Scattering Amplitudes and Null Polygon Wilson Loops

Operator Product Expansion and Integrability

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Outline

- The 4D/2D map for Wilson Loops and Scattering Amplitudes
- The Operator Product Expansion (OPE) at...
  - Tree-level
  - One-loop
  - All loops
  - Strong coupling
- Integrability
- Conclusions and future directions
4d Gauge Theories are
2d String Theories

- This is just a cartoon of course. Sometimes this cartoon can be made precise (AdS/CFT)
- In practice the message is that if we are smart enough we should be able to describe 4d observables in a purely 2d language. If we are lucky the 2d description might be Integrable.
Scattering in 4d...

... is intimidating.
Scattering in 2d...

... can be deliciously simple...

... if we have Integrability
Scattering in 2d...

... can be deliciously simple ...

... if we have *Integrability*
4d Gauge theories with an Integrable 2d dual description deserve a special name like *Harmonic Oscillators of Gauge Theories* or something like this.

**N=4 SYM** is one such theory.

(Certainly in the planar limit; probably/hopefully beyond as well)
Scattering Amplitudes

4d description

\[ k_1 + k_2 + k_3 + k_4 - k_5 + k_6 = ? \]

2d description

\[ = \]

?
Null Wilson Loops

4d description

Another 4d description

[Alday, Maldacena], [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]
[Drummond, Henn, Korchemsky, Sokatchev], [Berkovits, Maldacena]
Super Amplitudes

\[ \Phi(k, \bar{\eta}) = G^+(k) + \bar{\eta}^A \Psi_A(k) + \bar{\eta}^A \bar{\eta}^B \Phi_{AB}(k) + \bar{\eta}^A \bar{\eta}^B \bar{\eta}^C \epsilon_{ABCD} \bar{\Psi}^D(k) + \bar{\eta}^A \bar{\eta}^B \bar{\eta}^C \bar{\eta}^D \epsilon_{ABCD} G^-(k) \]

\[ \mathcal{A}(k_i, \bar{\eta}_i) = \mathcal{A}^{\bar{\eta} \bar{\eta} \bar{\eta} \bar{\eta} \bar{\eta} \bar{\eta} \bar{\eta}}(k_i) \bar{\eta}_1^1 \bar{\eta}_1^2 \bar{\eta}_2^3 \bar{\eta}_2^4 \bar{\eta}_2^1 \bar{\eta}_2^2 \bar{\eta}_2^3 \bar{\eta}_2^4 + \ldots \]

\[ = \mathcal{A}^{\text{MHV} (\bar{\eta})^8}(k_i) \left( 1 + \mathcal{R}^{\text{NMHV} (\bar{\eta})^4}(k_i) + \mathcal{R}^{\text{N}^2 \text{MHV} (\bar{\eta})^8}(k_i) + \ldots \right) \]

---++++ and SUSY friends

---++++ and SUSY friends

---++++ and SUSY friends
At tree level we know all super amplitudes.
Super Wilson Loops

For edge $i$: $A + \eta_i^A \bar{\Psi}_A + \eta_i^A \eta_i^B D\Phi_{AB} + \ldots$

For vertex $i$: $1 + \eta_i^A \eta_{i+1}^B \Phi_{AB} + \ldots$

NMHV ratio function. Conformal invariant finite quantity.

Usual bosonic Wilson loop. (Contains UV divergences which one knows how to isolate)

$W(k_i, \eta_i) = W_{\text{MHV}}^0(\eta)(k_i) \left( 1 + R_{\text{NMHV}}^4 (k_i) + R_{\text{N}^2 \text{MHV}}^8 (k_i) + \ldots \right)$
Super duality

\[ A(k, \tilde{\eta}) = A_{\text{tree level}}^{\text{MHV}}(k, \tilde{\eta}) \mathcal{W}(k, \eta) \]

(The tilded and non-tilded eta’s are simple linear combinations of each other.)
A remarkable property

Increasing MHV degree is a window into higher loop physics

![Feynman diagrams](image-url)
The 2 dim map

4d description

2d description

?
An excitation is created in the **bottom**, propagates in a **1+1 dim flux tube** and is absorbed at the **top**.
The simplest example

No cross-ratios for $n=4$ or $n=5$. The Amplitude/Wilson loop is fixed by conformal symmetry.

