Chern-Simons Contact Terms & 3D RG Flows

Guido Festuccia

IAS

RG Flows, Holography, and Entanglement Entropy
Michigan Center for Theoretical Physics
Ann Arbor, September 2012

Based on arXiv:1206.5218, arXiv:1205.4142

C. Closset, T. Dumitrescu, Z. Komargodski, N. Seiberg, GF.

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Contact Terms

Correlation functions at coincident points

$$\langle \mathcal{O}(x), \mathcal{O}(0) \rangle = \cdots \alpha\, \delta(x)$$

Generically they are arbitrary. They depend on physics at the cutoff scale Λ and hence on the regularization scheme.

It is convenient to promote all coupling constants to classical background fields and consider a combined Lagrangian $\mathcal{L}$ for the dynamical fields and the classical backgrounds.

Contact are shifted by local counterterms whose coefficients are not fixed

$$\mathcal{L} = \phi(x)\mathcal{O}(x) + \alpha\, \phi^2(x) + \cdots$$

There are cases in which contact terms are not arbitrary for example:

- The seagull term in scalar electrodynamics is required by gauge invariance
- In a 2d CFT the trace of the energy momentum tensor T_μ^μ is redundant (its correlation functions are zero at separated points). Its two point function has a contact term fixed by conservation of $T_{\mu\nu}$.

We will consider a third case, contact terms which are physical modulo integer multiples of a set amount.

Chern-Simons Contact Terms

Consider in 3D a theory with a compact $U(1)$ global symmetry. Two point functions of the corresponding conserved current j_μ admit a contact term

$$\langle j_\mu(x), j_\nu(y)\rangle = \cdots + \frac{i\kappa}{2\pi}\epsilon_{\mu\nu\rho}\partial^\rho \delta^{(3)}(x-y)$$

We can couple j_μ to a background gauge field a_μ. The contact term above can be shifted by adding a Chern-Simons counter term to the effective action for a_μ

$$\delta\mathcal{L} = \frac{i\delta\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

$$\delta\mathcal{L} = \frac{i\delta\kappa}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$$

is not the integral of a gauge invariant local density. Nevertheless for trivial $U(1)$ bundles it is well defined and invariant under infinitesimal gauge transformations.

For arbitrary bundles on arbitrary (spin) manifolds $\mathcal{M}_3$ we can define it via an auxiliary four-manifold $\mathcal{M}_4$ whose boundary is $\mathcal{M}_3$.

$$\delta\mathcal{L} = \frac{i\delta\kappa}{4\pi}\int_{\mathcal{M}_3} d^3x\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho = \frac{i\delta\kappa}{16\pi}\int_{\mathcal{M}_4} d^4x\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

A different choice of $\mathcal{M}_4$ can shift $\delta\mathcal{L}$ by $2\pi i\delta\kappa$ hence, for $e^{-\delta\mathcal{L}}$ to be well defined $\delta\kappa$ must be an integer.

Consider the two point correlator of the conserved current j_μ.
There two structure functions compatible with current conservation

$$\langle j_\mu(p) j_\nu(-p)\rangle = \tau\left(\frac{p^2}{\mu^2}\right)\frac{p_\mu p_\nu - p^2\delta_{\mu\nu}}{16|p|} + \kappa\left(\frac{p^2}{\mu^2}\right)\frac{\epsilon_{\mu\nu\rho}p^\rho}{2\pi} \ .$$

- $\tau\left(\frac{p^2}{\mu^2}\right)$ is physical as is the p dependence of $\kappa\left(\frac{p^2}{\mu^2}\right)$.
- The structure functions are real
- In a unitary CFT τ and κ are constants and $\tau > 0$.

Adding a constant to $\kappa\left(\frac{p^2}{\mu^2}\right)$ shifts the contact term. When the flavor symmetry is compact this ambiguity is quantized.

$$\kappa_{UV} = \lim_{p\to\infty}\kappa\left(\frac{p^2}{\mu^2}\right) \qquad \kappa_{IR} = \lim_{p\to 0}\kappa\left(\frac{p^2}{\mu^2}\right)$$

They are not physical but their difference $\kappa_{UV} - \kappa_{IR}$ is.

Conclusions so far

The difference $\kappa_{UV} - \kappa_{IR}$ is universal. It does not depend on counterterms in the UV.

For compact global symmetries the possible counterterms have quantized coefficients.

In this case the Chern-Simons contact terms are also universal modulo integer shifts.

Their fractional parts are good intrinsic observables. e.g. κ_{CFT}.

