Scattering Amplitudes

and the

Positive Grassmannian

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\+ P. Deligne
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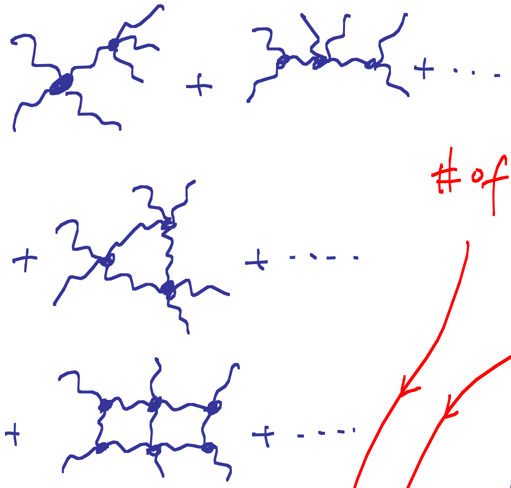
M. Goresky

Goal: A deeper understanding of the relationship between Grassmannians and ($\mathcal{N}=4$ SYM) Scattering Amplitudes.

Amplitudes



# of particles

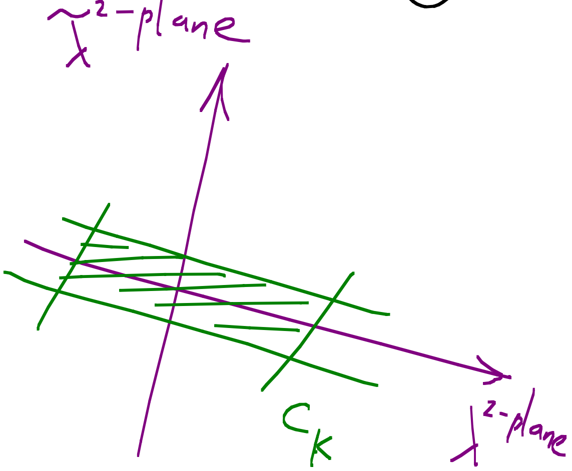
#_ hel. gluons

$$\longrightarrow \mathcal{M}_{n,k}[\lambda,\tilde{\lambda},\tilde{\eta}]$$

Manifestly Local, Unitary

Horrendously Complicated

Grassmannian Integral



$$\int \frac{d^{k\times n}C_k}{(12\cdots k)\cdots(n1\cdots k-1)}/GL(k)$$

Non-manifest locality, Unit

Simple

Yangian Invariance Manifest

Grassmannian $G(k,n)$: k- planes in n-dimensions.

- $$C_{\alpha a} = \begin{array}{c} \xleftrightarrow{\quad n \quad} \\ \left[\begin{array}{c} (\qquad\qquad\qquad) \\ (\qquad\qquad\qquad) \\ (\qquad\qquad\qquad) \\ \vdots \\ (\qquad\qquad\qquad) \end{array}\right] \end{array} \updownarrow k \; , \quad C_{\alpha a} \sim L_\alpha^\beta C_{\beta a}$$

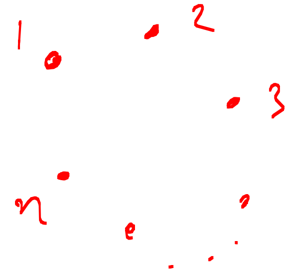
  GL(K) redund.

- $(m_1 \cdots m_k) = \epsilon^{\alpha_1 \cdots \alpha_k} C_{\alpha_1 m_1} \cdots C_{\alpha_k m_k} =$ "minor"

- $C_{\alpha a} = \left[\vec{c}_1 \; \vec{c}_2 \cdots \vec{c}_n \right]$ Little group $c_i \to t_i c_i$

  each $c_i \in P^{k-1}$

So, $C_{\alpha a} \sim$



Space of points in $P^{k-1} / GL(K)$

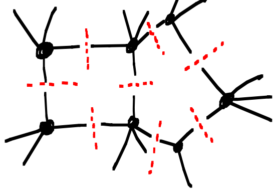
$$\int \frac{d^{k(n-k)} C_{\alpha a}}{C(12\cdots k)\cdots C(n1\cdots k\text{-}1)} \prod_{\alpha=1}^{k} \delta^{4|4}[C_{\alpha a} W_a] \longleftrightarrow$$

All-loop planar integrand in manifestly Yangian Invariant form.

$\Big\downarrow$ Residues

Grassmannian Residues ⟷

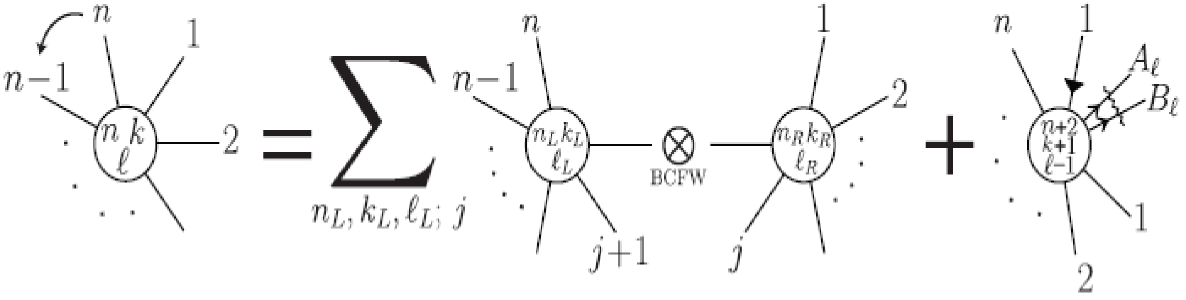
"Leading Singularities"



Generate all Yangian invariants

+ Relations between them ⟷ Locality, Unitarity

All-Loop Recursion



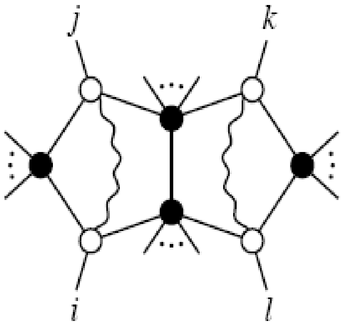
"Classical"

"Quantum"

Complete definition with Yangian symmetry manifest.

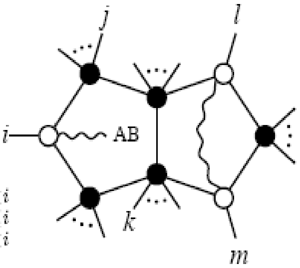
The words "spacetime", "Lagrangian", "Path Integral", "Gauge Symmetry" .... make no appearance.

$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$



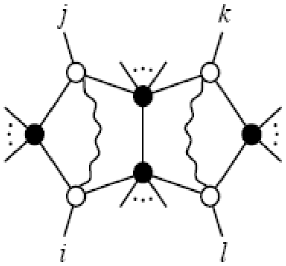
← Momentum Twistor Integrals

$$\mathcal{A}^{2-\text{loop}}_{\text{NMHV}} = \sum_{\substack{i<j<l<m\le k<i\\ i<j<k<l<m\le i\\ i\le l<m\le j<k<i}}$$



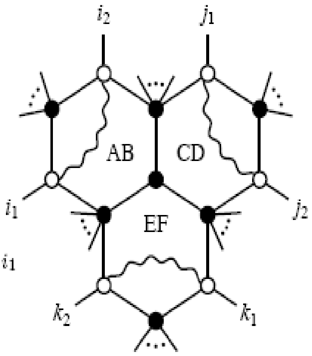
$$\times\,[i, j, j+1, k, k+1]$$

$$+\frac{1}{2}\sum_{i<j<k<l<i}$$

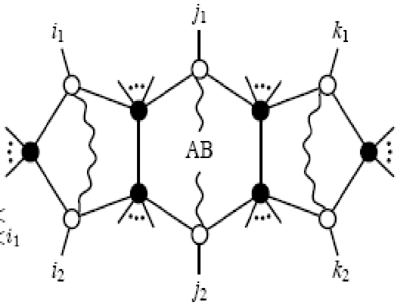


$$\times\left\{\begin{matrix}\mathcal{A}^{\text{tree}}_{\text{NMHV}}(j,\ldots,k;\,l,\ldots,i)\\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(i,\ldots,j)\\ +\mathcal{A}^{\text{tree}}_{\text{NMHV}}(k,\ldots,l)\end{matrix}\right\}$$

$$\mathcal{A}^{3-\text{loop}}_{\text{MHV}} = \frac{1}{3}\sum_{\substack{i_1\le i_2<j_1\le\\ \le j_2<k_1\le k_2<i_1}}$$



$$+\frac{1}{2}\sum_{\substack{i_1\le j_1<k_1<\\ <k_2\le j_2<i_2<i_1}}$$



The "Positive Part" of the Grassmannian

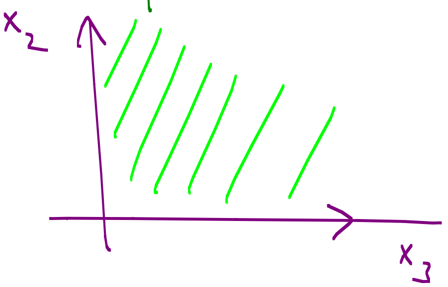
[c.f. Lustig, Postnikov et. al., Fock + Goncharov, Knutson, Lam, Speyer]

