Properties of entropy in holographic theories

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1 Properties of entropy

One system:

$$S(A) \geq 0\,, \qquad S(A) = 0 \quad \text{iff } \rho_A \text{ is pure}$$

Hence $S(A)$ is a measure of mixedness

. . . .

Two systems:

(1) Araki-Lieb:

$$|S(A) - S(B)| \leq S(AB) \leq S(A) + S(B)$$

in particular if AB is pure then $S(A) = S(B)$

(2) Subadditivity:

$$S(AB) \leq S(A) + S(B)\,, \qquad S(AB) = S(A) + S(B) \quad \text{iff } \rho_{AB} = \rho_A \otimes \rho_B$$

Hence mutual information

$$I(A:B) = S(A) + S(B) - S(AB)$$

is a measure of correlation (classical + quantum) between A, B. Key quantity in (classical + quantum) information theory. Motivations/applications:

- In terms of conditional entropy

$$S(A|B) = S(AB) - S(B) = \text{expected entropy of } A \text{ conditioned on } B,$$

$$\begin{aligned} I(A:B) &= S(A) - S(A|B) \\ &= \text{how much your ignorance about } A \text{ decreases if you know state of } B \end{aligned}$$

- $I(A:B)$ gives the rate at which information can reliably be sent over a noisy channel (Shannon)
- Bound on correlators between normalized operators: (Wolf, Verstraete, Hastings, Cirac '07):

$$(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2 \leq 2I(A:B)$$

Examples:

- Classical correlation:

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$
$$\rho_A = \rho_B = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$
$$S(A) = S(B) = S(AB) = \ln 2\,, \qquad I(A:B) = \ln 2$$

- Entanglement:

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$
$$\rho_A = \rho_B = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$
$$S(A) = S(B) = \ln 2\,, \qquad S(AB) = 0\,, \qquad I(A:B) = 2\ln 2$$

. . . .

Three systems:

Strong subadditivity:

$$S(ABC) + S(B) \leq S(AB) + S(BC)$$
$$S(A) + S(C) \leq S(AB) + S(BC)$$

SSA implies monotonicity of mutual information:

$$I(A:BC) \geq I(A:B)$$

. . . .

Four systems:

Constrained inequality (Linden, Winter '04): If

$$I(A:BC) = I(A:B) = I(A:C)\,, \qquad I(B:CD) = I(B:D)$$

then

$$I(C:D) \geq I(C:AB)$$

. . . .

Five or more systems:

Infinite hierarchy of constrained inequalities (Cadney, Linden, Winter '11)

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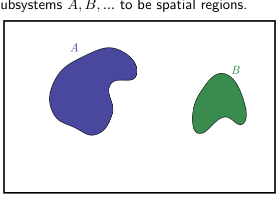
2 EE in QFT

In a QFT, we can take subsystems $A, B, \ldots$ to be spatial regions.



The EE is UV-divergent along boundary of A,

$$S(A) = \epsilon^{2-D}\text{area}(\partial A) + \cdots$$

or, in two dimensions,

$$S(A) = -\frac{c_{\text{UV}}}{6}\ln\epsilon\ \#(\partial A) + \cdots$$

Mutual information is divergent along common boundary,

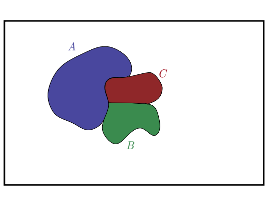
$$I(A:B) = 2\epsilon^{2-D}\,\text{area}(\partial A \cap \partial B) + \cdots\,,$$

finite if separated

. . . .

Area-law divergence manifestly obeys SSA, since

$$\text{area}\left(\partial A \cap \partial(BC)\right) = \text{area}(\partial A \cap \partial B) + \text{area}(\partial A \cap \partial C) \geq \text{area}(\partial A \cap \partial B)$$



Also obeys Linden-Winter constrained inequality, since if

$$I(A:BC) = I(A:C)\,, \qquad I(B:CD) = I(B:D)$$

then C is separated from A, B, hence $I(C:AB) = 0$, hence

$$I(C:D) \geq I(C:AB)$$

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3 RT formula

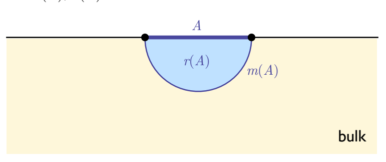
Ryu-Takayanagi formula for EE

- of a spatial region
- in a holographic theory
- dual to classical Einstein gravity ("large N, strong coupling")
- in a state described in the bulk by a static, classical field configuration ($\Rightarrow$ distinguished constant-time surfaces):

$$S(A) = \frac{1}{4G_N}\min_{m \sim A}\Big(\text{area}(m)\Big)$$

where

- $\text{area}(m)$ is computed w.r.t. spatial, Einstein-frame metric
- $m \sim A$ means $\exists$ bulk region r s.t. $\partial r = m \cup A$
- call minimizer $m(A)$, $r(A)$