[Drummond, Henn, Korchemsky, Sokatchev]

So the simplest object is the **Hexagon NMHV polygon** which starts at tree level

3 cross-ratios for the Hexagon:

\[
\begin{align*}
    u_2 &= \frac{(x_2 - x_4)^2(x_1 - x_5)^2}{(x_2 - x_5)^2(x_1 - x_4)^2}, \\
    u_{1,3} &= u_2 \big|_{x_i \to x_{i \pm 1}}
\end{align*}
\]
More precisely

\[
\frac{A(k_i, \eta_i)}{A_{\text{MHV tree-level}}(k_i, \eta_i)} = \mathcal{W}^{\text{MHV}}(\eta)^0(k_i) \left(1 + \mathcal{R}^{\text{NMHV}}(\eta)^4(k_i) + \mathcal{R}^{\text{N}^2\text{MHV}}(\eta)^8(k_i) + \ldots\right)
\]

\[
\mathcal{R}^{\text{NMHV}}(\eta)^4(k_i) = \mathcal{R}^{(2356)}(k_i) \eta_2^A \eta_3^B \eta_5^C \eta_6^D \epsilon_{ABCD} + \mathcal{R}^{(2256)}(k_i) \eta_2^A \eta_2^B \eta_5^C \eta_6^D \epsilon_{ABCD} + \ldots
\]

At tree level:

\[\eta_2 \eta_3 \eta_5 \eta_6\]

\[\eta_2 \eta_2 \eta_5 \eta_6\]

+ 

+ \ldots
OPE variables

We want a good three parameter family Hexagons

\[ k_i(\sigma, \tau, \phi) \]

suitable to describe the 2d picture:
The OPE

\[ \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle \]
The OPE

\[ \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle \]

Top group

Bottom group
The OPE

\[ \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle \]

Top group

Bottom group
The OPE

\[ \left\langle O(x_1) O(x_2) O(x_3) O(x_4) \right\rangle \]
The OPE

\[ \langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(1)\mathcal{O}(\infty) \rangle = \sum_{\Delta, l} C_{\Delta, l} \frac{1}{|x|^{2\Delta}} \left( \frac{x}{\bar{x}} \right)^l = \sum_p C_{12p} C_{p34} F_p (1, 2, 3, 4; x) \]

\( \Delta = \) energy

\( \log |x| = \) time

Top group

Bottom group
The OPE

Correlation Functions: Null Polygon Wilson Loops:
Two null lines preserve $SL(2) \times R_\sigma \times SO(2)_\phi$

The OPE

Correlation Functions:

Null Polygon Wilson Loops:
The OPE

Correlation Functions:

Null Polygon Wilson Loops:

Two null lines preserve

$$SL(2) \times R_\sigma \times SO(2)$$

Reference Square = Conformal frame
It preserves

$$R_\tau \times R_\sigma \times SO(2)$$
The OPE

Correlation Functions:

Null Polygon Wilson Loops:

Two null lines preserve $SL(2) \times R_\sigma \times SO(2)\phi$

Reference Square = Conformal frame
It preserves $R_\tau \times R_\sigma \times SO(2)\phi$
Two null lines preserve
\( SL(2) \times R_\sigma \times SO(2)_\phi \)

Reference Square = Conformal frame
It preserves
\( R_\tau \times R_\sigma \times SO(2)_\phi \)

The OPE

Correlation Functions:

Null Polygon Wilson Loops:
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The OPE

Correlation Functions:

Null Polygon Wilson Loops:

Two null lines preserve $SL(2) \times R_\sigma \times SO(2)_\phi$

Reference Square = Conformal frame
It preserves $R_\tau \times R_\sigma \times SO(2)_\phi$

$$\langle W \rangle = \int dp \, e^{ip_\sigma} \sum_m e^{im_\phi} \sum_{E(m,p)} \mathcal{C}(m,p) \, e^{-E(m,p)\tau} + \text{Two particles} + \ldots$$
The OPE family

We act with the $\mathbb{R}_\tau \times \mathbb{R}_\sigma \times SO(2)_\phi$ symmetries of a ref. square on the bottom of the hexagon. In this way we generate a family of Hexagons $\mathcal{W}(\tau, \sigma, \phi)$

E.g. $\mathcal{R}^{(2356)}_{\text{tree-level}} = \frac{1}{\cosh(\tau) \cosh(\sigma) + \cos(\phi)}$

Choice of zero and infinity in the usual OPE are the analogue of the choice of reference square.
Tree level and the OPE

\[ R_{\text{tree-level}}^{(2356)} = \frac{1}{\cosh(\tau) \cosh(\sigma) + \cos(\phi)} \]

\[ = \sum_{m=-\infty}^{\infty} \int dp \, e^{i m \phi - i p \sigma} C_m(p) F_{|m|+1,p}(\tau) \]

Simple ratio of Gamma functions

Simple Hypergeometric function:

\[ F_{E,p}(\tau) = e^{-E \tau} + (\ldots) e^{-(E+2) \tau} + \ldots \]
The SL(2) physics

\[ \mathcal{R}_{\text{tree-level}}^{(2356)} = \sum_{m=-\infty}^{\infty} \int dp \, e^{im\phi - ip\sigma} C_m(p) \mathcal{F}_{|m|+1,p}(\tau) \]

From the picture we expect the propagation of a scalar primary plus its SL(2) descendants. The primaries are

\[ \partial^m_z \phi_{AB}, \quad \partial^m_{\bar{z}} \phi_{AB}, \quad \text{where } m = 0, 1, 2, \ldots \]

and have energy \( E = |m| + 1 \) which is precisely the argument of the “Conformal blocks” \( F \). Indeed, these obey

\[ (SL(2) \text{ Casimir}) \circ (\mathcal{F}_{E,p} e^{-ip\sigma}) = E(E-2) (\mathcal{F}_{E,p} e^{-ip\sigma}) \]

energy of the primary \quad descendants

\[ \mathcal{F}_{E,p}(\tau) = e^{-E\tau} + (\ldots) e^{-(E+2)\tau} + \ldots \]

\( C_m(p) \) is the amplitude for creating the primary at the bottom and absorbing it at the top.
One loop and the OPE

At tree level the particles do not feel the flux tube; they are free. That is why their energies are quantized.

At one loop the energy gets shifted by the anomalous dimension of the scalar primaries. This gives rise to a term linear in \( \tau \).

\[
\nu_2 = \frac{1}{\cosh^2(\tau)}, \quad \tau \sim \log(\nu_2)
\]
OPE Discontinuities

\[ u_2 = \frac{1}{\cosh^2(\tau)} \]

\[ R^{(2356)}_{\text{one-loop}} = \log(u_2) D_2^{(1)} + \tilde{D}_2^{(1)} \]

The anomalous dimension of the scalar primaries can be computed from integrability. They are the scalar excitations around the so called GKP string.

With these we can immediately get \( D_2 \) as a sum which can easily be done, yielding simple logs agreeing with the known results.

