Strong subadditivity of entanglement entropy and quantum field theory

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Plan of the talk

Strong subadditivity + Euclidean symmetry
+ Lorentz symmetry

Entropic c-theorem in 1+1 dimensions

RG-running of the constant term of the circle entropy in 2+1
$\Longleftrightarrow$ F-theorem

Generalizations to more dimensions?

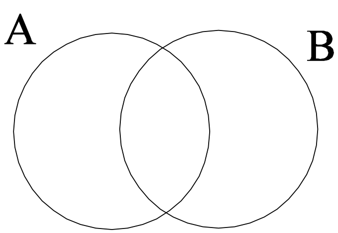
Strong subadditivity and reflection positivity

Strong subadditivity of entropy

$S(\rho) = -\mathrm{tr}(\rho \log \rho)$ von Neumann entropy

Strong subadditivity Lieb, Ruskai (1973)

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \longrightarrow S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123})$ $\rho_{12} = \mathrm{tr}_{H_3}(\rho)$



$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

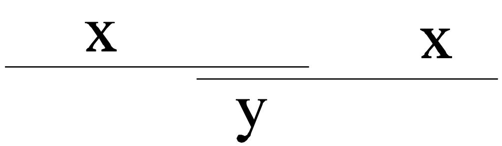
Additivity

$\rho_{AB} = \rho_A \otimes \rho_B \rightarrow S(AB) = S(A) + S(B)$

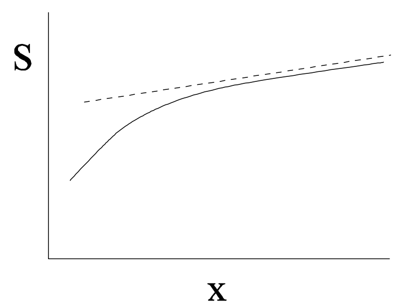
Subadditivity

$S(AB) \leq S(A) + S(B)$

SSA + translational invariance (Robinson-Ruelle 1967): existence of entropy density



$$2S(X) \geq S(Y) + S(2X - Y) \rightarrow S''(X) \leq 0$$



$$\lim_{x \to \infty} \frac{S(x)}{x} \to s$$

- Generalized to more dimensions (in the limit of large volume/area ratio)
- S~volume, but entropy density can be zero

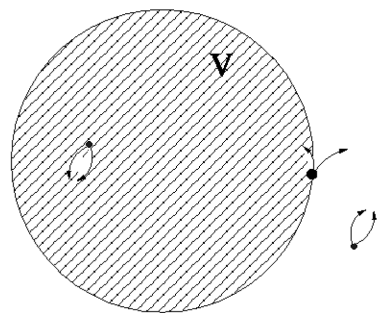
SSA + Lorentz symmetry: the vacuum state

Reduced density matrix $\rho_V = \mathrm{tr}_{-V} |0\rangle \langle 0|$ $\longrightarrow$ S(V) entanglement entropy

S(V)=S(-V) pure global state

S(V) meassures the entropy in vacuum fluctuations

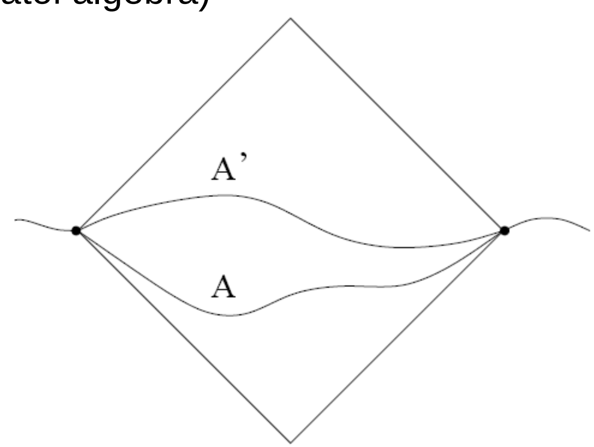
No boundary artificial conditions: S(V) property of the QFT



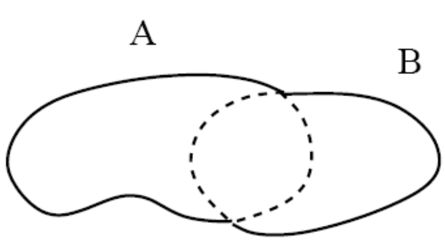
Region V in space-time:

Causality $S(A) = S(A')$ $(\rho_A = \rho_{A'})$

S is a function of the "diamond shaped region"
of equivalently the region boundary (vacuum state
on an operator algebra)



Conditions for use of SSA:
"diagonalizing" Cauchy surface for A and B.
Boundaries must be spatial to each other