Notes:

- Is $r(A)$ the holographic dual (in some sense) of ρ_A? (See also Czech, Karczmarek, Nogueira, Van Raamsdonk '12)
- Corrections believed to take general form (α' = classical higher-derivative; G_N = quantum $= 1/N^2$):

$$S(A) = \frac{1}{4G_N}\min_{m\sim A}\Big(\text{area}(m) + \mathcal{O}(\alpha')\Big) + \mathcal{O}(G_N^0)$$

See Hung, Myers, Smolkin '10, de Boer, Kulaxizi, Parnachev '10

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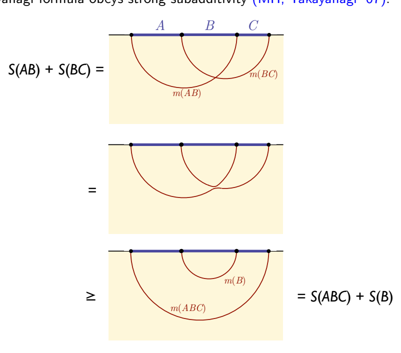
Ryu-Takayanagi formula obeys strong subadditivity (MH, Takayanagi '07):



Formal proof: Take $\tilde{r}(B) = r(AB) \cap r(BC)$, $\tilde{r}(ABC) = r(AB) \cup r(BC)$

Notes:

- Easily generalizes to prove other form of SSA, $S(AB) + S(BC) \geq S(A) + S(C)$, as well as subadditivity and Araki-Lieb
- Proof is far simpler than proof of SSA in quantum mechanics. General relativity knows some sophisticated quantum information theory!
- This proof (along with other tests of RT formula) automatically still holds in presence of higher-derivative (α') corrections, but not quantum ($1/N$) corrections

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4 Monogamy

We saw that mutual information is monotonic:

$$I(A:BC) \geq I(A:B)$$

(adjoining a system cannot decrease correlations).

But what do we expect for

$$I(A:BC) \qquad \text{vs.} \qquad I(A:B) + I(A:C) \quad ?$$

How are correlations correlated with each other? In principle three possibilities:

1. correlations are uncorrelated: $I(A:BC) = I(A:B) + I(A:C)$
2. correlations are shared: $I(A:BC) < I(A:B) + I(A:C)$
3. correlations are distributed: $I(A:BC) > I(A:B) + I(A:C)$

All three behaviors are possible in general. Example of shared correlations:

$$\rho(ABC) = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right)$$
$$I(A:BC) = I(A:B) = I(A:C) = \ln 2$$

Example of distributed correlations:

$$\rho(ABC) = \frac{1}{4}\left(|000\rangle\langle 000| + |110\rangle\langle 110| + |101\rangle\langle 101| + |011\rangle\langle 011|\right)$$
$$I(A:BC) = \ln 2\,, \qquad I(A:B) = I(A:C) = 0\,.$$

. . . .

Define "tripartite information":

$$\begin{aligned} I_3(A:B:C) &= I(A:B) + I(A:C) - I(A:BC) \\ &= S(A) + S(B) + S(C) - S(AB) - S(BC) - S(AC) + S(ABC) \end{aligned}$$

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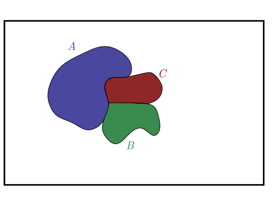
What happens in QFTs?

Area-law divergence in QFT has $I_3 = 0$; reflects purely pairwise correlations across entangling surfaces



In gapped $2+1$ theories (for A connected region),

$$S(A) = \epsilon^{-1}\,\text{area}(\partial A) - \gamma$$

In states with topological order, $\gamma > 0$, "topological entanglement entropy" (Kitaev & Preskill, Levin & Wen '05). Hence $I_3(A:B:C) = -\gamma < 0$: correlations are distributed non-locally

Free $1+1$ examples (Casini, Huerta '08):

- Massive fermion or boson: I_3 can be positive or negative depending on A, B, C
- Fermion in massless limit: $I_3 \to 0$ for all A, B, C
- Boson in massless limit: $I_3 \to +\infty$ for all A, B, C; after tracing over complement of ABC, long-wavelength modes lead to shared correlations

. . . .