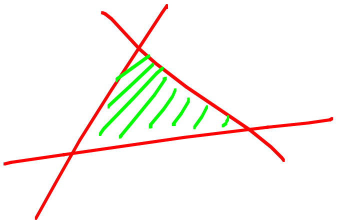
- Generalize Simplex in $P^{n-1} = G(1,n)$:

Choose co-ordinates

$X = (x_1\ x_2\ x_3)$

"positive part" $x_i > 0$ → Closure is Simplex (123)





- Denote by (123..n), boundaries just put diff. $x_i \to 0$, $\partial(123..n) = (23..n) - (13..n) + ..(12..n-1)$ is just "deletion".

For $G(k,n)$:

$$C = \overset{\longleftarrow\; n \;\longrightarrow}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} \; \overset{\uparrow}{\underset{\downarrow}{k}} \qquad n \text{ } k\text{-vectors.}$$

"Positive Part": $(c_{i_1} \cdots c_{i_k}) > 0$ for $i_k > \cdots > i_1$.

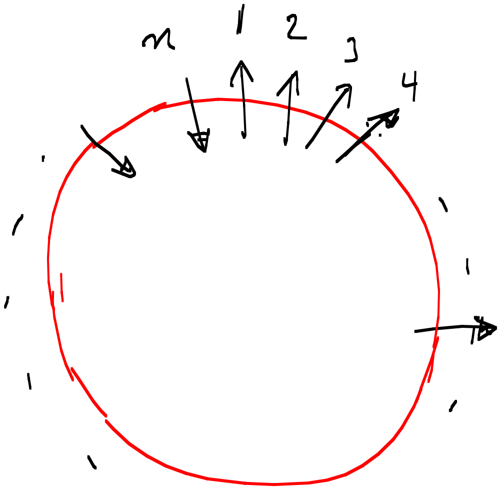
["All minors positive"].

Note: (twisted) cyclic structure

$$c_1 \to c_2,\; c_2 \to c_3,\; \ldots,\; c_n \to (-1)^{k+1} c_1$$

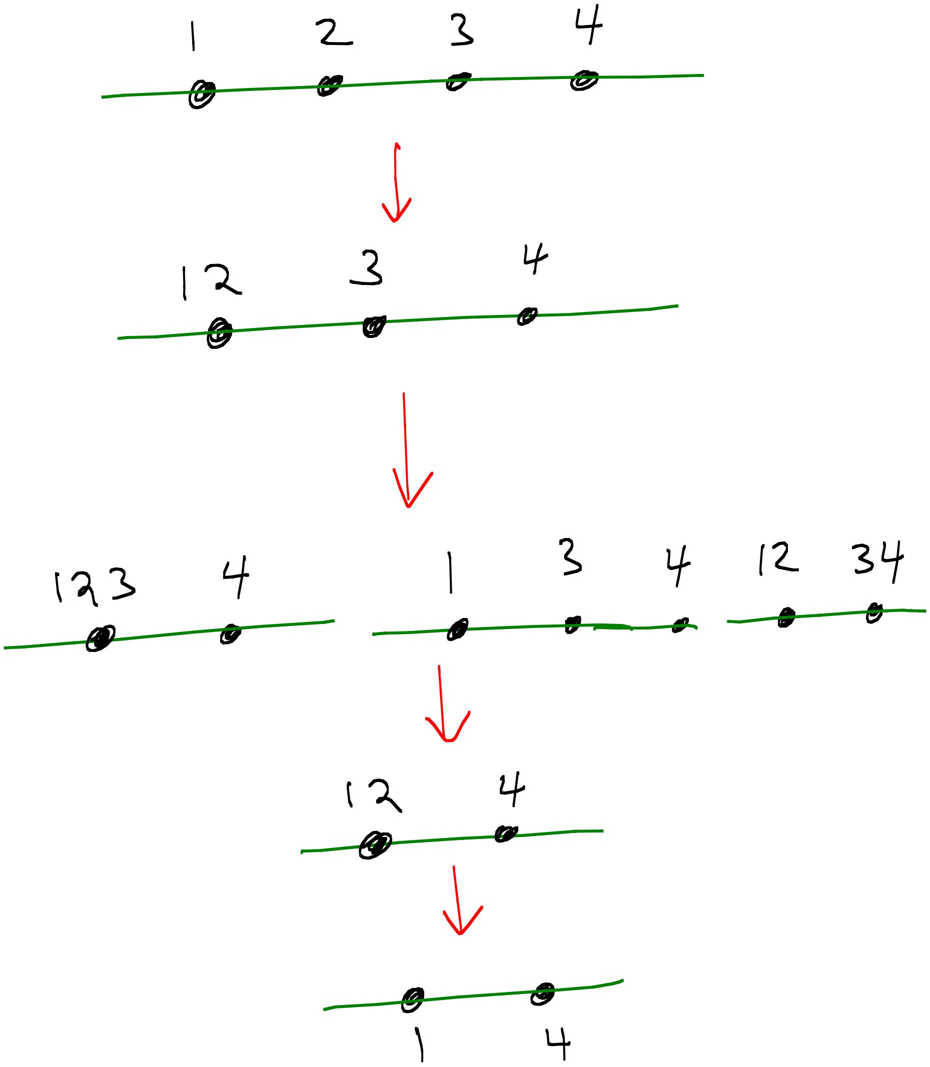
- In e.g. $G(2,n)$:

$C = (c_1 \cdots c_n)$, pos. part $(ij) > 0$ for $j > i$



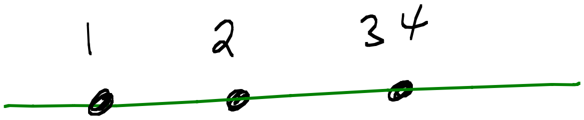
Closure vastly richer than a simplex!

"Grassmannian Polytope"

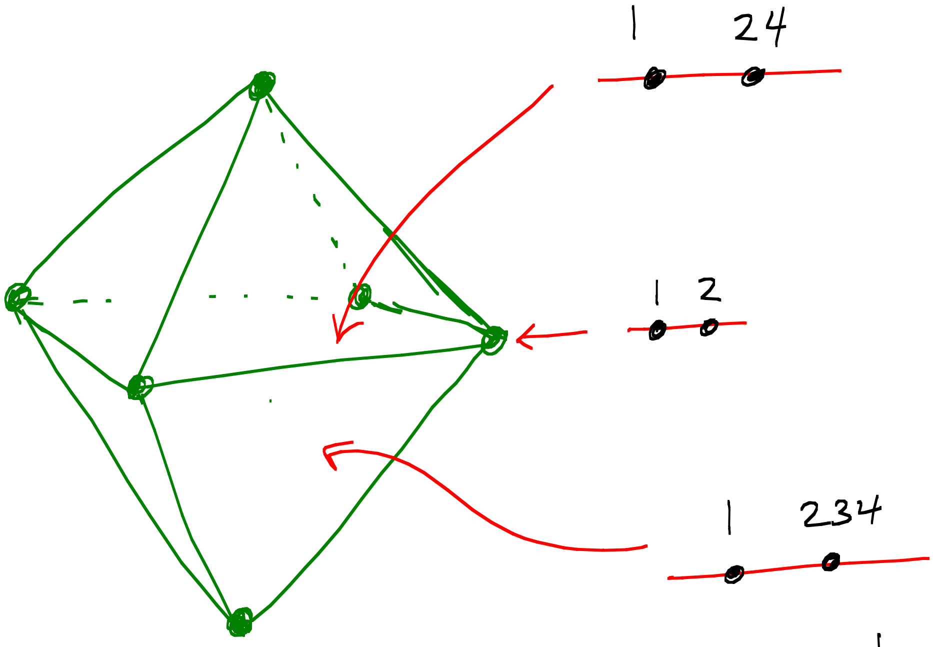


∂ operator is merge + delete

[not just delete!]