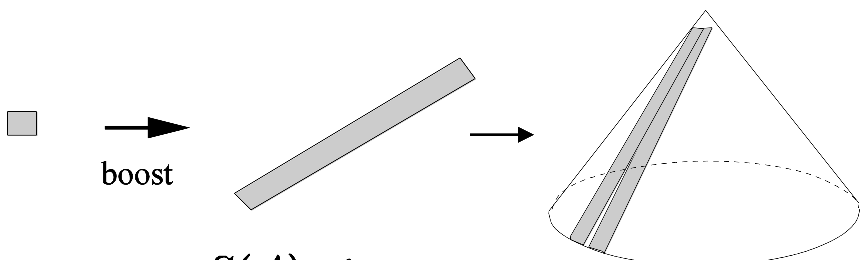
A geometric theorem

Entanglement entropy:

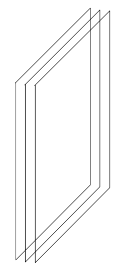
- Finite   Regularized entropy?, BH entropy finite
- Poincare invariant
- Causality
- Strong subadditivity on Cauchy surfaces
- Positivity $S(A) \geq 0$

$$S(A) = s \ \text{area}(A) + \alpha$$

Entropy linear in boundary area for relativistic polyhedrons



$S(A) \leq \text{area}$



+ translation invariance:

S ~ c N area

$I(A,B) = S(A) + S(B) - S(A \cup B)$ mutual information is zero

$$I(A,B) \geq \frac{1}{2} \frac{\left(\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle\right)^2}{\|O_A\|^2 \|O_B\|^2}$$

is upper bound to correlations: no correlations: trivial theory

S divergent in QFT, regularized S? : finite versus positive (bounded below), i.e. in 1+1 CFT:

$$S(x) = \frac{c}{3} \log(x/\epsilon) \rightarrow \frac{c}{3} \log(x)$$

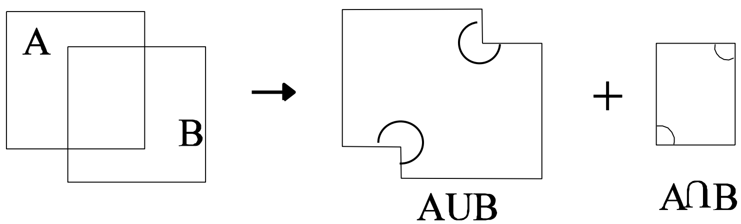
The proof requieres arbitrarily large and small distances: other geometries dS, AdS?

Divergent terms on entanglement entropy

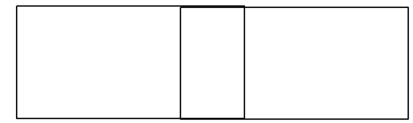
$$S(V) = g_2[\partial V]\,\epsilon^{-(2)} + g_1[\partial V]\,\epsilon^{-1} + g_0[\partial V]\,\log(\epsilon) + S_0(V) \qquad \text{(d=4)}$$

The functions g are local and extensive on the boundary due to UV origin of divergences.

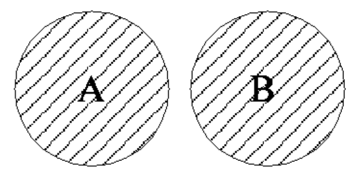
SSA trivial due to corner log divergences



SSA non trivial



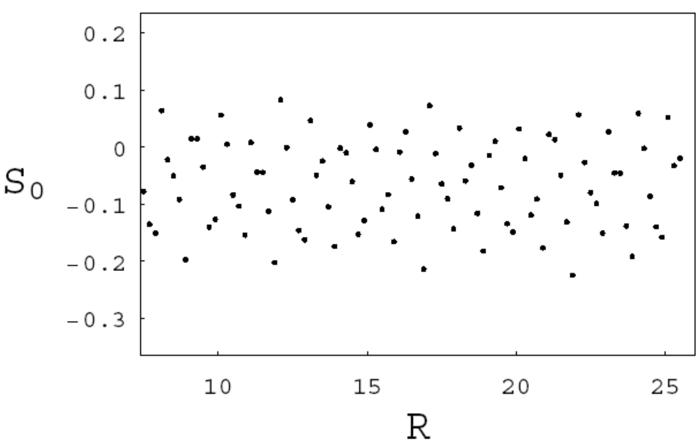
Mutual information $I(A,B) = S(A) + S(B) - S(A \cup B)$

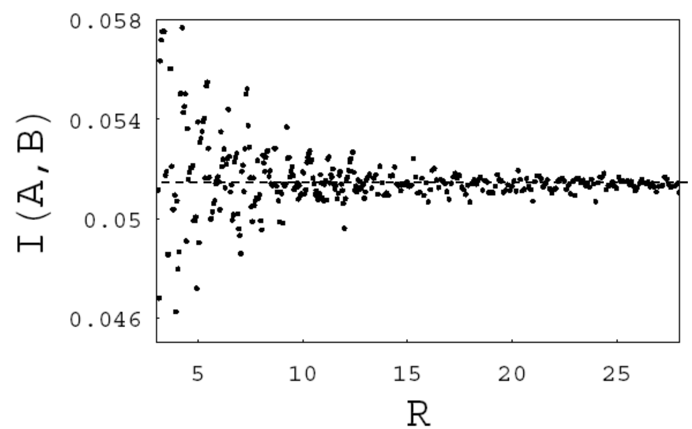


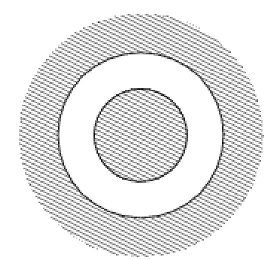
The divergences cancel in I(A,B) which is finite and well defined

