Status of the a-Theorem in $d = 6$.

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$d = 4$	1107.3987	Komargodski + Schwimmer	[KS]
	1112.4538	Komargodski	[K2]
$d = 6$	1205.3994		
	Elvang, DZF, Hung, Kiermaier, Myers, Theisen		

Outline:

I. Introduction – Historical background and significance

II. The 8-minute a-theorem in $d = 4$

III. Overview of a-theorem in $d = 6$

IV. Some details for $d = 6$

I. Introduction

The *a*-theorem: In any RG flow from $\text{CFT}_{\rm UV} \longrightarrow \text{CFT}_{\rm IR}$, the Euler central charges satisfy $a_{UV} > a_{IR}$.

1. A universal and fundamental property of interacting QFT_2 and QFT_4. Interpreted as the decrease of the number of degrees of freedom of the QFT as more and more are integrated out in the flow toward IR.
2. An inequality of this type can help determine the strong coupling IR realization of an asymp. free gauge theory in $d = 4$.

Brief history

1. 1986 Zamolodchikov c-theorem for $d = 2$.
2. 25 year effort to extend to $d = 4$, from which we learned:
   1988 Cardy's conjecture that the Euler central charge is the right quantity.
   1997 Convincing evidence in SUSY gauge theories.
   1999 a-theorem holds for any RG flow with an AdS/CFT dual.

2011 Komargodski + Schwimmer formulated a concise, insightful and vigorous proof, but not without subtleties.

Key idea: recast the trace anomaly as a low energy theorem for scattering amplitude of the dilation field $\tau(x)$.

Distinguish two classes of RG flows

i) spontaneously broken conformal symmetry, as on Coulomb branch of $\mathcal{N} = 4$ SYM.
Dilaton is a physical mode of the theory, the Goldstone boson of the broken symmetry.
A physical scale f appears due to the breaking.

ii) Explicitly broken conf. sym. Simplest model is free massive scalar.
Dilaton is added to theory as a fictitious massless compensator which restores conf. sym.

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}[(\partial\Phi)^2 + M^2\Phi^2] \quad \rightarrow \\
\mathcal{L}' &= \frac{1}{2}[f^2 e^{-2\tau}(\partial\tau)^2 + (\partial\Phi)^2 + M^2 e^{-2\tau}\Phi^2]
\end{aligned}$$

The new theory $\mathcal{L}'$ has a traceless stress tensor, so we are essentially back to spont. broken case. BUT we choose $f >> M$ to make coupling of dilaton very weak.

II. The 8-minute a-theorem in $d = 4$

1. A CFT_4 is invariant under conf. trfs. of SO(4,2) in flat space: $T^\mu_\mu = 0$.
2. In curved background geometry there is always an anomaly of the corresp. Weyl trf: $g_{\mu\nu}(x) \to g_{\mu\nu}(x)e^{-2\sigma(x)}$.

$$\langle T^\mu_\mu \rangle = cW^2 - aE_4$$

3. We need an action $S_{anom}[g]$ whose variation is

$$\delta S = \int d^4x \sqrt{-g}\, \sigma(x)\, (cW^2 - aE_4)$$

4. If we add the dilation $\tau(x)$ which trfs as Goldstone boson, i.e. $\tau(x) \to \tau(x) + \sigma(x)$, we can write the local action:

$$S_{anom} = \int d^4x \sqrt{-g} \Big\{ c\, \tau\, W^2 -$$

$$a \left[\tau\, E_4 + 4 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \tau\, \partial_\nu \tau - 4(\partial\tau)^2 \Box\tau + 2(\partial\tau)^4 \right] \Big\}.$$

No "extra terms" for Weyl, but needed for Euler bc.
$\delta E_4 \sim \partial(R\partial\sigma)$.

5. Full $S_{eff}[g,\tau]$ also contains Weyl invariant terms constructed from curvature invariants of $\hat{g}_{\mu\nu} = g_{\mu\nu}e^{-2\tau}$. Up to order ∂^4, these are:

$$
\begin{aligned}
S_{inv}[\hat{g}] &= \int d^4x\sqrt{-\hat{g}}[-\frac{1}{12}f^2\hat{R} + \kappa\hat{R}^2] \\
\hat{R} &= e^{-2\tau}[R + 6\Box\tau - 6(\partial\tau)^2]\,.
\end{aligned}
$$

