Holographic Uniformization

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Outline

Motivation

- Is there a relation between RG flows in field theory and geometric flows? Ricci flow arose from the RG flow equations of the 2D nonlinear sigma model [Friedan], [Tseytlin]. Later it was a recognized as a useful tool in Mathematics [Hamilton], [Perelman], ...
- A simple idea - study RG flows "across dimensions". Put a D-dimensional field theory on $\mathbb{R}^{1,d-1} \times M_{D-d}$. What is the effect of the RG flow on the geometry of M_{D-d}?
- Hard to address in general via field theory methods. Study theories with maximal supersymmetry, known holographic duals and with a topological twist on M_{D-d} for better calculational control.

Introduction

- A large class of interacting $\mathcal{N}=2$ SCFTs in four dimensions from the $(2,0)$ theory on a Riemann surface Σ_g [Gaiotto-Moore-Neitzke], [Gaiotto].
- The UV theory is well defined for any metric on Σ_g. The IR CFT has information only about the complex structure deformations of Σ_g [Gaiotto].
- What happens to the rest of the metric degrees of freedom of Σ_g? Gaiotto: "They are washed out by the RG flow". There are other possibilities - the flow exists only for a specific choice of metric, other CFTs, massive theory, free theory... More generally, what happens with the metric degrees of freedom of topologically twisted supersymmetric field theories on curved manifolds?
- The same picture emerges also for a large class of $\mathcal{N}=1$ SCFTs in four dimensions [Benini-Tachikawa-Wecht], [Bah-Beem-NB-Wecht] as well as for the two-dimensional SCFTs obtained from $\mathcal{N}=4$ SYM compactified on a Riemann surface [Bershadsky-Johansen-Sadov-Vafa].
- Direct RG analysis in the field theory is hard. For many twisted field theories there is a useful holographic description [Maldacena-Núñez], [Gauntlett-Kim-Waldram], . . .

Twists and branes

Topological twists - I

A supersymmetric field theory on a generic curved manifold is no longer supersymmetric

$$\nabla_\mu \epsilon = (\partial_\mu + \omega_\mu)\, \epsilon \neq 0 \ .$$

To preserve supersymmetry perform a "topological twist", i.e. use the R-symmetry to cancel the space-time curvature [Witten]

$$A_\mu^{(R)} = -\omega_\mu \ , \qquad \to \qquad \widetilde{\nabla}_\mu \epsilon = \left(\partial_\mu + \omega_\mu + A_\mu^{(R)}\right) \epsilon = \partial_\mu \epsilon = 0 \ .$$

Branes in string/M theory wrapping curved cycles preserve supersymmetry in this way [Bershadsky-Sadov-Vafa] .

Here - topological twists of the $(2,0)$ theory in 6D and $\mathcal{N} = 4$ SYM in 4D on a closed Riemann surface of genus g Σ_g. Focus on $g > 1$ but this is not essential.

Topological twists - II

Put the A_{N-1} (2,0) theory on a Riemann surface Σ_g. Decompose the supercharges under $SO(1,3) \times SO(2)_{\Sigma_g} \times U(1)_1 \times U(1)_2 \subset SO(1,5) \times SO(5)_R$

$$\mathbf{4} \otimes \mathbf{4} \to \left[(\mathbf{2},\mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\right] \otimes \left[(\tfrac{1}{2},\tfrac{1}{2}) \oplus (-\tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},-\tfrac{1}{2}) \oplus (-\tfrac{1}{2},-\tfrac{1}{2})\right]$$

and define

$$SO(2)' = SO(2)_{\Sigma_g} + aU(1)_1 + bU(1)_2 \ .$$

For $a \pm b = \pm 1$ there are 4 invariant supercharges (1/4 BPS). For $b = 0$ (or $a = 0$) enhancement to 1/2 BPS, i.e. $\mathcal{N} = 2$ in 4D.

These twists are realized in M-theory by N M5 branes wrapping a calibrated 2-cycle in a Calabi-Yau manifold - CY_2 for 1/2 BPS and CY_3 for 1/4 BPS. [Maldacena-Núñez]

Topological twists - III

One can also study $\mathcal{N} = 4$ SYM on a Riemann surface in a similar fashion [Bershadsky-Johansen-Sadov-Vafa], [Maldacena-Núñez] .