SSA $\longrightarrow$ $I(A,B) \geq 0$ Positivity

$I(A,B)$ Monotonously increasing with size: smooth

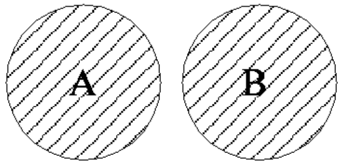




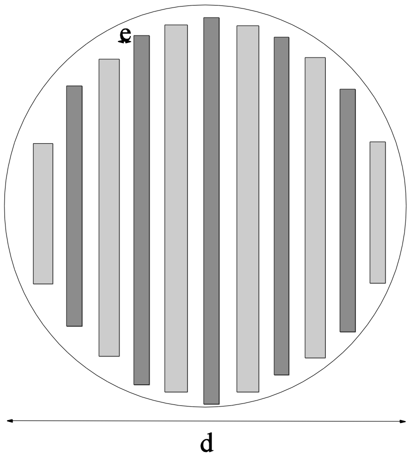


Circles in a square lattice, massless scalar in 2+1.

More mutual information ideology…



$I(A,B)$ ~1 (little entanglement, local theory)



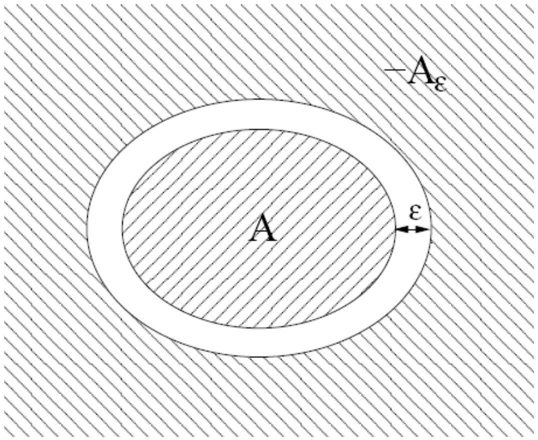
$$I(A,B) \sim n\frac{d^2}{\epsilon^2} \sim \frac{d^3}{\epsilon^3}$$

$$I(A,B) \leq 2\min(S(A),S(B))$$

Lower bound to entropy

This is a cutoff free statement indicating that the number of degrees of freedom and the information capacity grow as the volume in QFT.

Entropy bound violated by vacuum fluctuations in flat space



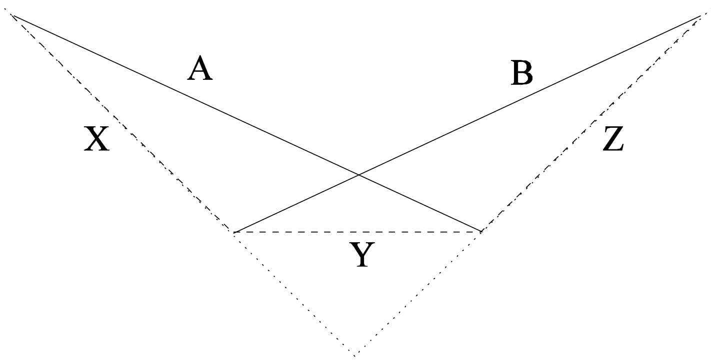
$$I(A,-A) = S(A) + S(-A) - S(A\cup -A) = 2S(A)$$