In perfect agreement with these expectations, we have indeed

\[ R^{(2356)}_{\text{tree-level}} = \sum_{m=-\infty}^{\infty} \int dp \ e^{im\phi - ip\sigma} c_m(p) F_{|m|+1,p}(\tau) \]

\[ D_2^{(1)} = \sum_{m=-\infty}^{\infty} \int dp \ e^{im\phi - ip\sigma} c_m(p) F_{|m|+1,p}(\tau) \gamma_{|m|+1}(p) \]
Bootstrap

(like for correlation functions)

- Take the tree level data from BCFW or whatever
- OPE promote it to get a one loop discontinuity in a good channel
- Use SUSY Ward identities \([\text{Elvang,Freedman,Kiermaier}]\), ciclicity and parity to extract the discontinuities any of the OPE three channels for any amplitude
- Bootstrap the final result by imposing the corresponding discontinuities in all OPE channels
Higher loop OPE predictions

\[ \mathcal{R}^{(2356)}_{n\text{-loops}} = \log(u_2)^n D_2^{(n)} + \log(u_2)^{n-1} \tilde{D}_2^{(n)} + \ldots \]

where:

\[ D_2^{(0)} = \sum_{m=-\infty}^{\infty} \int dp \, e^{im\phi - ip\sigma} c_m(p) F_{|m|+1,p}(\tau) \]

\[ D_2^{(1)} = \sum_{m=-\infty}^{\infty} \int dp \, e^{im\phi - ip\sigma} c_m(p) F_{|m|+1,p}(\tau) \gamma_{|m|+1}(p) \]

\[ D_2^{(n)} = \sum_{m=-\infty}^{\infty} \int dp \, e^{im\phi - ip\sigma} c_m(p) F_{|m|+1,p}(\tau) \frac{(\gamma_{|m|+1}(p))^n}{n!} \]
At 5 loops:

\[
S \left( R_{\text{5 loops}}^{(2356)} \right) = u_2 \otimes u_2 \otimes u_2 \otimes u_2 \otimes u_2 \otimes S \left( \sum_{m=-\infty}^{\infty} dp e^{im\phi-ip\sigma} C_m(p) F_{|m|+1,p}(\tau) \frac{(\gamma|m|+1(p))^5}{5!} \right)
\]

\[
+ u_2 \otimes u_2 \otimes u_2 \otimes u_2 \otimes S \text{(Next-to-simplest OPE contribution. It contains first two particle contribution.)}
\]

\[
+ u_2 \otimes u_2 \otimes u_2 \otimes S \text{(Next}^2\text{-to-simplest OPE contribution. It contains one, two and three particles.)}
\]

\[
+ \ldots
\]

\[
C_m(p) = (-1)^m \frac{\Gamma \left( \frac{|m|+1+ip}{2} \right) \Gamma \left( \frac{|m|+1-ip}{2} \right)}{\Gamma(|m|+1)}
\]

\[
F_{E,p}(\tau) = \text{sech}^E(\tau) \binom{E-ip}{2} F_1 \left( \frac{E-ip}{2}, \frac{E+ip}{2}, E, \text{sech}^E(\tau) \right)
\]

\[
\gamma_E(p) = 2g^2 \left( \psi \left( \frac{E+ip}{2} \right) + \psi \left( \frac{E-ip}{2} \right) - 2\psi(1) \right)
\]

where \( \psi = d/dx \log \Gamma(x) \)
Summary of the logic

Data from tree level

\[ = \sum \]

Bottom/top transition amplitude pictures from the WL (or from null correlation functions)

In fact, since we have the Casimir operators at our disposal we can use them instead of the pictures to derive what is flowing

From these we **bootstrap** the full one loop NMHV amplitudes. We also get an infinite amount of all loop predictions.

Dynamics of the propagation can be studied from the GKP string/chain

**Purely 2d approach.** No Feynman diagrams. Only conformal invariance used.
Integrability...

... is not yet being used. It is certainly present at strong coupling where the classical integrability of the string sigma model is used to find the minimal area problem

$$\mathcal{A} \sim e^{-\sqrt{\lambda} \text{Area}}$$

where the Area is given by a Y-system

$$\log Y_s(\theta) = \log (u_s) \cosh(\theta) + i \log (u'_s) \sinh(\theta) + \int d\theta' K_{s,s'}(\theta, \theta') \log(1 + Y_s(\theta'))$$

$$\text{Area} = \sum_s \sum_n \int d\theta' f_{n,s}(\theta') [Y_s(\theta')]^n$$

$$\simeq \int d\theta' e^{-\tau \cosh(\theta) + i \sigma \sinh(\theta)} C(\theta) + \ldots$$

Density of OPE relativistic particles propagating.