The supercharges decompose under $SO(1,1) \times SO(2)_{\Sigma_g} \times U(1)_1 \times U(1)_2 \times U(1)_3 \subset SO(1,3) \times SO(6)_R$ and the twist is implemented by the choice

$$SO(2)' = SO(2)_{\Sigma_g} + a_1 U(1)_1 + a_2 U(1)_2 + a_3 U(1)_3$$

For $a_1 \pm a_2 \pm a_3 = \pm 1$ there are 2 invariant supercharges (1/8 BPS or $(0,2)$ susy in 2D). For $a_3 = 0$ enhancement to 1/4 BPS, i.e. $(2,2)$ in 2D. For $a_1 = a_2 = 0$ enhancement to 1/2 BPS, i.e. $(4,4)$ in 2D.

These twists are realized in IIB string theory by N D3 branes wrapping a calibrated 2-cycle in a Calabi-Yau manifold - CY_2 for 1/2 BPS, CY_3 for 1/4 BPS and CY_4 for 1/8 BPS.

M5 branes on Σ_g

The focus here will be on the 1/2 BPS twist of the 6D $(2,0)$ theory. The IR theory should be conformal and preserve $\mathcal{N}=2$ supersymmetry [Klemm-Lerche-Mayr-Vafa-Warner], [Witten], [Gaiotto].

The geometric realization of the twist is by wrapping N M5 branes on a holomorphic 2-cycle in a Calai-Yau two-fold. In the field theory limit the Calabi-Yau geometry is non-compact

$$T^*(\Sigma_g)$$

In other words two of the five transverse directions to the M5 branes are the normal bundle to Σ_g and the other three are flat, realizing the $SU(2)$ R-symmetry of the IR theory.

Holography is one of the few available tools to study this theory. The brane realization provides the necessary input information to construct the supergravity dual [Madacena-Núñez].

Supergravity

Setup - I

In the large N limit the A_{N-1} $(2,0)$ theory is dual to 11D supergravity on $AdS_7 \times S^4$. To study the twist we need only modes that lie within the maximal seven-dimensional gauged supergravity [Maldacena-Núñez].

We need only the metric, two Abelian gauge fields, $A_\mu^{(i)}$, and two neutral real scalars λ_i. This is a consistent truncation of the maximal 7D theory [Liu-Minasian], [Cvetič, et al.].

The Ansatz for the supergravity fields is specified by the twist of the field theory and its realization in terms of M5 branes

$$ds^2 = e^{2f}(-dt^2 + dz_1^2 + dz_2^2 + dz_3^2) + e^{2h}dr^2 + e^{2g}\frac{dx^2 + dy^2}{y^2} \ ,$$

$$A^{(i)} = A_x^{(i)}dx + A_y^{(i)}dy + A_r^{(i)}dr \ ,$$

$$\lambda_i = \lambda_i(x, y, r) \ , \qquad i = 1, 2 \ .$$

Setup - II

In the UV the solutions have the following asymptotics

$$f \to -\log(r)\ , \qquad h \to -\log(r)\ , \qquad g \to -\log(r) + g_0(x,y)\ ,$$

$$\lambda_i \to 0\ , \qquad A_\mu^{(1)} + A_\mu^{(2)} = -\omega_\mu\ .$$

The UV boundary is $\mathbb{R}^{1,3} \times \Sigma_g$ with $g_0(x,y)$ the metric of the Riemann surface.

The Riemann surface is a quotient of $\mathbb{H}_2$ by $\Gamma \in PSL(2,\mathbb{R})$. One can treat also S^2 and T^2 with the same approach.

In previous work there was no dependence on the coordinates of the Riemann surface [Maldacena-Núñez]

For the 1/2 BPS twist set $A^{(2)} = 0$ and $3\lambda_1 + 2\lambda_2 = 0$ and define $A \equiv A^{(1)}$ and $\lambda \equiv \lambda_2$. One then derives the conditions for existence of supersymmetry.