$$S(A) = \frac{1}{2}\, I(A,-A_\epsilon)$$

For pure global states

Regularized entropy: all coefficients on the expansion are universal

The simplest unidimensional case: two intervals and the c-theorem in 1+1



H.C., M. Huerta, 2004

$$S(XY) + S(YZ) \geq S(Y) + S(XYZ)$$

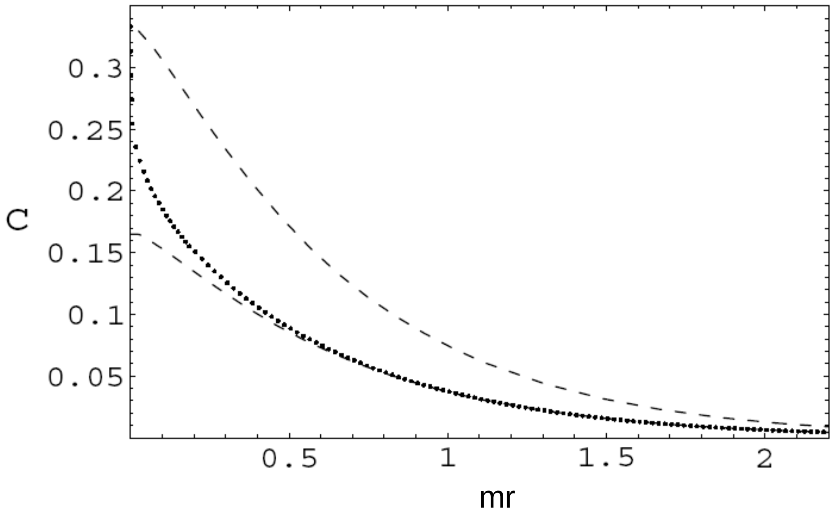
$$2\,S(\sqrt{rR}) \geq S(R) + S(r)\,.$$

$$rS''(r) + S'(r) \leq 0\,. \qquad C(r) = rS'(r) \longrightarrow C'(r) \leq 0$$

C(r) dimensionless, well defined, decreasing. At conformal points

$$S(r) = \frac{c}{3}\log(r/\epsilon) + c_0 \longrightarrow$$ C(r)=c/3

The central charge of the uv conformal point must be larger than the central charge at the ir fixed point: the same result than Zamolodchikov c-theorem



Myers-Sinha (2010) Holographic c-theorems

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2_{d-1}) + dr^2$$

$A(r) = \text{cons}\ \ r$ at fixed points (AdS space)

Higher curvature gravity lagrangians:
a(r) function of A(r) and coupling constants
$a(r) = a^* = \text{constant}$ at fixed points

$$a'(r) \sim (T^t_t - T^r_r) \geq 0$$

null energy condition

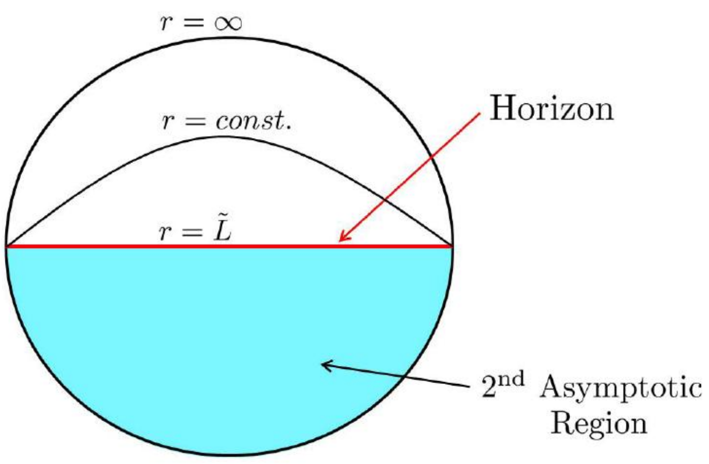
$$a^*_{uv} \geq a^*_{ir}$$

QFT interpretation: For even spacetime dimensions a^* is the coefficient of the Euler term in the trace anomaly (coincides with Cardy proposal for the c-theorem)

In any dimensions: asymptotic boundary of "hyperbolic black holes" in AdS $\longleftrightarrow$ thermal $H^{d-1} \times R$ $\longleftrightarrow$ Spherical "diamond" in vacuum Minkowski

BH entropy $\longleftrightarrow$ Thermal entropy in $H^{d-1} \times R$

$\longleftrightarrow$ Entanglement entropy for a sphere

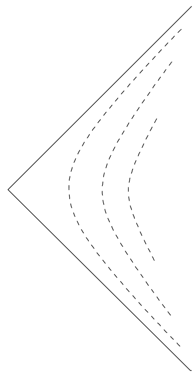


The constant term of the circle entropy proportional to a^*

Result coincides with, but is independent of the Ryu-Takayanagi proposal for holographic EE

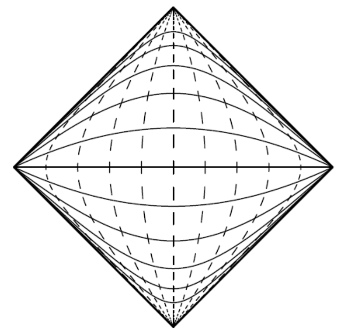
F-theorem (Jafferis, Klebanov, Pufu, Safdi (2011)): propose finite term in the free energy F=–log(Z) of a three sphere decreases under RG. Non trivial tests for supersymmetric and non-susy theories.