6. In flat spacetime, the entire $S_\tau = S_{inv} + S_{anom} \rightarrow$

$$
\begin{aligned}
S_\tau &= \int d^4x\Big[-\frac{1}{12}f^2e^{-2\tau}(\partial\tau)^2 + \kappa(\Box\tau - (\partial\tau)^2)^2 \\
&+ 2a[2(\partial_\tau)^2\Box\tau - (\partial\tau)^4]\Big]
\end{aligned}
$$

i) Weyl and diffeo. inv. lead to highly constrained form of dilaton action, even in flat spacetime.
ii) κ-term vanishes by dilation EOM: $\Box\tau - (\partial\tau)^2 = 0$

7. Calculate scattering amplitude using "physical dilaton" with std. kinetic term $e^{-\tau} = 1 - \phi/f$. On shell, $\Box\phi = 0$.

$$S_\tau \to S_\varphi = \int d^4x \left[-\frac{1}{2}(\partial\varphi)^2 + 2a \left(\frac{1}{4f^4}\varphi^2\Box^2\varphi^2 + \mathcal{O}(\Box^2, \varphi^k) \right) \right], \qquad k \geq 5\,.$$

The anomaly determines ∂^4 terms in the low energy expansion of n-point amplitudes for all $n \geq 4$.

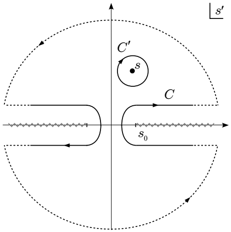
8. K+S use anomaly matching to show that on-shell dilaton 4-point amplitude obeys low energy theorem: as $s \to 0$,

$$\begin{aligned} A(s,t) &\to \frac{1}{2f^4}\Delta a\langle -p_3, -p_4|\varphi^2\Box^2\varphi^2|p_1, p_2\rangle \\ &= \frac{4}{f^4}\Delta a(s^2 + t^2 + u^2), \end{aligned}$$

where $\quad \Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}}$.

9. Analyticity: (Cauchy Thm) $\Longrightarrow$

$$\frac{1}{2\pi i}\oint_{C\cup C'}\frac{ds\,A(s,t)}{s^3}=0\,.$$



Crossing: $\mathrm{Im}A(s,0)=\mathrm{Im}A(-s,0)$

Unitarity: Rearrange the furniture to get:

$$a_{UV}-a_{IR} \;=\; \frac{f^4}{4\pi}\int_0^\infty ds\,\frac{\mathrm{Im}A(s,0)}{s^3}$$

$$=\frac{f^4}{4\pi}\int_0^\infty ds\,\frac{\sigma_{Tot}(s)}{s^2}\;>\;0\,!$$

This is the 8-minute a-theorem, a universal truth for any RG flow in $d=4$.

Some subtleties:
1) anomaly matching

2) convergence of sum rule in UV and IR, discussed in
ZK 1112.4538, Luty, Polchinski, Rattazzi 1204.5221

Comment: in models of flows in which a_{UV} and a_{IR} are known, one can check the K+S low energy thm, independent of the sum rule. These models include:
1. free massive scalar K+S
2. free massive spinor, DZF, private communication.
3. spont. broken $\mathcal{N} = 4$ SYM checked at large N by dual D3-brane calc. in our $d = 6$ paper.

III. Overview of a-theorem in $d = 6$

Motivation for our $d = 6$ work:
Initially posed as a learning problem to understand new ideas of K+S, and we expected a straightforward extension of their result. Instead, we found important differences.

Summary:

1. We construct $S_\tau = S_{inv} + S_{anom}$
2. New order ∂^4 term does not vanish on-shell

$$S_{inv} \sim b \int d^6x \sqrt{\hat{g}}(\hat{R}_{\mu\nu})^2 \to b \int d^6x e^{-\tau} \Box^2 e^{-\tau}$$

3. This affects matrix elements of S_τ to order p^4 and p^6, e.g.

$$A_4(s,t) = \frac{b}{2f^8}(s^2 + t^2 + u^2) + \frac{3}{f^8}[\frac{3}{2}\Delta a - \frac{b^2}{f^4}]stu$$

Anom. contrib. in order p^6 is "polluted" by b^2 term for spont. breaking in which $b^2 \sim f^4$ cannot be suppressed, but not for explicit breaking in which $b^2 \sim m^4$ and term is suppressed for $f >> m$.