BPS equations

The set of BPS equations looks unwieldy

$$
\begin{aligned}
&\partial_r\lambda + \frac{2m}{5}e^{h-3\lambda} - \frac{2m}{5}e^{h+2\lambda} + \frac{2}{5}e^{h-2g+3\lambda}F_{xy} = 0 \ , \\
&(\partial_x + i\partial_y)\lambda + \frac{2}{5}e^{-h+3\lambda}(F_{yr} - iF_{xr}) = 0 \ , \\
&\partial_r\left(f - \tfrac{1}{2}\lambda\right) + \frac{m}{2}e^{h+2\lambda} = 0 \ , \\
&(\partial_x + i\partial_y)\left(f - \tfrac{1}{2}\lambda\right) = 0 \ , \\
&\partial_r(g + 2\lambda) + me^{h-3\lambda} - \frac{m}{2}e^{h+2\lambda} = 0 \ , \\
&\partial_r\partial_y(g + 2\lambda) + 2mF_{rx} = 0 \ , \\
&\partial_r\partial_x(g + 2\lambda) - 2mF_{ry} = 0 \ , \\
&(\partial_x^2 + \partial_y^2)(g + 2\lambda) + \frac{1}{y^2} - 2mF_{xy} = 0 \ .
\end{aligned}
$$

The parameter m is the coupling of the gauged supergravity with $R_{AdS} \sim m^{-1}$.

BPS equations redux

However there is a drastic simplification!

Define

$$F(r) \equiv f(r,x,y) - \tfrac{1}{2}\lambda(r,x,y) \ , \qquad H(r) \equiv h(r,x,y) + 2\lambda(r,x,y) \ , \qquad \rho \equiv \tfrac{2}{m}F(r) \ ,$$

and

$$\varphi(\rho,x,y) \equiv 2g(\rho,x,y) + 4\lambda(\rho,x,y) \ , \qquad \Phi(\rho,x,y) \equiv \varphi(\rho,x,y) - 2\log y \ .$$

Remarkably the BPS equations reduce to a single second order nonlinear elliptic PDE

$$\boxed{(\partial_x^2 + \partial_y^2)\Phi + \partial_\rho^2 e^{\Phi} = m^2 e^{\Phi}}$$

All background fields are determined by $\Phi(\rho,x,y)$.

For $m=0$ this is the $SU(\infty)$ Toda equation. It is well-known and integrable [Saveliev]. It arises in various places in Physics - continuum limit of the Toda lattice, self-dual gravitational solutions in 4D. It is crucial also in the analysis of 1/2 BPS solutions of 11D supergravity [Lin-Lunin-Maldacena]. The connection with our setup is unclear.

Covariant flow equation

The flow equation can be rewritten covariantly as

$$\partial_\rho^2 g_{ij} - 2R_{ij} - m^2 g_{ij} = 0$$

with g_{ij} the metric on an auxiliary Riemann surface $ds^2_{\Sigma'} = e^{\Phi}(dx^2 + dy^2)$. This is the "right" metric for which the geometric flow happens.

Clearly this is very different from Ricci flow

$$\partial_\tau g_{ij} = -2R_{ij}$$

However, it seems to have similar effects on the metric of the Riemann surface.

IR analysis

Even when one allows for an arbitrary metric on the Riemann surface there is a unique AdS_5 IR solution. The Riemann surface has constant curvature

$$e^g = \frac{2^{1/10}}{m} \ , \qquad e^\lambda = 2^{1/5} \ , \qquad e^f = e^h = \frac{2^{3/5}}{m}\frac{1}{r} \quad \to \quad e^{\varphi_{IR}} = \frac{2}{m^2}$$

The constant curvature IR solution exhibits a local attractor behavior in the space of metrics on Σ_g. Note that in the IR $\rho \to -\infty$

$$\varphi = \varphi_{IR} + \epsilon \sum_{n=0}^{\infty} \widetilde{\varphi}_n(\rho) Y^{(n)}(x,y) \ , \qquad \epsilon \ll 1$$

Since the Riemann surface is compact and hyperbolic, we have

$$y^2(\partial_x^2 + \partial_y^2) Y^{(n)}(x,y) = -\mu_n Y^{(n)}(x,y) \ , \qquad \mu_0 = 0 \ , \quad \mu_n > 0 \ , \quad n > 1 \ .$$

$$\widetilde{\varphi}_n(\rho) = a_n e^{\alpha_n^{(+)} m\rho} + b_n e^{\alpha_n^{(-)} m\rho} \ ,$$

where

$$\alpha_n^{(\pm)} = \pm\sqrt{1 + \tfrac{1}{2}\mu_n} \ ,$$

We need to set $b_n = 0$ for regularity.