Log(z) for a three sphere= entanglement entropy of a circle



Conformal mapping

$\longleftrightarrow$



S=entanglement entropy

Causal domain of dependence of a sphere in Minkowski

Rindler Wedge
Unruh temperature
(any QFT)

$$\rho \sim e^{2\pi K}$$

Conformal mapping

$\updownarrow$

Static patch in de Sitter space

$$ds^2 = -\left(1 - \frac{\hat{r}^2}{R^2}\right) d\tau^2 + \frac{d\hat{r}^2}{1 - \frac{\hat{r}^2}{R^2}} + \hat{r}^2\, d\Omega^2_{d-2} \longrightarrow$$

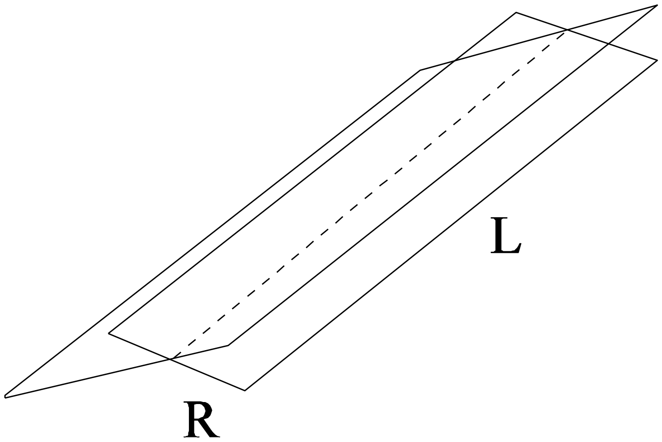
three sphere

$$T = 1/(2\pi R)\,, \qquad \langle T^\mu_\nu \rangle = \kappa\, \delta^\mu_\nu = 0 \quad \longleftrightarrow \quad S = \beta E + \log(Z) = \log(Z)$$

Good definition of c_0 as the cutoff independent term in log(Z)?

Returning to strong subadditivity: 2+1 dimensions

H.C., M. Huerta, 2012



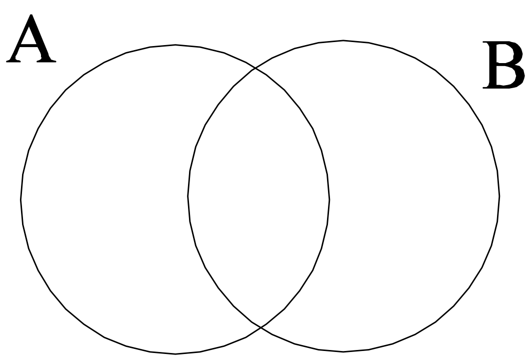
$$\lim_{L\to\infty} \frac{S(L,R)}{L} = G(R)$$

Dimensional reduction

$$(RG'(R))' \leq 0$$

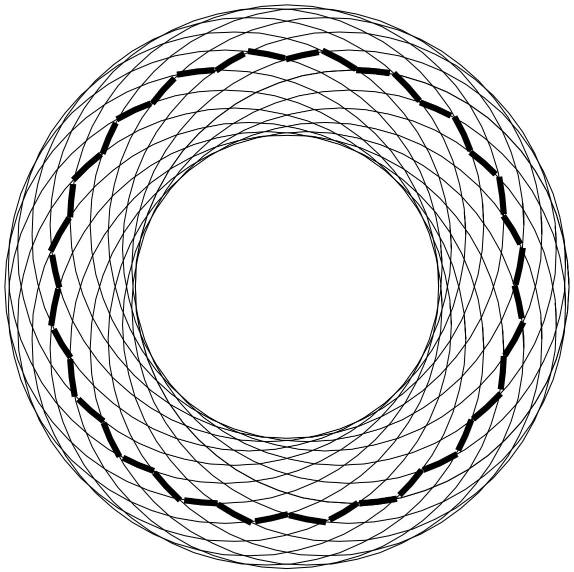
However a dimensionfull quantity does not converges to a number in the conformal limit

$$RG'(R) \sim \frac{c}{R}$$



Two problems: different shapes and divergent angle contributions

Multiple regions



$$S(A) + S(B) + S(C)$$

$$\begin{aligned}
&\geq S(A \cap B) + S(A \cup B) + S(C) \\
&\geq S(A \cup B \cup C) + S((A \cup B) \cap C) + S(A \cap B) \\
&\geq S(A \cup B \cup C) + S(((A \cup B) \cap C) \cup (A \cap B)) + S(A \cap B \cap C) \\
&= S(A \cup B \cup C) + S((A \cap C) \cup (A \cap B) \cup (B \cap C)) + S(A \cap B \cap C)
\end{aligned}$$