4. This low energy structure for A_4, A_5, A_6 is confirmed in full detail in two models:
i) explicit breaking— free massive scalar
ii) spont. breaking– Coulomb branch of (2,0) CFT_6 computed from M5-brane probe action in $\mathrm{AdS}_7 \times \mathrm{S}_4$

Confirms general ideas of K+S that
a) low energy couplings of dilation are determined by diffeo and (broken) Weyl symmetry.
b) anomaly matching.

5. Can we apply analyticity + crossing + unitarity to derive positivity? Anomaly term in 4-point matrix el. $\Delta a\, stu$ vanishes in forward direction, $t = 0$. A red flag against simple applic. of unitarity by optical thm.

Results for free massive scalar

1. from low energy thm $\Delta a = a_{UV} = 1/((4\pi^2)2^4 3^4 7)$, in agreement with heat kernel results in literature.

2. Same fwd disp. relation used for $d = 4$ now gives

$$\frac{1}{2}A''(s,0) = \frac{8}{f^8}b = \frac{2}{\pi}\int_{4m^2}^{\infty} ds \frac{ImA(s,0)}{s^3} > 0$$

Positivity of b correct but uninteresting, b/c b is not an anomaly, but rather a dimensionful parameter $b = cm^2$.

3. Isolate Δa by writing a disp. del. for $(\partial/\partial t)A''(s, t = 0)$, which gives sum rule:

Sum rule for Δa.

In general model with explicit breaking

$$\Delta a = \frac{2f^8}{9\pi}\int_0^\infty \left[\frac{3}{s^4} ImA(s,0) - \frac{2}{s^3}\frac{\partial}{\partial t} ImA(s,0)\right]$$

Integrand is difference of two +ve terms, so sign is unclear. In free boson model, we compute these terms from unitarity of box graph. Numerical integration agrees with previous value of a, BUT integrand is not +ve definite.

Thus, even the simplest model which obeys the a-theorem does not have the manifest positivity.
Positivity from forward $A_6(p_1, p_2, p_3, -p_1, -p_2, -p_3)$ is an open question, but analyticity, crossing, unitarity for 6-point function are very complicated!

IV. Some details for $d = 6$

A. Construction of $S_\tau = S_{anom} + S_{inv}$
B. probe M5-brane as an RG flow of the (2,0) CFT_6

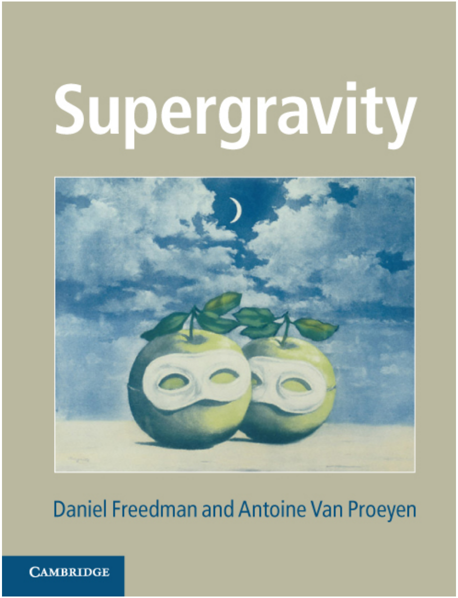
1. Construction of A_{anom} in $d = 6$

$$\langle T^\mu_\mu \rangle = \sum_{i=1-3} c_i I_i + aE_6$$

3 indep Weyl tensor invariants I_i vanish in flat space; drop them.
Euler density E_6 in cubic in $R^{ab}{}_{\mu\nu}$.
We seek $S_{anom}[g, \tau]$ such that

$$\delta S_{anom} = \int d^6 x \sigma(x) E_6$$

We constructed S_{anom} by two methods i) std. Wess-Zumino method – arguably shorter
ii) supergravity-inspired method– arguably sweeter



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Brief discussion of SG-inspired method

a. Start with $S_0 = \int d^6x\sqrt{-g}\tau(x)E_6$. Its infinitesimal Weyl variation is

$$\delta S_0 = \int \sqrt{-g}[\sigma(x)E_6 + \tau(x)\delta E_6]$$

"direct" $\delta\tau = \sigma$ variation is good; but "indirect" $\delta E_6 \sim \partial(R^2\partial\sigma)$ is bad.

b. Cancel it by inventing $S_1 = \frac{1}{2}\int \partial\tau\partial\tau RR$ whose direct $\delta\tau$ variation cancels bad term from δS_0, but whose indirect δR variation requires still another term.