UV analysis

The UV expansion is ($\zeta \equiv e^{-\frac{m}{2}\rho}$ and $\zeta \to 0$)

$$g(\zeta, x, y) = -\log(\zeta) + g_0(x, y) + g_2(x, y)\zeta^2 + g_{4\ell}(x, y)\zeta^4 \log\zeta + g_4(x, y)\zeta^4 + \mathcal{O}(\zeta^5)$$

$$\lambda(\zeta, x, y) = \lambda_2(x, y)\zeta^2 + \lambda_{4\ell}(x, y)\zeta^4 \log\zeta + \lambda_4(x, y)\zeta^4 + \mathcal{O}(\zeta^5)$$

The scalar λ is dual to a dimension 4 operator in the $(2, 0)$ CFT, λ_2 is related to the source and λ_4 controls the vev. All other functions are fixed in terms of g_0 and λ_4. Note that λ_2 is controlled only by g_0. The logarithmic terms vanish for $g_0 = \text{const}$.

The metric on the Riemann surface is arbitrary in the UV!

Holographic RG flows are somewhat different from Wilsonian RG flows. We need to choose the "correct" λ_4 and λ_2 to flow to the IR fixed point. Generic choices will lead to a singular solution. This is a well-known feature of holographic RG flows [Gubser-Freedman-Pilch-Warner], [Gubser] . . .

Exact solution and linearized analysis - I

The exact solution of Maldacena-Núñez is for a constant negative curvature metric on Σ_g

$$e^{\varphi_{\mathrm{MN}}} = \frac{e^{2m\rho} + 2e^{m\rho} + C}{m^2 e^{m\rho}}$$

Only for $C = 0$ the solution flows to the fixed point in the IR. For other values of C the RG flow is to the Coulomb/Higgs branch, i.e. C is proportional to the vev of the dimension 4 operator.

One can expand around the MN solution, linearize and solve for the small perturbations ($\epsilon \ll 1$)

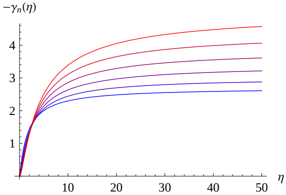
$$\lambda = \lambda_{\mathrm{MN}}(\rho) + \epsilon \sum_n \ell_n(\rho) Y^{(n)}(x,y) \ ,$$

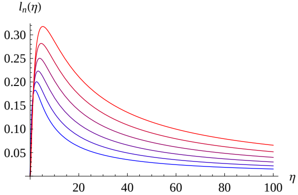
$$g = g_{\mathrm{MN}}(\rho) + \epsilon \sum_n \gamma_n(\rho) Y^{(n)}(x,y)$$

An analytic solution for $\ell_n(\rho)$ and $\gamma_n(\rho)$ in terms of hypergeometric functions.

Exact solution and linearized analysis - II

It is convenient to work with $\eta = e^{m\rho}$. The UV is at $\eta \to \infty$ and the IR is at $\eta \to 0$.





This is the expected uniformizing behavior! Any (small) local perturbation of the metric on Σ_g is damped and the constant curvature metric is an IR attractor.

The proof

There is a rigorous proof that the uniformizing flows exist globally, i.e. for any metric on the Riemann surface in the UV. The proof uses some technology from nonlinear functional analysis and has the following structure:

- Show that the space of solutions is a smooth (Banach) manifold.
- Show that the boundary map (from the space of solutions to the space of boundary conditions) is smooth and proper. This implies that it has a well defined degree.
- Show that the degree is not zero.
- Smooth maps with non-zero degree are surjective. Therefore for any initial condition in the UV there is a flow to the prescribed IR solution with constant curvature on Σ_g.

Monotonicity of the area

The area of the auxiliary Riemann surface $ds^2_{\Sigma'} = e^{\Phi}(dx^2 + dy^2)$ decreases monotonically along the flow.

This can be proven for all cases we studied. For the 1/2 BPS flow for M5 branes one finds

$$\mathcal{A}(\rho) = \int_{\Sigma'} e^{\Phi} = c_1 e^{\rho} + c_2 e^{-\rho} + 4\pi\chi(\Sigma')$$

For the solution which flows to the IR AdS_5 vacuum $c_2 = 0$ and $\mathcal{A}$ is clearly monotonic.