Gravitatonal Chern-Simons

Similarly there can be a contact term in the two point function of the energy momentum tensor $T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \cdots - i\frac{\kappa_g}{192\pi}\left(\epsilon_{\mu\rho\lambda}\partial^\lambda(\partial_\nu\partial_\sigma - \partial^2\delta_{\nu\sigma}) + symm\right)$$

$T_{\mu\nu}$ couples to the background metric and the contact term above can be shifted by the following counterterm:

$$\frac{i\delta\kappa_g}{192\pi}\int_{\mathcal{M}_3}\sqrt{g}\, d^3x\, \epsilon^{\mu\nu\rho}\, Tr\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)$$

Again this can be defined precisely going to $\mathcal{M}_4$ and independence on the choice of (spin) $\mathcal{M}_4$ implies that $\delta\kappa_g \in Z$.

example I: Free Fermion

Consider one free Dirac fermion of mass m. The theory has a $U(1)$ global symmetry.

Integrating out the massive fermion results in

$$\kappa_{IR} = \kappa_{UV} - \frac{1}{2} sgn(m)$$

In the IR the effective action for the background field a_μ is proportional to $\int a \wedge da$. The IR is completely gapped hence

$$\kappa_{IR} \in Z$$

To ensure consistency we must add a Chern-Simons counterterm in the UV with fractional coefficient so that [Redlich]

$$\kappa_{UV} = \frac{1}{2} \mod Z \ .$$

Example II: Topological theory

$$\mathcal{L} = \frac{i}{4\pi}\epsilon^{\mu\nu\rho}\left(k\, A_\mu\partial_\nu A_\rho + 2p\, a_\mu\partial_\nu A_\rho + q\, a_\mu\partial_\nu a_\rho\right), \quad k,\ p,\ q\ \in Z$$

- A_μ is a dynamical U(1) gauge field
- a_μ is a classical background $U(1)$ gauge field coupled to the topological current $j^\mu = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$.

Integrating out A_μ results in an effective action for a_μ

$$\mathcal{L} = \frac{i}{4\pi}\left(q - \frac{p^2}{k}\right)\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \quad \Rightarrow \quad \kappa_{IR} = \kappa_{UV} = q - \frac{p^2}{k}$$

- The expression above is not valid in all topological sector, there are remaining topological degrees of freedom.
- The theory is topological, all correlation functions of local operators vanish at separated points. However the fractional κ above is captured by nonlocal operators and is observable.

Example III

Consider a UV free theory with two crossover scales $m \ll M$.

Asume that the IR theory is fully gapped (not even topological d.o.f.) then κ_{IR} must be quantized.

For $m \ll E \ll M$ the theory is approximately conformal. The fractional part of $\kappa_{CFT} = \kappa(E)$ is an observable of the CFT.

In the UV the theory is free. κ_{UV} is determined by the number of fermions and possible couplings to topological degrees of freedom.

We can determine κ_{CFT} mod 1 either flowing out to the IR, or flowing in from the UV.

Adding an R-symmetry

For a non-supersymmetric theory with a $U(1)$ global flavor symmetry we considered two kinds of Chern-Simons terms

- Flavor-Flavor: $a \wedge da$
- Gravitational: $\omega \wedge d\omega$

they correspond to contact terms in the two point functions of j_μ and $T_{\mu\nu}$ respectively.

For a $\mathcal{N} = 2$ theory with a conserved R-current j_μ^R coupled to a background gauge field A_μ we also have:

- Flavor-R: $a \wedge dA$
- R-R: $A \wedge dA$

corresponding to contact terms in correlation functions of j_μ and j_μ^R.

All these four Chern-Simons terms are conformal.
The fractional parts of these contact terms are universal when the corresponding global symetries are compact.

$\mathcal{N} = 2$ Susy

We will consider theories with $\mathcal{N} = 2$ susy. A conserved flavor current j_μ is part of a linear superfield:

$$(J,\ j_\mu,\ K,\ \cdots)$$

Supersymmetry relates the correlation functions of the components of the multiplet e.g:

$$\langle J(p), K(-p)\rangle = \frac{1}{2\pi}\kappa\left(p^2/\mu^2\right)$$

This multiplet can be coupled to a background vector superfield with components:

$$(\sigma,\ a_\mu,\ D,\ \cdots)$$

$$\delta\mathcal{L} = -j_\mu a^\mu - K\sigma - JD + \cdots$$

R-multiplet

In a $\mathcal{N}=2$ theory with a $U(1)$ R-symmetry the R-current j^R_μ is in a multiplet with the energy momentum tensor: [Dumitrescu, Seiberg]

$$(j^R_\mu,\ T_{\mu\nu},\ j^Z_\mu,\ J^Z,\ \cdots)$$

This multiplet couples to the fields in the (new minimal) supergravity multiplet;

$$(A_\mu,\ g_{\mu\nu},\ V_\mu,\ H,\ \cdots), \qquad \nabla^\mu V_\mu = 0$$

$A_\mu,\ V_\mu,\ H$ are auxiliary fields. Here we will regard them as arbitrary background fields.