- 

: 3-dim face

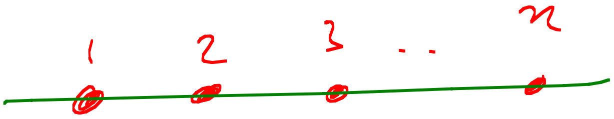


- [Whole 4d Polytope is topologically a ball].

Note: without being put in by hand, boundaries associated with linear dependencies between 2 consecutive vectors!

[Loosely, with "$(i\, i+1) \to 0$"].

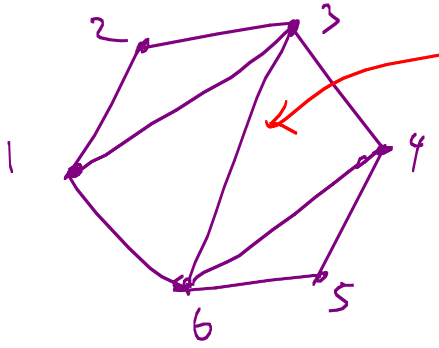
- A variety of nice "positive co-ordinates":
- Most obvious "polygon co-ordinates"



$$C = \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \;\; \underbrace{y_3 \begin{pmatrix} x_3 \\ 1 \end{pmatrix}}_{u_3} \;\; \underbrace{y_4 \begin{pmatrix} x_3 + x_4 \\ 1 \end{pmatrix}}_{u_4} \cdots \right)$$

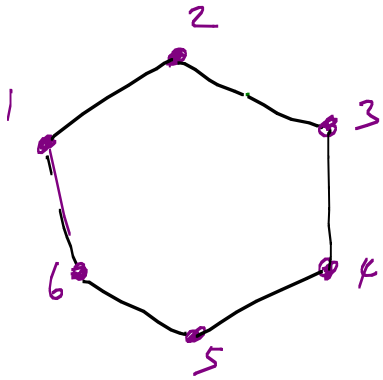
[Generalizes to all $G(k,n)$].

- For $G(2,n)$ – "Fock–Goncharov" co-ordinates



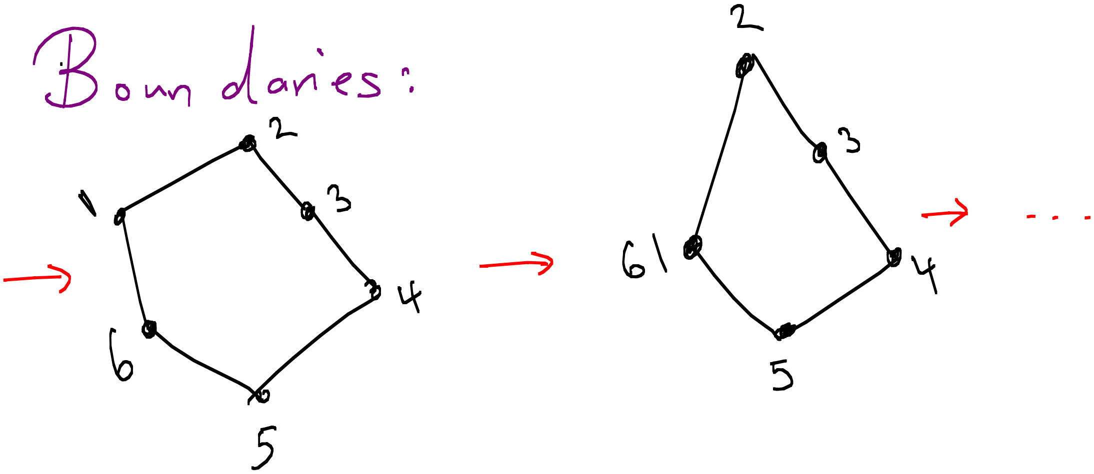
$(36) > 0, \ldots$
guarantees all minors positive

For $G(3,n)$: $(i_1 i_2 i_3) > 0$



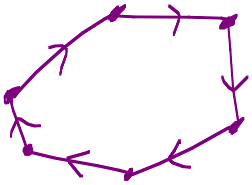
→ Convex Polygon

Boundaries:



Again: boundaries associated with linear dependencies between 3 consecutive vectors! [Loosely, "$(i\ i{+}1\ i{+}2) \to 0$"].

Nice "polygon co-ordinates" here too :



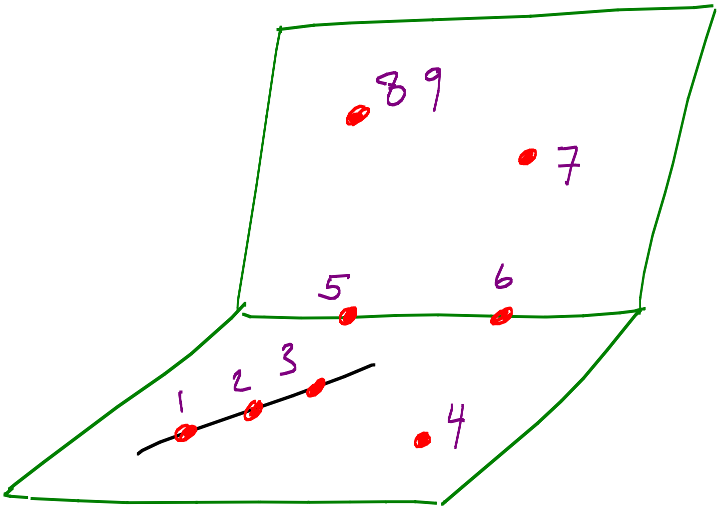
Note $(\vec{c}_{i+1} - \vec{c}_i)_{\text{proj.}}$ are $G(2,n)$ co-ordinates!

$$C_{3,n} = \left[\begin{matrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & z_4\begin{pmatrix}\boxed{u_4}\\ 1\end{pmatrix} & z_5\begin{pmatrix}\boxed{u_4}+\boxed{u_5}\\ 1\end{pmatrix} & \cdots \\ 0 & 0 & 1 & & & \end{matrix}\right]$$

$u_{4,5,\ldots}$ are the $G(2,n)$ polygon co-ordinates.

Simply generalizes for all k, n

Classifying All Configurations



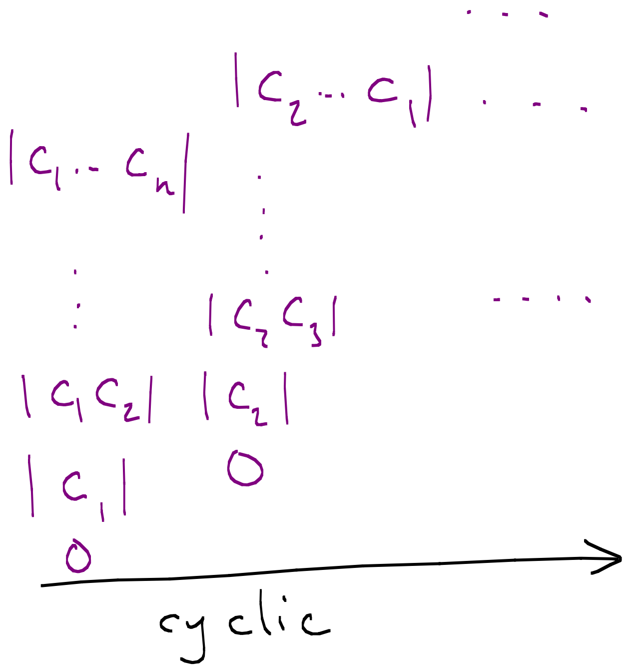
example in $G(4,9)$ .....

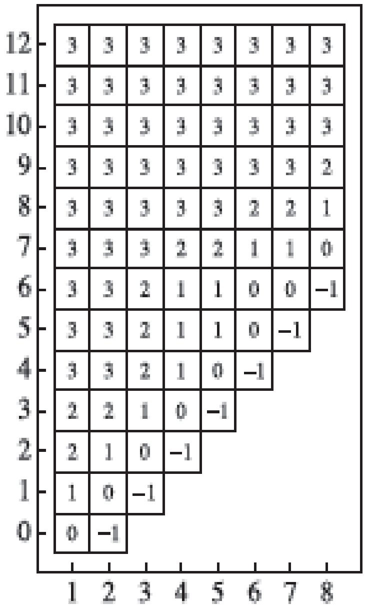
clearly need to be systematic!