Note: This is the interpretation, not the derivation!
So the key is understanding 2 particles. More than 2 should then follow from the factorizability of Integrable models.
One remarkable property of the Super Wilson Loop/Scattering Amplitude duality is that the $k$-particle MHV scattering amplitude at $l$ loops is given, roughly, by a $k^l$ Wilson loop computation. This means that the known tree and one loop data from the scattering amplitude present an excellent window towards higher loop physics from the Wilson loop point of view. In particular, multi-particle flux tube excitations can in principle be studied with the already available data.

The scattering of $N$ particles then factorizes in a sequence of $N-1$ two-body scattering events. From the MHV point of view, the critical missing point of data which, optimistically, should shed light over the full integrability structure is a three loop scattering amplitude. That seems like a very hardcore computation with the current Feynman diagramatic tools.

In this paper we initiate the study of non MHV amplitudes from the OPE point of view. Our main motivation resides on a remarkable feature of the Wilson Loop/Scattering amplitude duality which is that a $k$-particle MHV scattering amplitude at $l$ loops is given by a $k^l$. The simplest of such amplitudes is probably the three loop eight point amplitude in a restricted kinematics regime where all momenta of the external particles lie in the same plane.

Back to data gathering

Purest OPE two-particle data. Main hope: once we understand two particles we should be able to get any number of particles using integrability. They interaction factorizes.
Work in progress

Our best friend is a 2d $N^2$MHV Octagon:
(Thanks Jake and Freddy!)
(The OPE expansion corresponds to small $T$)

We already understand the first terms of the expansion ($T^0$, $T^1$ and $T^2$) in terms of the flux tube expansion :-)

On top of the energy and momentum of the 2 particles, there are infinitely more integrable charges that are encoded in the transfer matrix $T$. We are computing this object at weak coupling. It will be very interesting and illuminating to compare it to the strong coupling transfer matrices (out of which we defined the $Y$-functions).
Conclusions

• The OPE is a simple 2D approach for computing 4D null polygon Wilson loops in CFTs.

• No Feynman diagrams. Valid at any coupling. In particular, seen at strong coupling. [Alday,Gaiotto,Maldacena,Sever,PV]

• Infinite amount of predictions at any loop order:
  [Alday,Maldacena,Sever,PV]

  • Fixes all two loop MHV amplitudes and one loop NMHV amplitudes in N=4 SYM. [Gaiotto,Maldacena,Sever,PV], [Gaiotto,Maldacena,Sever,PV], [Sever,PV], [Sever,PV,Wang]

  • Almost fixes three loop MHV scattering amplitudes [Dixon,Drummond,Henn], [Heslop,Khoze]

  • Important constraint on two loop NMHV scattering amplitudes. [Dixon,Drummond,Henn]

• Related to BFKL high energy scattering [Bartels,Lipatov,Prygarin]

• Very natural formalism to embed Integrability. (We like the OPE sums better than Li2’s)