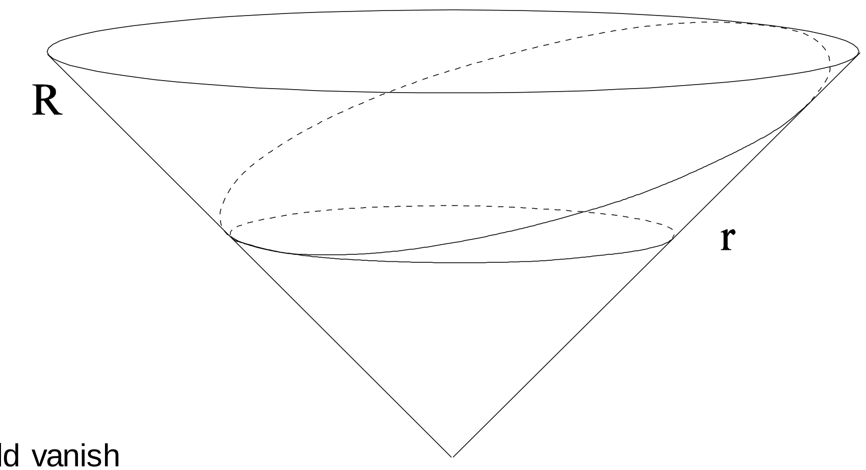
For rotated circles in a plane the angle contribution and the perimeter term do not match the ones of circles

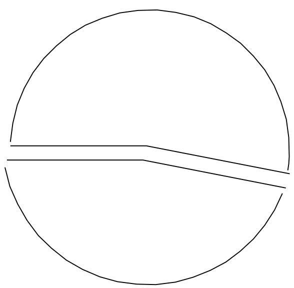
$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + ... + S(\cap_i X_i)$$

- Equal number of regions on both sides of the inequality
- Regions on right hand side ordered by inclusion
- Totally symmetrical with respect to permutation of the regions

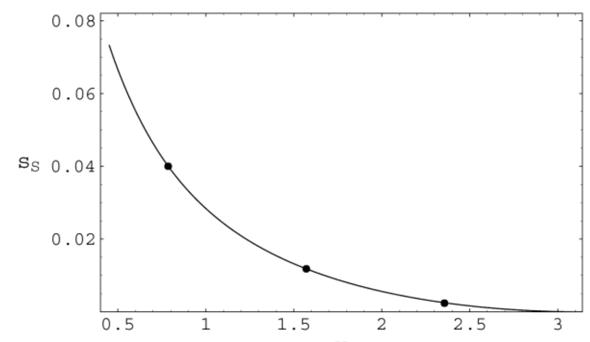
N rotated circles on the light cone

$$N\, S(\sqrt{Rr}) \geq \sum_{i=1}^{N} \tilde{S}\left(\frac{2rR}{R+r-(R-r)\cos(\frac{\pi i}{N})}\right)$$





Complementary angles have equal Logarithmic contribution which should vanish quadratically for angle π



As we approach the light cone the angles go to π and the perimeters of the wiggled regions approach the ones of circles of the same radius

Coefficient of the logarithmically divergent term for a free scalar field

$$\Rightarrow \quad S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^{\pi} dz\, S\left(\frac{2rR}{R+r-(R-r)\cos(z)}\right)$$

Equation for the area and constant terms

Infinitesimal inequality $S'' \leq 0$

At fix points $S(r) = c_0 + c_1 r$

Total variation of the area term $\Delta c_1 = c_1^{\mathrm{uv}} - c_1^{\mathrm{ir}} = S'^{\mathrm{uv}} - S'^{\mathrm{ir}} = -\int_0^\infty dr\, S'' \geq 0$

For free fields $\Delta c_1 = \frac{\pi}{6} m$

The area coefficient is not universal in QFT. Could this result be interpreted as a monotonous running of the Newton constant? SBH= area/(4 G)

Running of the constant term

Interpolating function $c_0(r) = S(r) - r\, S'(r)$ (coincides with c_0 at fix point (Liu-Mezei 2012))

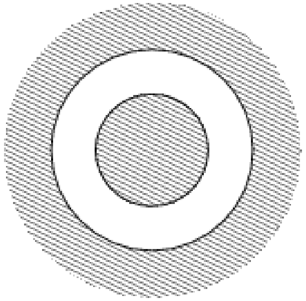
$$\Delta c_0 = c_0^{\mathrm{ir}} - c_0^{\mathrm{uv}} = -\int_0^\infty dr\, r\, S''(r) \geq 0 \longrightarrow \Delta c_0 \geq 0$$

Free field, and holographic calculations indicate $S''(r) \sim r^{-3}$ at large r: finite Δc_0

Then c_0 is dimensionless increasing from uv to ir, and has a finite total variation
A c-theorem in 2+1 dimensions?

Intrinsic definition for c_0 at fixed points? Liu-Mezei 2012

H.C., M. Huerta, R.C. Myers, A. Yale, in progress $I(R+\epsilon/2, R-\epsilon/2) = a\frac{R}{\epsilon} - c_0 + \mathcal{O}(\epsilon)$



Is the choice of constant term in Log(Z) on a sphere natural in odd dimensions?
Dimensionaly continued c-theorem: some numerology

Normalize the c-charge to the scalar c-charge in any dimension. For the Dirac field we have for the ratio of c-charge to number of field degrees freedom

d	2	4	6	8	10	12	14
$\frac{c[\text{Dirac}]}{2^{d/2}c[\text{scalar}]}$	$\frac{1}{2}$	$\frac{11}{4}$	$\frac{191}{40}$	$\frac{2497}{368}$	$\frac{73985}{8416}$	$\frac{92427157}{8562368}$	$\frac{257184319}{20097152}$
approx.	0.5	2.75	4.775	6.7853	8.7909	10.7946	12.7971