{Completely Solved by Postnikov...}

Deligne's Notation

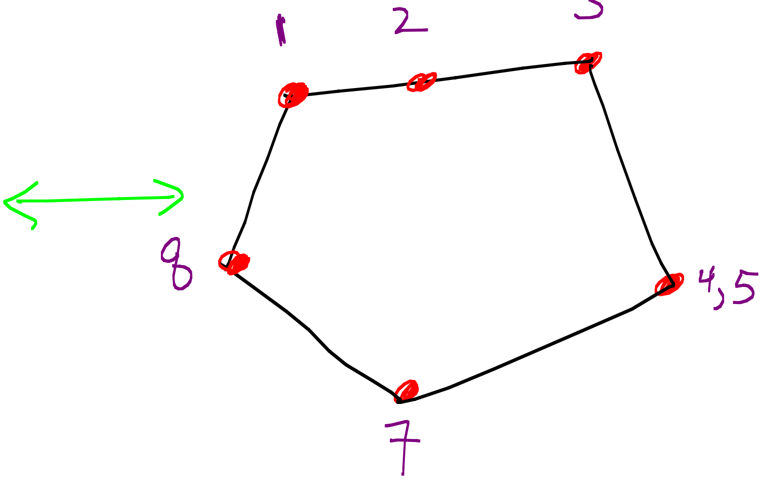
Given some configuration $(\vec{c}_1, \dots, \vec{c}_n)$ of k-vectors, draw the table



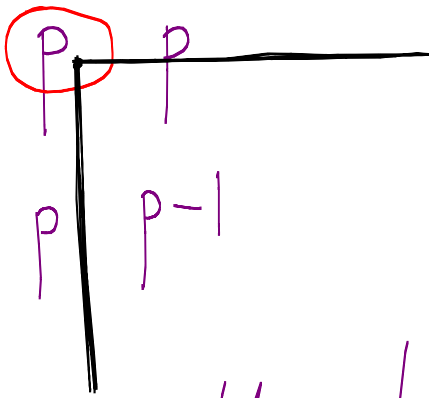


12	3	3	3	3	3	3	3	3
11	3	3	3	3	3	3	3	3
10	3	3	3	3	3	3	3	3
9	3	3	3	3	3	3	3	2
8	3	3	3	3	3	2	2	1
7	3	3	3	2	2	1	1	0
6	3	3	2	1	1	0	0	−1
5	3	3	2	1	1	0	−1	
4	3	3	2	1	0	−1		
3	2	2	1	0	−1			
2	2	1	0	−1				
1	1	0	−1					
0	0	−1						
	1	2	3	4	5	6	7	8

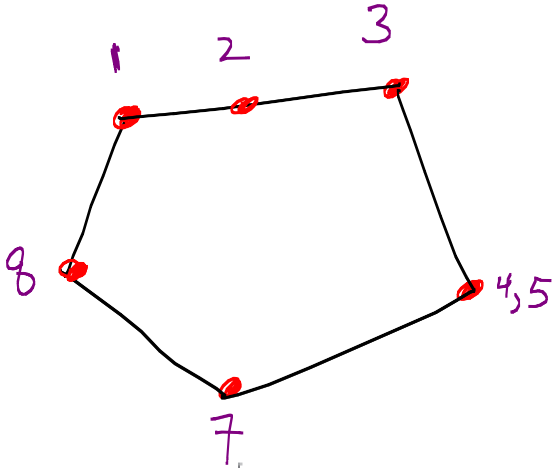
=



In any column, there is a unique spot that locally looks like

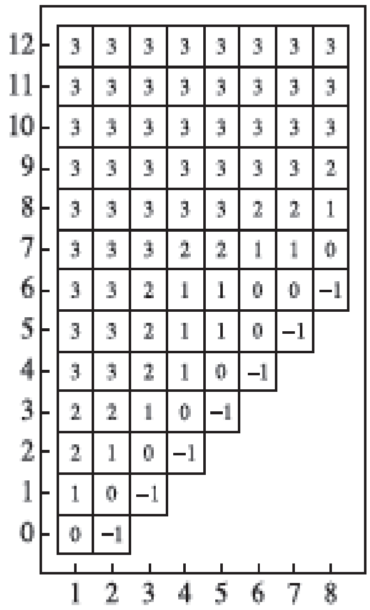


"Hook" diagram Uniquely labels configuration!



1 → 3
2 → 7
3 → 8
4 → 5
5 → 1 + 8
6 → 6
7 → 2 + 8
8 → 4 + 8

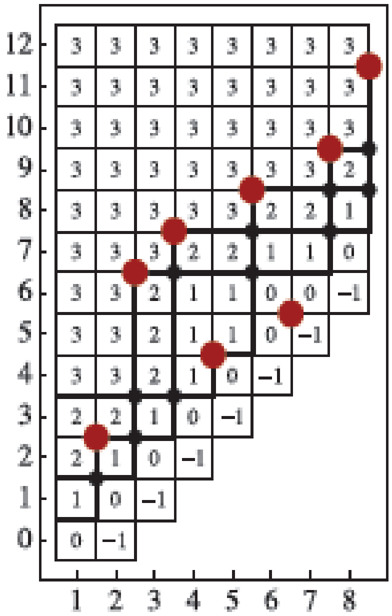
"Affine Permutation"



12	3	3	3	3	3	3	3	3
11	3	3	3	3	3	3	3	3
10	3	3	3	3	3	3	3	3
9	3	3	3	3	3	3	3	2
8	3	3	3	3	3	2	2	1
7	3	3	3	2	2	1	1	0
6	3	3	2	1	1	0	0	−1
5	3	3	2	1	1	0	−1	
4	3	3	2	1	0	−1		
3	2	2	1	0	−1			
2	2	1	0	−1				
1	1	0	−1					
0	0	−1						
	1	2	3	4	5	6	7	8

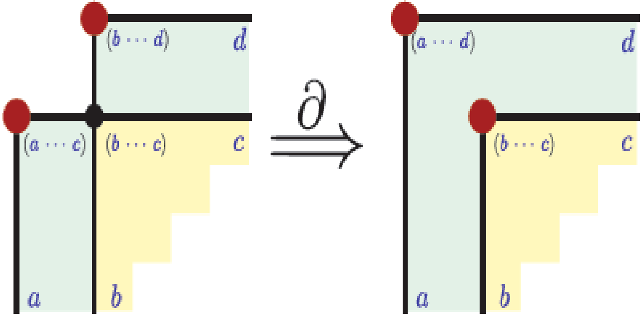
Avg height of ●'s is k

dim. of config = # ●'s + ●'s above "diagonal"



Labeled by f_i = height of red dot in column i.

The Boundary Operator



$[\partial^2 = 0]$

Polytope for $G(5,10)$:

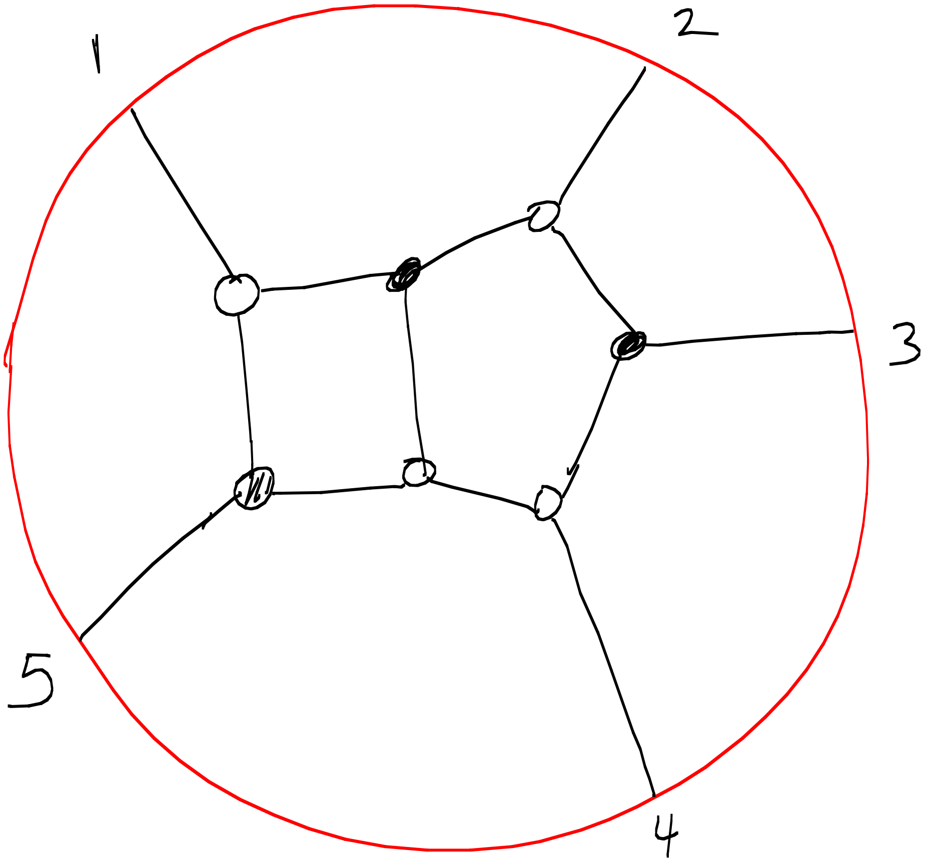
{Amazingly, combinatorially a ball!}

Face-Lattice of Contours for $\mathcal{L}_{10,5}$

25-planes	1	10	24-planes
23-planes	55	220	22-planes
21-planes	715	2002	20-planes
19-planes	4985	11240	18-planes
17-planes	23210	44220	16-planes
15-planes	78087	128100	14-planes
13-planes	195315	276450	12-planes
11-planes	362175	437112	10-planes
9-planes	482670	482940	8-planes
7-planes	432060	339360	6-planes
5-planes	228102	126420	4-planes
3-planes	54600	16800	2-planes
1-planes	3150	252	0-planes

Total: − 1865125 + 1865126 = 1

"Plabic Graphs"

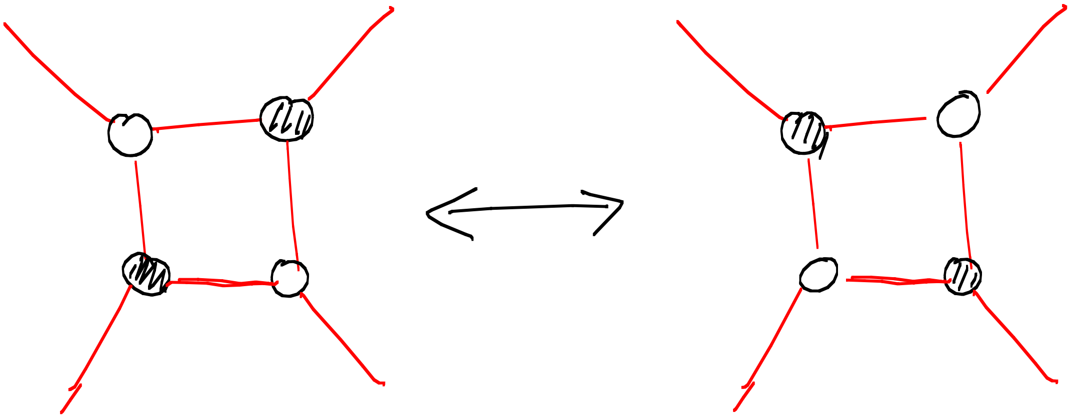


= LS diagrams

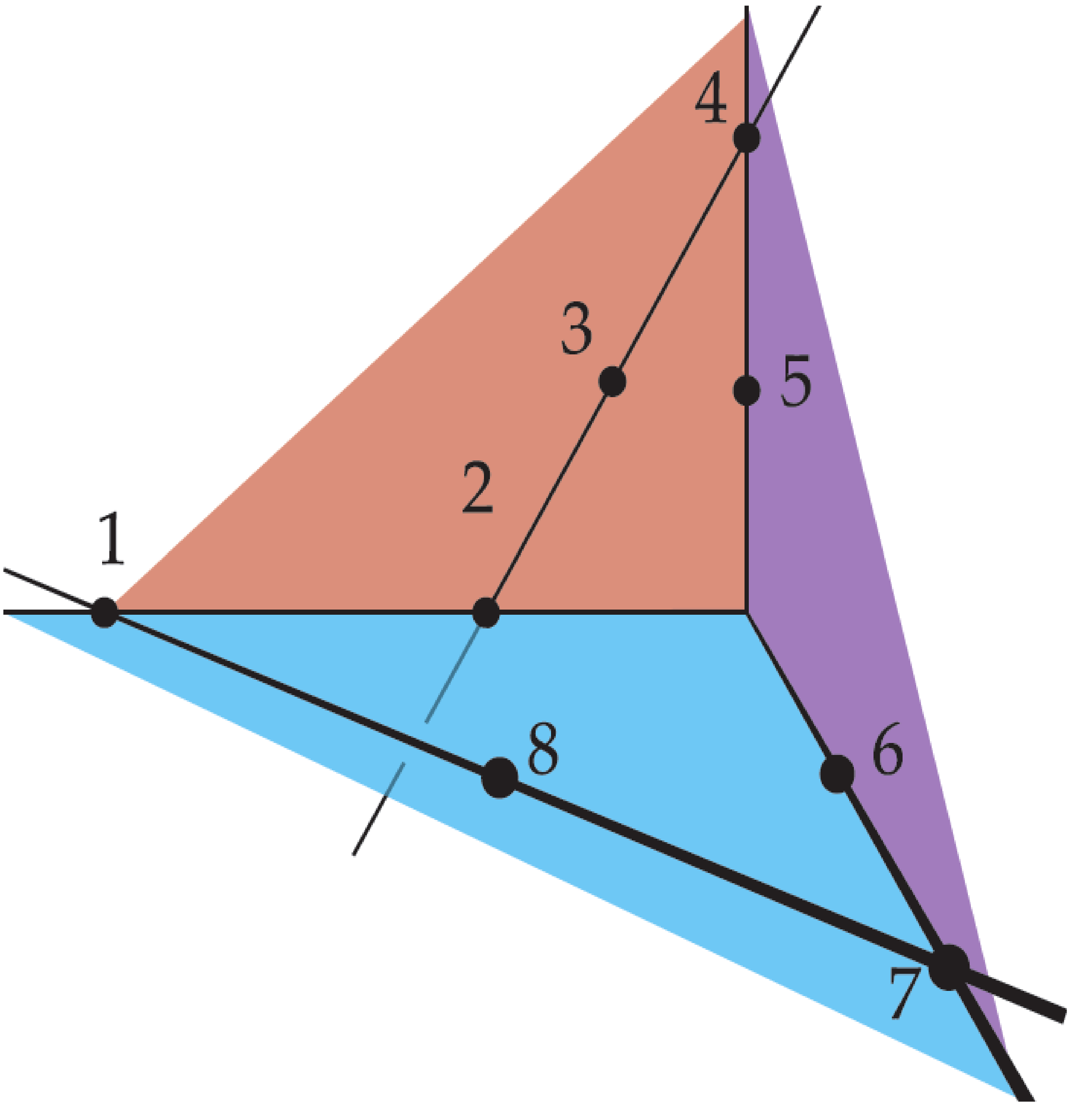
= Hodges diagrams / "Link rep"

!

...Many representatives...