  Quantum Y-system at any coupling re-summing the OPE expansion?...
Usual OPE

Bottom group of points

Top group of points
Parametrizing Hexagons

3 cross-ratios for the Hexagon:

\[ u_2 = \frac{(x_2 - x_4)^2 (x_1 - x_5)^2}{(x_2 - x_5)^2 (x_1 - x_4)^2}, \]

\[ u_{1,3} = u_2 |_{x_i \rightarrow x_{i\pm 1}} \]

\[
\begin{align*}
  \{k1, k2, k3, k4, k5\} &= \{\{1, 1, 0, 0\}, \{-1, 0, 1, 0\}, \{\sqrt{2}, 1, 0, -1\}, \{\sqrt{3}, 1, 1, -1\}, \{1, 0, 0, 1\}\}; \\
  \{0 = k1 \cdot k1, 0 = k2 \cdot k2, 0 = k3 \cdot k3, 0 = k4 \cdot k4, 0 = k5 \cdot k5\} &= \{\text{True, True, True, True, True}\}; \\
  k6 &= -k1 - k2 - k3 - k4 - k5 \\
  k6 \cdot k6 &= 0 \\
  \{-1 - \sqrt{2} - \sqrt{3}, -3, -2, 1\} 
\end{align*}
\]

Coming up with a three parameter family of momenta for closed null Hexagons is just too boring. We need better variables.
### Parametrizing Hexagons

\[ x_i \in \mathbb{R}^{1,3} \]

\[ x_i \rightarrow X^A = (1, x^2, x'^\mu) \rightarrow X^A(\sigma_A)_{ab} = Z_{[a} W_{b]} \]

**Light-cone directions**

**Momentum Twistors**

Conformal transf. for \( x = \) Lorentz for \( X = \text{SL}(4) \) for \( Z, W \)

\[ (x - y)^2 \propto X \cdot Y \propto \det(Z, W, \tilde{Z}, \tilde{W}) \]

so all we need to do is choose

\[ X_i = Z_i \wedge Z_{i-1} \]

\[ (X_i)^A(\sigma_A)_{ab} = (Z_i)_{[a}(Z_{i-1})_{b]} \]

**Null ray in 6d**

\[ X \in \mathbb{R}^{2,4} \]

\[ X^2 = 0 \]

\[ X \sim \lambda X \]
Parametrizing Hexagons

3 cross-ratios for the Hexagon:

\[
\begin{align*}
  u_2 &= \frac{(x_2 - x_4)^2(x_1 - x_5)^2}{(x_2 - x_5)^2(x_1 - x_4)^2}, \\
  &= \frac{\det(Z_1, Z_2, Z_3, Z_4) \det(Z_6, Z_1, Z_4, Z_5)}{\det(Z_1, Z_2, Z_4, Z_5) \det(Z_6, Z_1, Z_3, Z_4)} \\
  u_{1,3} &= u_2 |_{Z_i \rightarrow Z_i \pm 1}
\end{align*}
\]

\[
Zs = \text{RandomInteger}[\{-500, 500\}, \{6, 4\}]
\]

\[
\{\{442, 280, -213, 231\}, \{-481, -473, -343, 391\}, \{-375, -363, 59, -328\}, \\
\{-86, 318, 306, -498\}, \{-73, 310, 337, -98\}, \{-52, 212, 139, 365\}\}
\]
Finally, the object

\[
\frac{A(Z_i, \eta_i)}{A_{\text{tree-level}}^{\text{MHV}}(Z_i, \eta_i)} = \mathcal{W}^{\text{MHV}}(\eta)^0(Z_i) \left( 1 + R^{\text{NMHV}}(\eta)^4(Z_i) + R^{\text{N}^2\text{MHV}}(\eta)^8(Z_i) + \ldots \right)
\]

\[
R^{\text{NMHV}}(\eta)^4(Z_i) = R^{(2356)}(Z_i) \eta_2^A \eta_3^B \eta_5^C \eta_6^D \epsilon_{ABCD} + R^{(2256)}(Z_i) \eta_2^A \eta_2^B \eta_5^C \eta_6^D \epsilon_{ABCD} + \ldots
\]

At tree level:

\[\eta_2 \eta_3 \eta_5 \eta_6 + \eta_2 \eta_2 \eta_5 \eta_6 + \ldots\]
First round of facts

\[ \mathcal{R}_{\text{tree-level}}^{(2356)} = \frac{1}{\det(Z_2 Z_3 Z_5 Z_6)} \]