c. Continue the process until it closes

Result for S_{anom}

After one week of work, one finds structure containing 7 terms

$$\begin{aligned} S_{anom} &= \sum_{i=1,\ldots,7} S_i \\ &= \int d^6x\sqrt{-g}\Big[\tau E_6 \\ &+ \text{ 5 lines of terms involving } R^2 \text{ and } R \\ &+ \text{ terms surviving in flat space}\Big]. \end{aligned}$$

In flat limit:

$$\begin{aligned} S_{anom} &\to \Delta a \int d^6x \Big[24(\partial\tau)^2(\partial_\mu\partial_\nu\tau)^2 - 24(\partial\tau)^2(\Box\tau)^2 \\ &+ 36\Box\tau(\partial\tau)^2 - 24(\partial\tau)^6\Big] \end{aligned}$$

S_{inv}

We need to find all curvature invariants of order $\partial^2, \partial^4, \partial^6$ from

$$\hat{R}iem = Riem(\hat{g}) = Riem(g_{\mu\nu}e^{-2\tau})$$

Summary of results:

i) unique term of order ∂^2

$$\begin{aligned}\int d^6x\sqrt{\hat{g}}\hat{R} &\rightarrow \int d^6x 10 e^{-4\tau}(\Box\tau - 2(\partial\tau)^2) \\ &= 20\int d^6x e^{-4\tau}(\partial\tau)^2\,.\end{aligned}$$

This is the dilation kinetic term in $d=6$. Again $\hat{R}=0$ by EOM $\Box\tau - 2(\partial\tau)^2 = 0$. The physical dilation is $e^{-2\tau} = 1 - \phi/f^2$.

ii) At order $(\partial)^4$, the only term which is non-vanishing on shell and in flat limit is (as already discussed)

$$S_{inv} \sim b\int d^6x\sqrt{\hat{g}}(\hat{R}_{\mu\nu})^2 \rightarrow b\int d^6x e^{-\tau}\Box^2 e^{-\tau}$$

iii) order $(\partial)^6$, among 11 terms in a basis of $(\hat{R}iem)^3$ invariants, none survive on shell and in flat space.
SUMMARY:
In flat spacetime, the on-shell dilaton action becomes (after non-trivial algebra)

$$S_\tau = \int d^6x \left[4b e^{-\tau} \Box^2 e^{-\tau} + \Delta a \, \tau \Box^3 \tau \right]$$

From this one computes matrix els. of physical dilation:

$$e^{-2\tau} = 1 - \phi/f^2 \qquad e^{-\tau} = \sqrt{1 - \phi/f^2} \qquad \tau = -\frac{1}{2} \ln(1 - \phi/f^2)\,.$$

SUBTLETY in matrix els. of ∂^4 term:

$$S_{(\partial)^4} = b \int d^6x \left[\sum_{m+n\geq 4} \phi^m \Box^2 \phi^n + \frac{1}{2f^6} \phi^2 \Box^2 \phi \right]$$

It may *seem* that one can ignore $\Box\phi$ b/c dilaton is on-shell. *FALSE* because $\exists$ tree diagrams,

$$A_{4,tree} \sim \frac{b^2}{b^{12}} s^2 (1/s) s^2 = \frac{b^2}{b^{12}} s^3 \,.$$

$\exists$ similar induced order p^6 terns in all $A_{n,tree}$, $n \geq 4$.

This is the reason why complete order p^6 term is:

$$A_4(s,t) = [\frac{3}{2}\Delta a - \frac{b^2}{f^4}] \frac{3}{f^8} stu \,.$$

i. This effect is essential for spont. broken conf sum, since f is a fixed physical scale, and $b \sim f^2$

ii. It is suppressed for explicit breaking if we take $f >> M$, where M is largest scale of flowing QFT_6,

iii. From p^4 and p^6 matrix els, one can determine both b and Δa.

iv. Full detailed structure for A_4, A_5, A_6 is confirmed in our work on probe M5-brane.

B. probe M5-brane as an RG flow of the (2,0) CFT_6

i. The $AdS_7 \times S_4$ soln of D=11 SG describes stack of N coincident M5-branes.

$$ds^2 = \frac{L^2}{z^2}[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2] \qquad L = (\pi N)^{1/3}\ell_p$$

Its Euler central charge is $a = (1/576\pi^3)N^3$.