"Cluster Transformation" ↔ "Square Identity" of twistor diagrams


1
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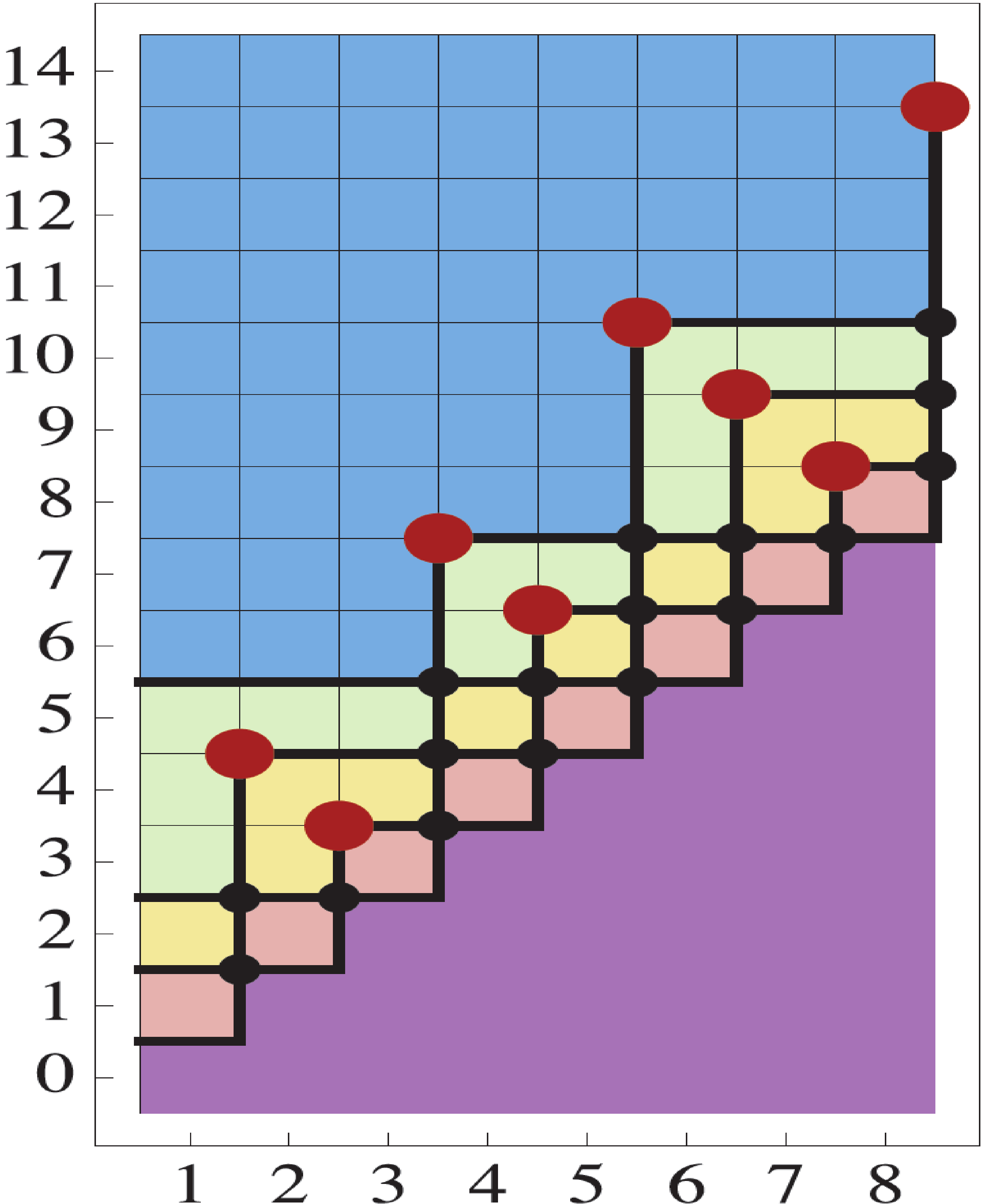
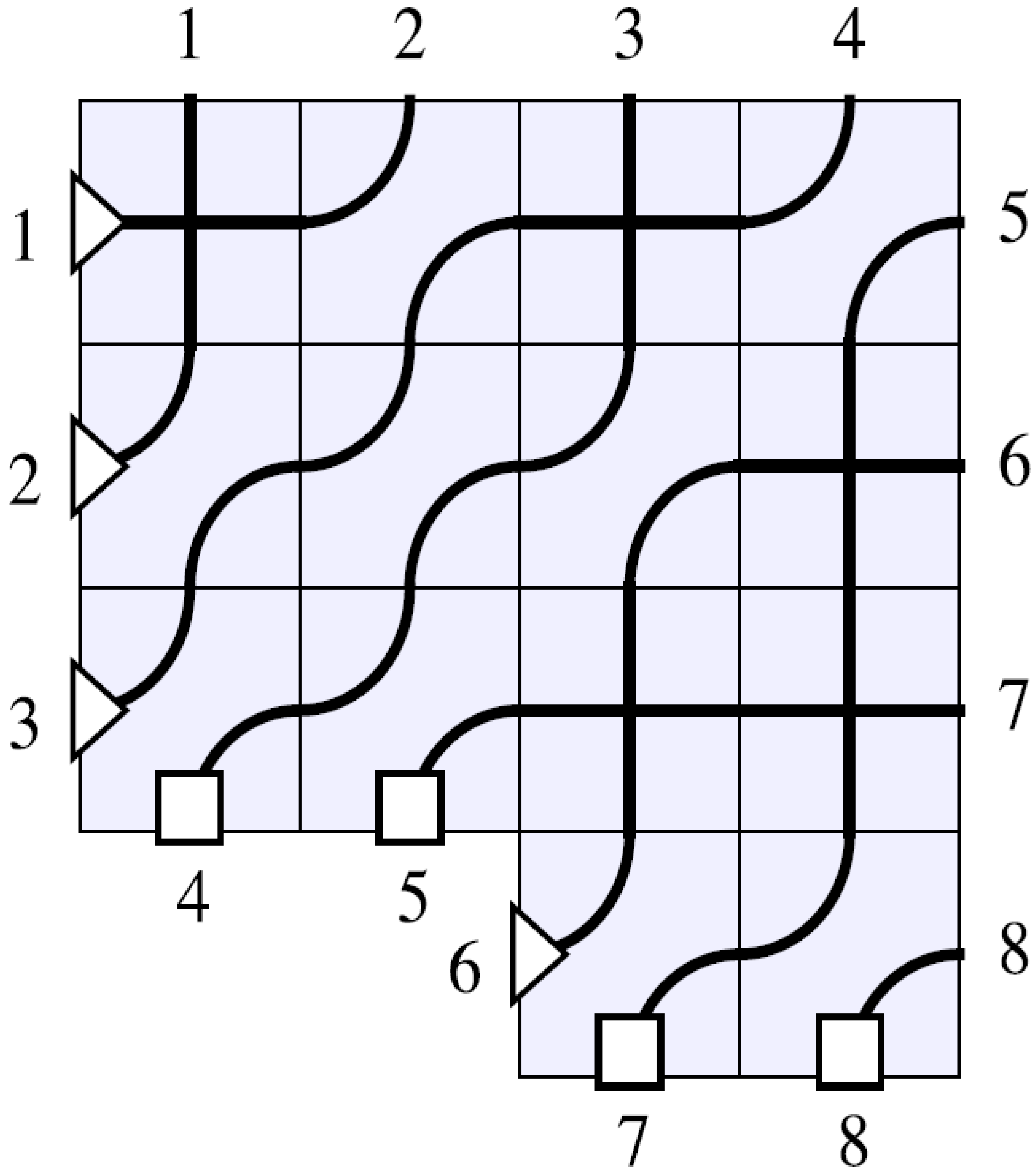