Maybe this is a general feature of all holographic RG flows "across dimension"? Generalized holographic c-theorem?

Other twists - I

- A similar analysis can be performed for a $1/4$ BPS twist of the $(2,0)$ theory as well as twists of $\mathcal{N}=4$ SYM.
- In all cases the gravitational description is given by a metric, an Abelian gauge field and a single neutral scalar. The BPS equations always imply the equations of motion and reduce to a single, second order, nonlinear elliptic PDE.
- When there is an AdS IR vacuum it is unique, behaves as a local attractor in the IR and the holographic RG flow uniformizes the metric on the Riemann surface.
- For all cases we studied there is a rigorous proof for the existence of the uniformizing solutions and the area of the auxiliary Riemann surfaces decreases along the flow.

Other twists - II

For the other flows we studied the nonlinear flow equations look less familiar.

- $1/4$-BPS twist of the $(2,0)$ theory

$$\partial_\rho^2 e^{\varphi} + y^2(\partial_x^2 + \partial_y^2)\varphi + 2 - \frac{3m^2}{2}e^{\varphi} - e^{\varphi}\left(\frac{1}{2}(\partial_\rho\varphi)^2 - m\partial_\rho\varphi\right) = 0$$

- $1/2$-BPS twist of $\mathcal{N}=4$ SYM

$$\partial_\rho^2 e^{\varphi} + 9y^2(\partial_x^2 + \partial_y^2)\varphi + 18 - 6\partial_\rho e^{\varphi} = 0$$

- $1/4$-BPS twist of $\mathcal{N}=4$ SYM

$$\partial_\rho^2 e^{\varphi} + 9y^2(\partial_x^2 + \partial_y^2)\varphi + 18 - 18e^{\varphi} - \frac{1}{2}e^{\varphi}(\partial_\rho\varphi)^2 = 0$$

Outlook

In progress

Study M5-branes on four-manifolds [Benini-NB]. The story is very rich and one can find large classes of 2D SCFTs with $(2,2)$, $(1,2)$ and $(0,2)$ supersymmetry. One can use anomalies to compute the exact central charges. Most progress can be made for the case $M_4 = \Sigma_{g_1} \times \Sigma_{g_2}$. The geometry is a CY_4 given by two complex line bundles over M_4. One finds a two-parameter family of $(0,2)$ SCFTs. This analysis led to a proof of the 2D analog of a-maximization, i.e. c-extremization.

Look for a family of 3D SCFTs by studying M5-branes wrapped on three-manifolds [Beem-NB]. There are two twists with $\mathcal{N}=2$ and $\mathcal{N}=1$ for which there is an IR SCFT [Gauntlett-Kim-Waldram]. Recent work on the field theory [Dimofte-Gaiotto-Gukov], [Cecotti-Cordova-Vafa].

Is there "holographic geometrization" for 3-manifolds [Anderson-Beem-NB-Rastelli]?

Future work

- Study Riemann surfaces with punctures along the lines of Gaiotto-Maldacena. There should be uniformizing holographic RG flows. This should be straightforward but technically involved.
- A similar picture should emerge for M2 branes wrapped on a Riemann surface.
- Can these holographic flows teach us anything new about geometry/topology? Can physics help with the familiar singularities of Ricci flow [Perelman], ...? String theory is very good at teaching us how to understand/resolve some singularities - geometric transition, enhançon,...

Questions

This is a workshop...

- Can one derive the NSVZ β-function from supersymmetric holographic RG flows? Is there some analog for supersymmetric RG flows across dimension?
- Is there some monotonic quantity for RG flows across dimension?
- Formulate the holographic RG flow as an initial value problem? Holographic analogue of Wilsonian RG flow? Recent attempts to address/revisit these questions [Heemskerk-Polchinski], [Faulkner-Liu-Rangamani].
- Any good criterion for distinguishing between physical and unphysical singularities for general holographic RG flows? This should be some generalization of the ideas of Gubser and Maldacena-Núñez.
- Any useful notion of entanglement entropy for a field theory that flows between two fixed points in different dimension?

THANK YOU!