From the 2d approach the tree level result is the seed...

\[ \mathcal{R}_{\text{one-loop}}^{(2356)} = \frac{1}{\det(Z_2 Z_3 Z_5 Z_6)} \left( \log(u_2) \log \left( \frac{u_1 u_3}{1 - u_2} \right) + \left[ - \log(u_1) \log(u_3) - \text{Li}_2(u_2) - \text{Li}_2(1 - u_1) - \text{Li}_2(1 - u_3) - \frac{\pi^2}{3} \right] \right) \]

\[ \mathcal{R}_{\text{one-loop}}^{(2356)} = \log(u_2) D_2 + \tilde{D}_2 \]
\[ = \log(u_1) D_1 + \tilde{D}_1 \]
\[ = \log(u_3) D_3 + \tilde{D}_3 \]

... while the discontinuities \( D_a \) are trivially computed given the seed.
Reference square

Without loss of generality

\[
\begin{pmatrix}
Z_{\text{left}} \\
Z_{\text{top}} \\
Z_{\text{right}} \\
Z_{\text{bottom}}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

so the symmetries of a null reference square are \( \mathbb{R}_\tau \times \mathbb{R}_\sigma \times SO(2)_\phi \)

\[
M(\tau, \sigma, \phi) = \begin{pmatrix}
e^{\sigma+\phi/2} & e^{-\sigma+\phi/2} \\
e^{\tau-\phi/2} & e^{-\tau-\phi/2}
\end{pmatrix}
\]
Example

\[ \mathcal{R}_{\text{tree-level}}^{(2356)} = \frac{1}{\det(Z_2, Z_3, Z_5, Z_6)} = \frac{1}{\cosh(\tau) \cosh(\sigma) + \cos(\phi)} \]
Back to geometry for a second.

The symmetry of two null lines (before it was the square)

Without loss of generality

\[
\begin{pmatrix}
Z_{\text{left}} \\
Z_{\text{top}} \\
Z_{\text{right}} \\
Z_{\text{bottom}}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

so the symmetries of the two null lines are \( SL(2)_\tau \times \mathbb{R}_\sigma \times SO(2)_\phi \)

\[
g_{SL(2)} =
\begin{pmatrix}
1 & 1 \\
\cdot & \cdot \\
\cdot & \cdot
\end{pmatrix}
\]

We will see that the Casimir of this \( SL(2) \) measures what is flowing from the top to the bottom.

\[
Z_{\text{left}} \wedge (tZ_{\text{top}} + (1 - t)Z_{\text{bottom}})
\]
Ward identities

- No cyclicity
- $R_{\text{one loop}}^{(2356)}$ have discontinuities in $u_1, u_3$ as well
- Other components are very similar in terms of $C$ and $F$
Ward identities

- No cyclicity
- \( R^{(2356)} \) have discontinuities in \( u_1, u_3 \) as well
- Other components are very similar in terms of \( \mathcal{C} \) and \( \mathcal{F} \)

\[ \Rightarrow \text{SUSY ward identities} \]
Ward identities

- No cyclicity
- $\mathcal{R}^{(2356)}_{\text{one loop}}$ have discontinuities in $u_1, u_3$ as well
- Other components are very similar in terms of $\mathcal{C}$ and $\mathcal{F}$

$\Rightarrow$ SUSY ward identities

$\Rightarrow$ Everything in the $u_2$ channel $\Rightarrow$ Everything in the $u_{1,3}$ channels

$$\frac{D_1^{(2356)}}{\mathcal{R}^{(2356)}} = \left( \frac{D_2^{(1245)}}{\mathcal{R}^{(1245)}} \right)_{u_i \rightarrow u_{i-1}}$$
Bootstrapping the full 1-loop amplitude

- We ``like'' the sums better (TBA)
- Systematic way is using symbols
Bootstrapping the full 1-loop amplitude