Fitting as
$$\frac{C[\text{Dirac}]}{2^{d/2}C[\text{scalar}]} = (d-2) + k_0 + \frac{k_1}{d} + \frac{k_2}{d^2} + \ldots$$

Fitting with 100 dimensions gives for d=3
$$\frac{C[\text{Dirac}]}{2^{[d/2]}C[\text{scalar}]} \to 1.7157936606$$

The correct value
$$\frac{C[\text{Dirac}]}{2^{[d/2]}C[\text{scalar}]} = \frac{\frac{\log(2)}{4} - \frac{3\zeta(3)}{8\pi^2}}{\frac{\log(2)}{4} + \frac{3\zeta(3)}{8\pi^2}} = 1.71579366494\ldots$$

Reason? The ratios of free energies on the sphere in zeta regularization
$$F = -\frac{1}{2}\lim_{s\to 0}\left[\mu^{2s}\zeta'(s) + \zeta(s)\log(\mu^2)\right]$$

$$\lim_{s\to 0}\frac{\zeta^1(s)}{\zeta^2(s)} = \begin{cases} \frac{\zeta^1(0)}{\zeta^2(0)} & \text{even dimensions} \\ \frac{\zeta^{1\,\prime}(0)}{\zeta^{2\,\prime}(0)} & \text{odd dimensions} \end{cases}$$

The same could be expected for the ratios of the entropies of spheres (taking out the most divergent terms)

More dimensions?

$$S(\sqrt{Rr}) \geq \frac{2^{d-2}\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_r^R dl \, \frac{(Rr)^{\frac{(d-1)}{2}}\left((l-r)(R-l)\right)^{\frac{d-3}{2}}}{(R-r)^{d-2}\, l^{d-1}} S(l)$$

symmetric configuration of boosted spheres in the limit of large number of spheres

$$r S''(r) - (d-2) S'(r) \leq 0$$

$$c_{d-1}^{\rm uv} - c_{d-1}^{\rm ir} = -\int_0^\infty dr \left(\frac{S'(r)}{(d-1) r^{d-2}}\right)' \geq 0$$

The coefficient of the area term decreases from uv to ir in any dimensions

The coefficient of the area term does not changes to leading order with the cutoff

Inequality is not correct for subleading terms in the cutoff: mismatch between curvatures of spheres and wiggled spheres, trihedral angles…

$$S(r) = c_2\, r^2 + c_1\, r + c_{\log}\, \log(R) + c_0$$

$C_{\log}$ negative in d=4 violates the inequality

For free fields

$$\Delta c_2 \;=\; \gamma_d \, {\rm vol}(s^{d-1})\, m^{d-1}\, \log(m\epsilon) \qquad {\rm for}\;\; d\;\; {\rm odd}$$

$$\Delta c_2 \;=\; \gamma_d \, {\rm vol}(s^{d-1})\, m^{d-1} \qquad {\rm for}\;\; d\;\; {\rm even}$$

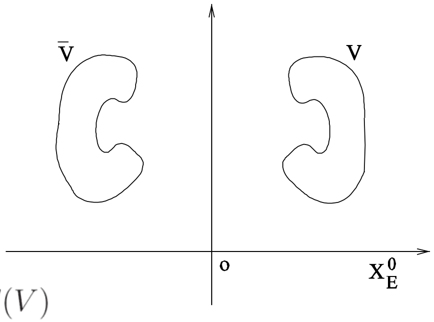
$$\gamma_d \;=\; (-1)^{\frac{d-1}{2}} [6 (4\pi)^{\frac{d-1}{2}} ((d-1)/2)!]^{-1} \;\; {\rm for}\;\; d\;\; {\rm odd}$$

Hertzberg-Wilczek (2010)

Suleading terms can have different signs

Strong subadditivity versus reflection positivity: Renyi entropies

Standard Zamolodchicov c-theorem uses reflection positivity on stress tensor correlation functions. Relation to SSA?