ii. holog. dual of a Coulomb branch flow describes N branes in the UV and $N-1$ branes in the IR. This RG flow has

$$\Delta a = (1/576\pi^3)[N^3 - (N-1)^3] = (1/192\pi^3)N^2. \qquad N >> 1$$

iii. This flow breaks conf. sym. spontaneously. Its dynamics is described by DBI action of 1 probe brane in AdS_7 in static gauge:

$$\begin{aligned} S_{DBI} &= -T_{M5}L^6 \int d^6x \frac{1}{z^6}\left[\sqrt{1+(\partial z)^2} - 1\right] \\ T_{M5} &= \frac{1}{(2\pi)^5 \ell_p^6} \end{aligned}$$

iv. Its derivative expansion is

$$S_{DBI} = -T_{M5} L^6 \int d^6x \frac{1}{z^6} \left[\frac{1}{2} (\partial z)^2 - \frac{1}{8} (\partial z)^4 + \frac{1}{16} (\partial z)^6 + \dots \right]$$

vi. Radial evolution in AdS corresponds to RG flow in dual CFT. We expect a relation between radial mode $z(x)$ and dilation $\tau(x)$. Establish this connection in 2 ways:

1. Calculate low energy S-matrix for S_{DBI} which is insensitive to exact relation of the fields and compare with physical dilaton matrix elements.
2. Find more exact relation $z \to \tau$ which transforms $S_{DBI} \to S_\tau$.

1. Postulate 'reasonable' relation $z = Le^\tau$ which kills $1/z^6$ sing.

$$S_{DBI} \to -T_{M5} \int d^6x \left[\frac{L^2}{2} e^{-4\tau} (\partial \tau)^2 - \frac{L^4}{8} e^{-2\tau} (\partial \tau)^4 + \frac{L^6}{16} (\partial \tau)^6 + \dots \right]$$

Dilaton S-matrix from S_{DBI}

We calculate order $p^4,\ p^6$ contribs to $A_4,\ A_5,\ A_6$ and match physical dilation matrix elements exactly, with parameters:

$$f^4 = \frac{1}{4} T_{M5} L^2 \qquad b = \frac{1}{2^5} T_{M5} L^4 \qquad \mathbf{\Delta a = (1/192\pi^3) N^2}$$

Correct anomaly flow!

2. $S_{DBI} \to S_\tau$

Not obvious how to achieve $z \to \tau$, such that

$$
\begin{aligned}
S_{DBI} &= -T_{M5}L^6 \int d^6x \frac{1}{z^6}\left[\sqrt{1+(\partial z)^2} - 1\right] \\
\to S_\tau &= \int d^6x \left[-2f^4 e^{-4\tau}(\partial\tau)^2 + 4be^{-\tau}\Box^2 e^{-\tau} + 3\Delta a\tau\Box^3\tau\right]
\end{aligned}
$$

Key: understand the symmetries of S_{DBI}.

i. The PBHFG diffeo of AdS, with parameter $\sigma(x)$ leaves the bulk invariant, but generates Weyl trf with same parameter on the brane:

$$\delta_\sigma x^\mu = -\frac{1}{2}z^2 g^{\mu\nu}\partial_\nu\sigma \qquad\qquad \delta_\sigma z = \sigma z$$

ii. This becomes a symmetry of S_{DBI} if we modify $\delta_\sigma z$ to restore static gauge:

$$\delta_\sigma z = \sigma z + \frac{1}{2}z^2 g^{\mu\nu}\partial_\mu z\partial_\nu\sigma$$

Strategy is now clear: find relation between z and τ such that $z \to z + \delta_\sigma z$ above produces the Weyl trf $\tau \to \tau + \sigma$ of the dilaton.

Postulate ansatz as a derivative expansion:

$$
\begin{aligned}
z &= Le^{\tau} + L^3 e^{3\tau}\Big[\alpha_1 \Box\tau + \alpha_2 (\partial\tau)^2\Big] \\
&+ \Big[\text{ analogous general expression for } \partial^4 \text{ terms}\Big]
\end{aligned}
$$

Matching of the Weyl trfs gives 5 relations among 9 parameters. Substitute results in S_{DBI}, the unfixed parameters drop out, and we find $S_{DBI} \to S_\tau$ with the same values of $f,\ b,\ \Delta a$ found in the calc.of on-shell amplitudes!

SUMMARY: status of a-thm in $d = 6$

There is evidence in two examples, one with explicit and one with spontaneous breaking of conformal symmetry, that the K+S low energy theorem produces correct +ve value of $a_{UV} - a_{IR}$.

Attempt to prove positivity for a general $d = 6$ theory using analyticity, crossing, and unitarity fails for the 4-point amplitude.

One needs a new idea or perhaps a counter-example to prove or disprove the theorem.