- We `like” the sums better (TBA)
- Systematic way is using symbols

\[
D_{u_2}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_1 + \log u_3 - \log(1 - u_2) \right]
\]
\[
D_{u_{1/3}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 - \log u_{3/1} - \log(1 - u_{1/3}) \right].
\]
Bootstrapping the full 1-loop amplitude

- We ``like'' the sums better (TBA)
- Systematic way is using symbols

\[
D_{u_2}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_1 + \log u_3 - \log(1 - u_2) \right]
\]
\[
D_{u_{1/3}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 - \log u_{3/1} - \log(1 - u_{1/3}) \right].
\]

\[\Rightarrow \mathcal{R}_{1\text{-loop}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log(u_2) \log(u_1u_3) - \begin{cases} 
\log(u_2) \log(1 - u_2) \\
\text{or} \\
\text{Li}_2(1 - u_2)
\end{cases} \right] + \ldots\]
Bootstrapping the full 1-loop amplitude

- We `like" the sums better (TBA)
- Systematic way is using symbols

\[ D^{(2356)}_{u_2} = 2 \mathcal{R}^{(2356)}_{\text{tree}} \left[ \log u_1 + \log u_3 - \log(1 - u_2) \right] \]

\[ D^{(2356)}_{u_{1/3}} = 2 \mathcal{R}^{(2356)}_{\text{tree}} \left[ \log u_2 - \log u_{3/1} - \log(1 - u_{1/3}) \right]. \]

\[ \Rightarrow \mathcal{R}^{(2356)}_{1-\text{loop}} = 2 \mathcal{R}^{(2356)}_{\text{tree}} \left[ \log(u_2) \log(u_1 u_3) - \left\{ \begin{array}{l}
\log(u_2) \log(1 - u_2) \\
\text{or} \\
\text{Li}_2(1 - u_2)
\end{array} \right\} \right] + \ldots \]
**Bootstrapping the full 1-loop amplitude**

- We "like" the sums better (TBA)
- Systematic way is using symbols

\[
\begin{align*}
D_{u_2}^{(2356)} &= 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_1 + \log u_3 - \log(1 - u_2) \right] \\
D_{u_1/3}^{(2356)} &= 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 - \log u_{3/1} - \log(1 - u_{1/3}) \right].
\end{align*}
\]

No cut at \( u_2 = 1 \) on the Euclidian sheet

\[
\Rightarrow \mathcal{R}_{1\text{-loop}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log(u_2) \log(u_1 u_3) - \begin{cases} 
\log(u_2) \log(1 - u_2) \\
\text{or} \\
\text{Li}_2(1 - u_2)
\end{cases} \right] + \ldots
\]

\[
\Rightarrow \mathcal{R}_{\text{one loop}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 \log(u_1 u_3) - \log u_1 \log u_3 - \sum_{a=1}^{3} \text{Li}_2(1 - u_a) + c \right].
\]
Bootstrapping the full 1-loop amplitude

- We \``like\'' the sums better (TBA)
- Systematic way is using symbols

\[
D_{u_2}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_1 + \log u_3 - \log(1 - u_2) \right]
\]
\[
D_{u_{1/3}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 - \log u_{3/1} - \log(1 - u_{1/3}) \right].
\]

\[\Rightarrow \mathcal{R}_{1\text{-loop}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log(u_2) \log(u_1 u_3) - \left\{ \log(u_2) \log(1 - u_2) \right\} \right] + \ldots\]

\[\Rightarrow \mathcal{R}_{\text{one loop}}^{(2356)} = 2 \mathcal{R}_{\text{tree}}^{(2356)} \left[ \log u_2 \log(u_1 u_3) - \log u_1 \log u_3 - \sum_{a=1}^{3} \text{Li}_2(1 - u_a) + c \right].\]

No cut at \(u_2 = 1\) on the Euclidian sheet

Collinear limits

\[c = -\pi^2 / 3\]