\frac{a_n^{(3)}}{T^2L^5}\cdots$$

only for $a_n^{k>3}$.

We see that $a_n^{k\leq 3}$ are all completely model independent. (Just depend on the number of dimensions of the original QFT.)

Effective Action on the Flux Tube

Paraphrasing Rob, every two graduate students who will attempt to compute $a_n^{k\leq 3}$ starting from the (non-renormalizable) $S=0$ theory, must agree. (This is if they both use regulators that preserve the symmetries, or carefully tune away explicitly broken symmetries due to the regulators.)

Various regularizations used in this context actually break some of the symmetries, and there is a very nice recent paper by Dubovsky,Flauger,Gobrenko. They explain how to do computations while not breaking the symmetries.

Effective Action on the Flux Tube

So indeed, graduate students were assigned this computation. It was found that

$$a_n^{(1)} = 4\pi \left(n - \frac{D-2}{24} \right) \;, \qquad a_n^{(2)} = -\pi \left(n - \frac{D-2}{24} \right)^2$$

This is completely consistent with the square-root formula $E_n = TL\sqrt{1 + \frac{8\pi}{TL^2}\left(n - \frac{D-2}{24}\right)}$ lattice people were using, and it explains why it works to such a high precision.

The computation of $a_n^{(3)}$, which are model independent and completely well defined, shows deviations from the square-root formula for $n > 0$. (It seems a little accidental that the ground state energy does not deviate from the square-root formula.) For instance, $\Delta a_1^{(3)} = \frac{4}{3}\pi^3(D-26)(D-3)$ for the scalar at level 1.

Another Formalism

There is the long string theory of Polchinski&Strominger. This is a theory with more than $D-2$ degrees of freedom, and a set of constraints that is supposed to eliminate the extra dofs.

If it is true that there is a unique realization of this nonlinear symmetry, and if it is true that they can impose their constraints consistently to all orders in $1/L$ in such a way that there are no ghosts, then their formulation should give the same answers as what we find.

Open Questions

- To verify experimentally these predictions.
- To extend this study of RG flows $D \to 2d$ for more exotic vortices, like those appearing in RG flows in supersymmetric theories. Those can have nontrivial theories living on them.
- Long strings can be naturally mapped via AdS/CFT to F strings or D strings in the bulk. One can try to verify these claims there. (Some of this was done for F strings but there are open questions. See Aharony&Karzbrun.)
- What can we learn from the model-dependent coefficients? interpolations to small L?
- Relation to the formulation of Polchinski&Strominger and a derivation of their formalism / proof of equivalence.