[5 4 8 7 3+8 2+8 1+8 6+8]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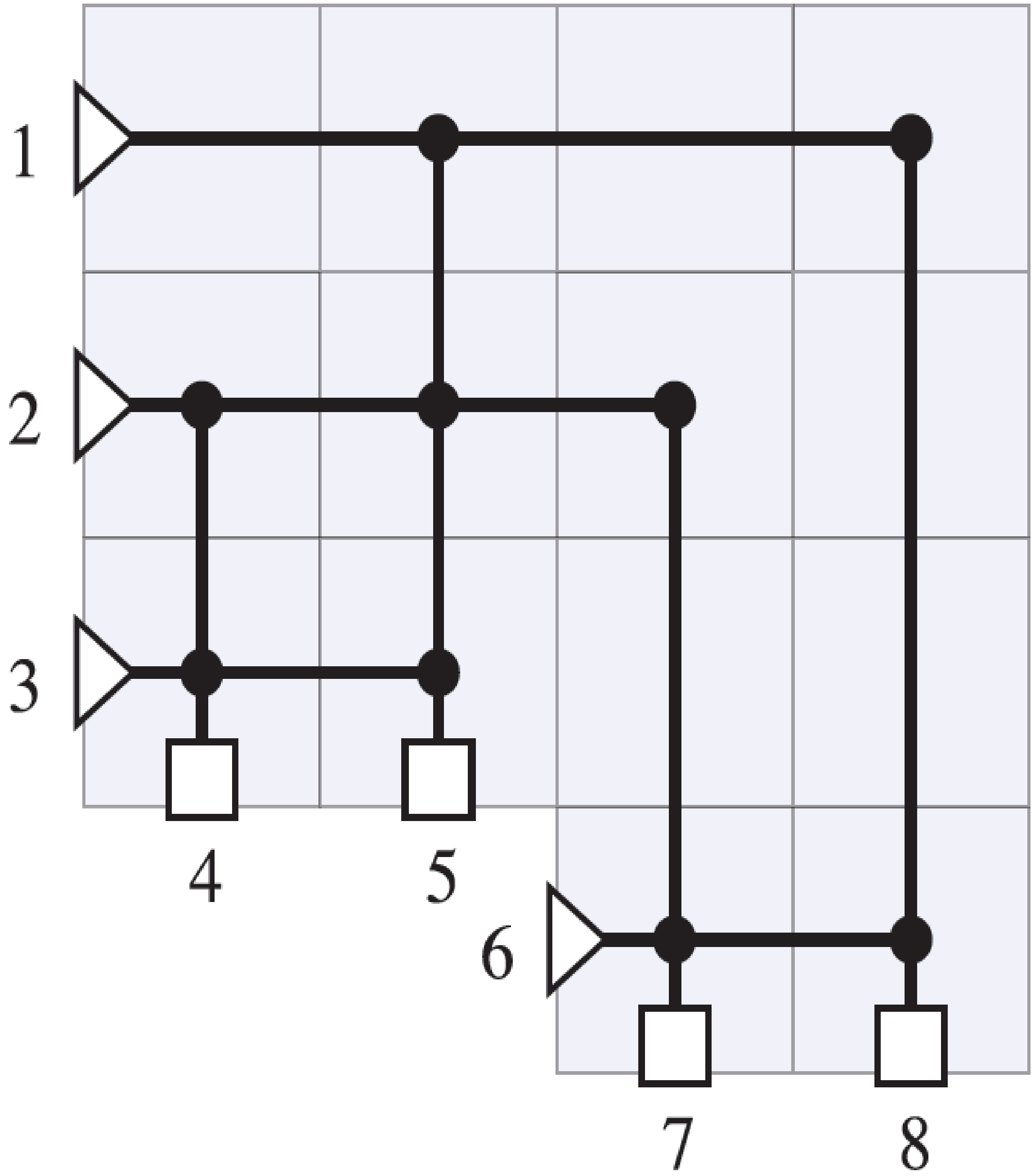
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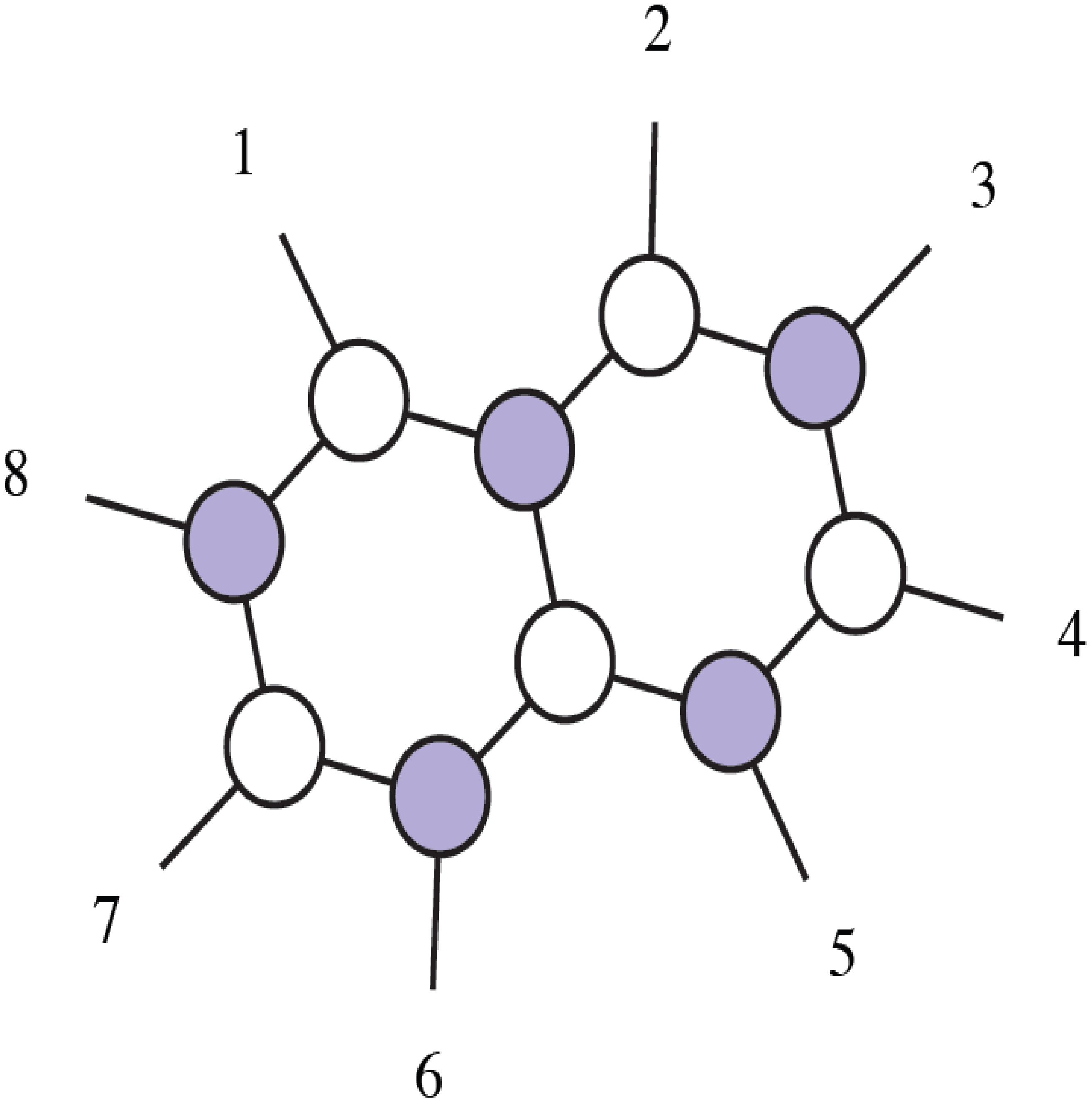