In holographic theories, RT formula implies $I_3(A:B:C) \leq 0$ for any A, B, C (Hayden, MH, Maloney '11). Proof is more complicated version of holographic SSA proof

Holds in presence of higher-curvature corrections, not quantum corrections

Implies Linden-Winter and Cadney-Linden-Winter inequalities. RT formula obeys every known applicable property of entropy

Interpretation is not clear, but it suggests that correlations are distribute non-locally, as in topological order

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5 SSA of covariant holographic EE

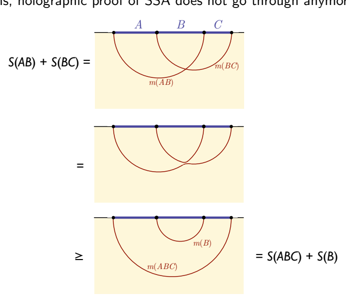
RT formula applies only to static bulk spacetimes, and region A lying in constant-time slice

Conjecture for covariant generalization by Hubeny, Rangamani, Takayanagi '07: replace minimal surface with minimal extremal surface. Has been applied to various systems, but subjected to fewer tests than static RT formula

Two key changes compared to static case:

- Surface $m(A)$ is co-dimension two in bulk spacetime; region $r(A)$ is co-dimension one
- $m(A)$ is not minimum but only extremum of area

For both reasons, holographic proof of SSA does not go through anymore:



Question: does covariant HEE formula obey SSA? A proof would

- Provide a crucial check on HRT formula
- Be an important new theorem in GR, at the level of the area theorem for black holes, and deepen our still-sketchy understanding of the entropy-area connection in GR

. . . .

Tests (Callan, He, MH '12; see also Allais, Tonni, '11)

We consider planar AdS_3, BTZ, AdS_3-Vaidya (preserve spatial translation symmetry)

Regions A, B, C are adjacent single intervals (so AB, BC, ABC are also single intervals). Two possibilities:

- constant-time:

- non-constant-time:

. . . .

Constant-time intervals:

Translation invariance: $S(A) = s(l_A)$, $S(AB) = s(l_A + l_B)$, etc

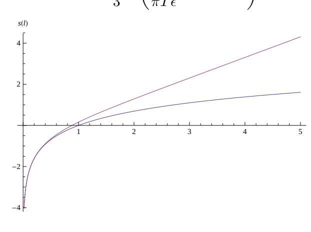
$$S(AB) + S(BC) \geq S(A) + S(C) \quad \Leftrightarrow \quad s'(l) \geq 0$$
$$S(AB) + S(BC) \geq S(ABC) + S(B) \quad \Leftrightarrow \quad s''(l) \leq 0$$

AdS_3:

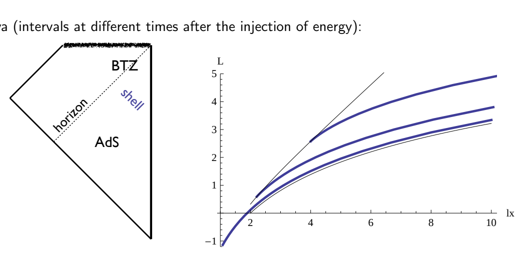
$$s(l) = \frac{c}{3}\ln\left(\frac{l}{\epsilon}\right)$$

BTZ:

$$s(l) = \frac{c}{3}\ln\left(\frac{1}{\pi T\epsilon}\sinh(\pi T L)\right)$$

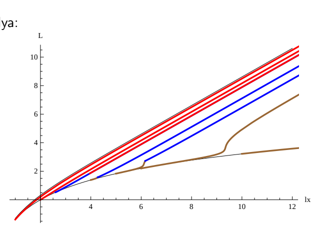


Vaidya (intervals at different times after the injection of energy):



UV thermalizes first, so short intervals show BTZ behavior while long intervals show AdS behavior

Negative-energy Vaidya:



Transition from BTZ to AdS behavior leads to loss of concavity. Violation of SSA is correlated with violations of null energy condition and second law (area theorem)

. . . .

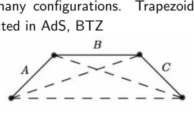
Non-constant-time intervals:

AdS, BTZ:

$$S(A) = \frac{1}{2}\left(s(\Delta x + \Delta t) + s(\Delta x - \Delta t)\right)$$

Since $s(l)$ obeys SSA, S obeys SSA

Vaidya: SSA was tested for many configurations. Trapezoid with A, C null provides most stringent test, since SSA saturated in AdS, BTZ



SSA violated precisely when NEC violated

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Proof?

A local proof for 2-dimensional theories can be obtained using geodesic deviation equation. Two assumptions required:

- Null energy condition
- Absence of conjugate points along geodesic

Still need global proof:

- Can absence of conjugate points be proven (N.B.: in positive signature, minimal geodesic never has conjugate points)?
- What about phase transitions?
- Higher dimensions?