If the theory is superconformal T^μ_μ, j^Z_μ and J^Z are redundant. j^R_μ couples to $A_\mu - \frac{3}{2}V_\mu$.

Supersymmetric Chern-Simons terms

The Chern-Simons terms can be supersymmetrized

- Flavor-Flavor:

$$\frac{\kappa_{ff}}{4\pi}\left(i\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - 2\sigma D + \cdots\right) \ .$$

  The contact term in $\langle j_\mu(x)\, j_\nu(0)\rangle$ is related by SUSY to a contact term in the two point function of K and J.

- Gravitational:

$$\frac{\kappa_g}{192\pi}\left(i\epsilon^{\mu\nu\rho}Tr\left(\omega_\mu\partial_\nu\omega_\rho+\frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)+4i\epsilon^{\mu\nu\rho}\left(A_\mu-\frac{3}{2}V_\mu\right)\partial_\nu\left(A_\rho-\frac{3}{2}V_\rho\right)+\cdots\right)$$

Both are superconformal.

- Flavor-R:

$$-\frac{\kappa_{fr}}{4\pi}\left(i\epsilon^{\mu\nu\rho}a_\mu\partial_\nu\left(A_\rho-\frac{1}{2}V_\rho\right)+\frac{1}{8}\sigma R-\frac{1}{2}DH+\cdots\right)$$

- R-R (Note that there is a $A\wedge dA$ piece also in the Lorentz CS):

$$-\frac{\kappa_{rr}}{2\pi}\left(i\epsilon^{\mu\nu\rho}\left(A_\mu-\frac{1}{2}V_\mu\right)\partial_\nu\left(A_\rho-\frac{1}{2}V_\rho\right)+\frac{1}{2}HR+\cdots\right)$$

These supersymmetric Chern-Simons terms are not superconformal.

R and the auxiliary fields $A_\mu-\frac{1}{2}V_\mu$ and H couple to redundant operators in a CFT for example R couples to T^μ_μ.

A new anomaly

In a CFT The Flavor-Flavor and Lorentz supersymmetric Chern-Simons term can be nonzero.

However the supersymmetric completion on the Flavor-R and R-R Chern Simons terms are not superconformal.

If the flavor symmetry and the R-symmetry are compact the fractional part of all Chern-Simons contact terms is observable.

A new anomaly

Whenever the R-R and Flavor-R Chern Simons terms are not quantized we cannot have all of the following

- Compactness of flavor and R-symmetries
- Supersymmetry
- Conformal Invariance

One possibility is to sacrifice the independence of the functional integral on the auxiliary manifold $\mathcal{M}_4$. We can then add fractional Chern-Simons counterterms to cancel the Flavor-Flavor and Flavor-R contact terms in the CFT. This is a new anomaly (similar to the framing anomaly. [Witten])

Partition function on S^3

It is possible to place an $\mathcal{N} = 2$ theory with an R-symmetry on certain Riemannian three-manifolds $\mathcal{M}_3$ preserving some Susy, for example S^3 [Kapustin, Willet, Yaakov, Jafferis,...]

We can interpret the resulting Lagrangians as arising from setting the fields in the gravity multiplet $A_\mu,\ V_\mu,\ H,\ g_{\mu\nu}$ to certain background values. [Seiberg, GF]

In the case of S^3 of radius r we must set $H = -\frac{i}{r}$.

Generally the Lagrangian is not reflection positive. If the theory is conformal, H decouples, and the theory on S^3 is reflection positive.

Consider a theory with a $U(1)_f$ global symmetry, flowing to a unitary SCFT in the IR.

We can turn on complex background gauge superfields (σ, a_μ, D) which couple to the conserved current multiplet.

To place such a theory on S^3 we need to make a choice of R-symmetry

$$R(t) = R_0 + t\, Q_f$$

Different choices correspond to shifting the imaginary part of σ

$$Im(\sigma) = \frac{t}{r}$$

$Re(\sigma) = m$ instead is a real mass term. The dependence on σ is holomorphic.

For a particular choice of $\sigma = i\frac{t^*}{r}$ the R-symmetry will correspond to the superconformal one.

We then expect the partition function $Z = e^{-F}$ of the resulting theory on S^3 to satisfy

- $F|_{t^*}$ is real by reflection positivity
- $\partial_\sigma F|_{t^*} = 0$
- $\frac{1}{r^2}\partial_\sigma^2 F|_{t^*} = \frac{\pi^2}{4}\tau > 0$

F_{t^*} can be computed exactly using localization techniques on the subspace $D = \frac{i}{r}\sigma$.