The path integral formula with the replica trick implies the exponentials of the integer index Renyi entropies obey reflection positivity inequalities (and hence define operator correlation functions)

$$S_n(V) = -(n-1)^{-1} \log(\mathrm{tr}\rho_V^n) \qquad \lim_{n\to 1} S_n(V) = S(V)$$

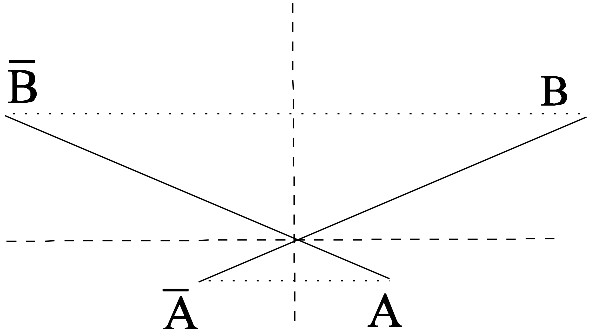
$$\det\left(\{\,\mathrm{tr}\rho^n_{V_i\bar{V}_j}\}_{i,j=1\ldots m}\right) = \det\left(\{e^{-(n-1)\,S_n(V_i\bar{V}_j)}\}_{i,j=1\ldots m}\right) \geq 0 \quad \longrightarrow \quad \mathrm{tr}\rho^n_{V_i\bar{V}_j} = \langle \mathcal{O}_{V_i}\mathcal{O}_{\bar{V}_j}\rangle$$

Twisting operators

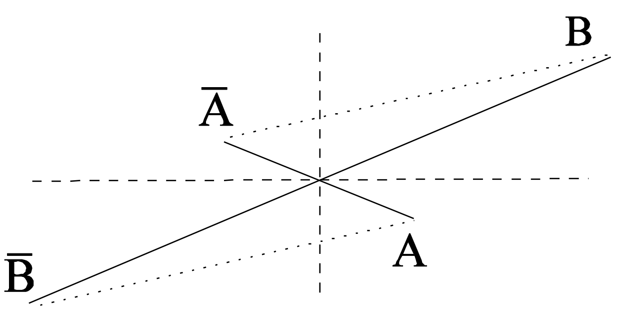
The inequality m=2 is linear and coincides with SSA for some coplanar symmetric cases (Renyi entropies are not SSA a priori and entropy need not be reflection positive). However, RP seems to be less powerfull than SSA:

$$2S_n(V_1\bar{V}_2) \geq S_n(V_1\bar{V}_1) + S_n(V_2\bar{V}_2)$$

SSA



Minkowskian reflection positivity

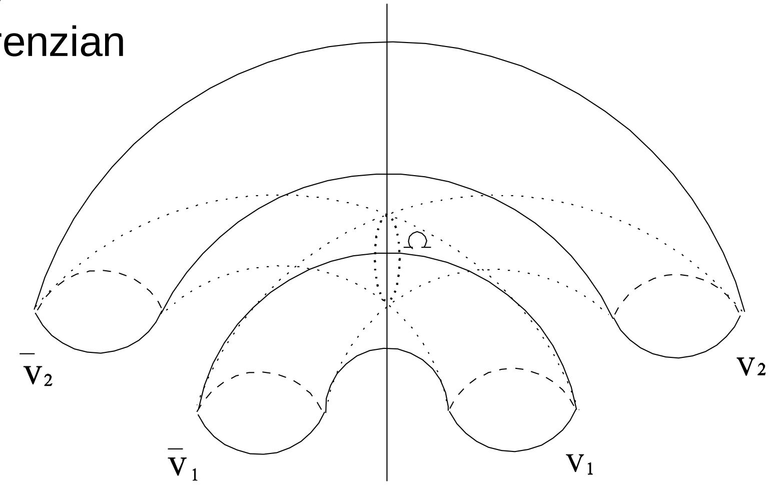


All the RP inequalities for an interval in 1+1 dimensions give a Kallen-Lehmann representation

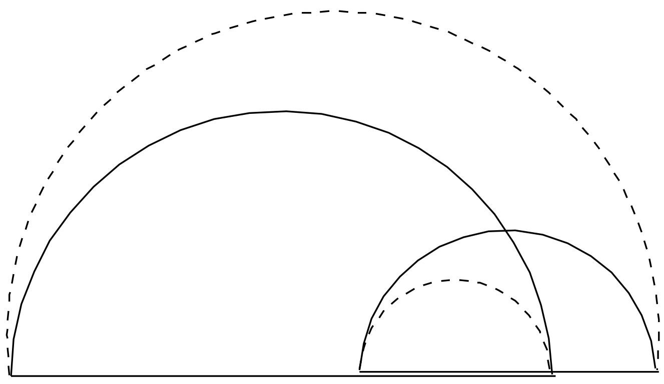
$$e^{-(n-1)S_n(r)} = \int_0^\infty dp^2\, g(p^2)\, K_0(pr)$$ which is not enough to give $$(rS_n(r)')' \leq 0$$

Strong subadditivity versus reflection positivity: Ryu-Takayanagi ansatz

Entanglement entropy of the CFT on the boundary
of AdS given by a minimal area in the bulk (for Lorenzian
geometries an extremal surface in AdS).



Linear RP inequalities guarranteed by
triangle inequality of the minimal area



SSA inequalities on a single hyperplane
hold because of triangle inequality of minimal area
Headrick-Takayanagi (2007)

Does the Ryu-Takayanagi ansatz give a SSA
entanglement entropy for general Lorenzian surfaces?

Needs at least energy conditions in the bulk, R. Callan, J.Y. He, M. Headrick 2012