The result does not satisfy the properties listed above.

WHY?

The Chern-Simons terms discussed above contribute to the partition function via the nonzero values for the various background fields we turn on.

The R-R and Flavor-R terms are not superconformal. When present they result in a partition function which is not compatible with conformal invariance. In particular because the auxiliary fields have non-standard reality conditions F is not real.

However we know the reality properties of the contact terms as these are the same as in flat space. This allows us to isolate their contribution to F.

We find that the imaginary part of $F|_{t^*}$ is entirely due to the non-conformal R-R contact term $Im(F)|_{t^*} = \pi\kappa_{rr}$.

The first derivatives of F with respect to t (or m by holomorphy) depends on the Flavor-R contact term:

$$\kappa_{fr} = -\frac{1}{2\pi}\frac{\partial}{\partial t} Im(F)\bigg|_{t^*}, \qquad \frac{\partial}{\partial t} Re(F)\bigg|_{t^*} = 0$$

Hence $Re(F)$ is extremized at $t = t^*$ [Jafferis]

The second derivatives of F depend on the constants τ_{ff} and κ_{ff} in the Flavor-Flavor two point functions:

$$\frac{\partial^2}{\partial t^2} Im(F)\bigg|_{t^*} = 2\pi\kappa_{ff} \ , \qquad \frac{\partial^2}{\partial t^2} Re(F)\bigg|_{t^*} = -\frac{\pi^2}{2}\tau_{ff} < 0$$

Hence $Re(F)$ is maximized [Jafferis,Klebanov,Pufu,Safdi]

Some Comments

- The values of the contact terms obtained by localization can be matched to perturbative calculations in flat space.
- κ_{ff}, κ_{fr} mod 1 and τ are independent of superpotential couplings
- To determine the gravitational Chern-Simons coefficient κ_g we need to consider a squashed S^3. [Hama Hosomichi Lee]

$$b^2(x_1^2 + x_2^2) + b^{-2}(x_3^2 + x_4^2) = r^2, \qquad \omega = \frac{i}{2}(b + b^{-1})$$

Then we have for $Im(F)|_{t^*}$

$$Im(F)|_{t^*} = \frac{\pi}{12}(\omega^2 + 1)\kappa_g - \pi\omega^2\kappa_{rr}$$

SQED example

Consider a $U(1)$ gauge theory with CS level k and N_f flavors pairs

$$Q_i, \ (q = 1, \ q_f = 1) \qquad \tilde{Q}_{\tilde{i}}, \ (q = -1, \ q_f = 1)$$

Charge conjugation which exchanges Q_i and $\tilde{Q}_{\tilde{i}}$ prevents the topological current to mix with $U(1)_f$.

There is a crossover scale at $M = \frac{\kappa e^2}{2\pi}$. Setting the contact terms to 0 in the UV we get for $E \ll M$

$$\kappa_{ff} = \frac{\pi^2}{4k} N_f + \mathcal{O}(k^{-3}), \qquad \kappa_{fr} = -\frac{1}{2k} N_f + \mathcal{O}(k^{-3})$$

The same can be obtained adding a small real mass $m \ll M$ and flowing out to a gapped theory.

These values agree with those obtained from the partition function on S^3 computed using localization. [Jafferis]

Matching contact terms across dualities

Consider two $\mathcal{N} = 2$ theories which flow to the same IR fixed point.

The partition function on S^3 should match on both sides of the duality up to the contribution of Chern-Simons counterterms. [Kapustin Willet Yaakov, Benini Closset Cremonesi;...]

These counterterms need to be properly quantized, hence the Chern-Simons contact terms should also match mod 1. This "anomaly matching" generalizes a similar condition for the parity anomaly. [Aharony, Hanany, Intriligator, Seiberg, Strassler]

For dual pairs, related by RG flows these quantized coefficients can be determined independently. If we are given the Chern-Simons counterterms needed for one theory we can determine them for the second theory by a one loop computation in flat-space.

Conclusions

Chern-Simons contact terms lead to new observables for 3D QFT's.

They are described naturally by coupling conserved currents to classical background fields.

For $\mathcal{N} = 2$ theories with an R-symmetry some of the contact terms are not superconformal and lead to a new anomaly.

When putting a theory on S^3 preserving SUSY we must give complex values to various classical backgrounds. The non-superconformal contact terms then violate reflection positivity.

This explains the features of the partition function computed on S^3 by localization and allows to prove F maximization.

Non trivial tests of various dualities.

Thank You!