2
1
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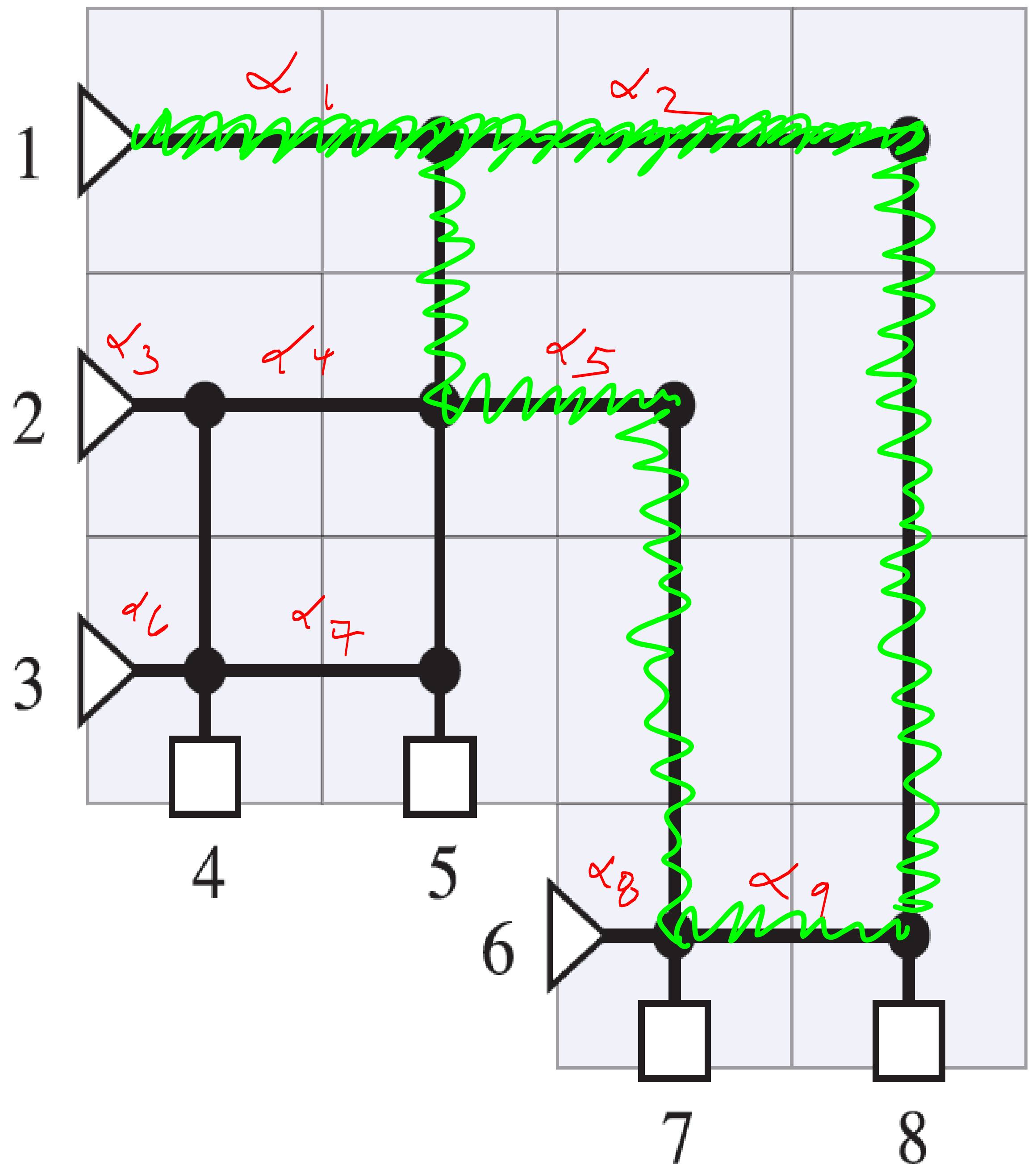
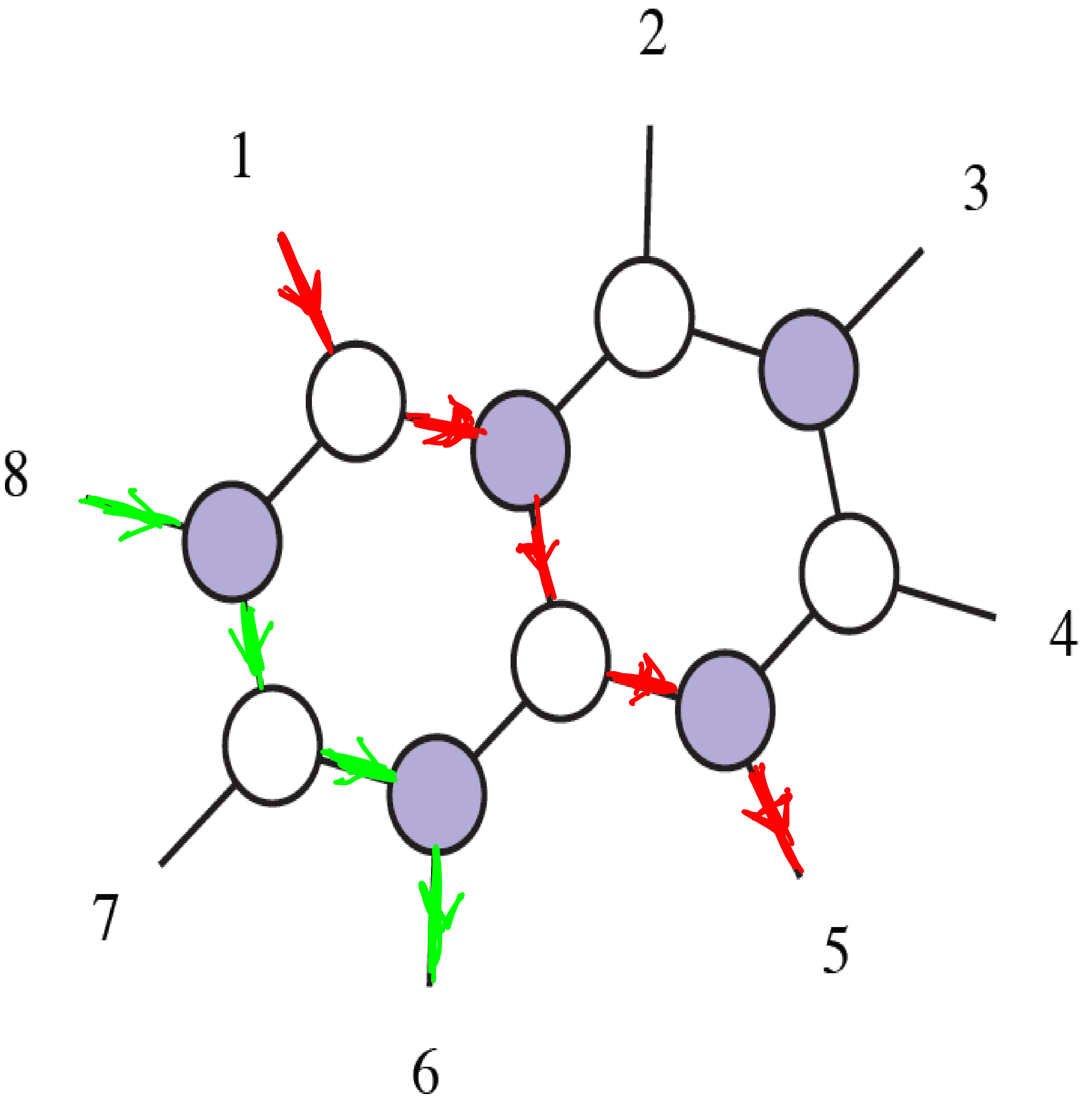
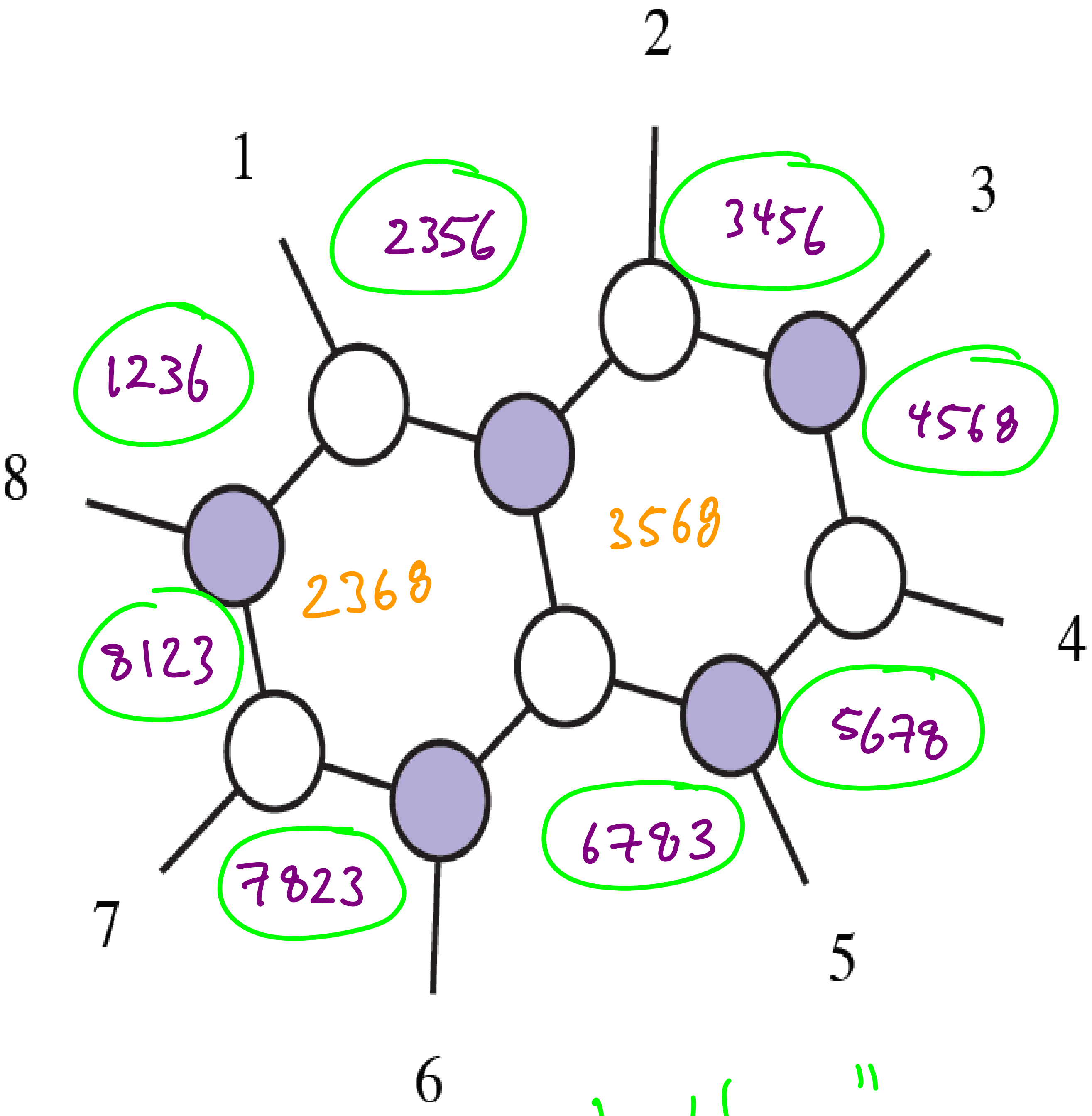


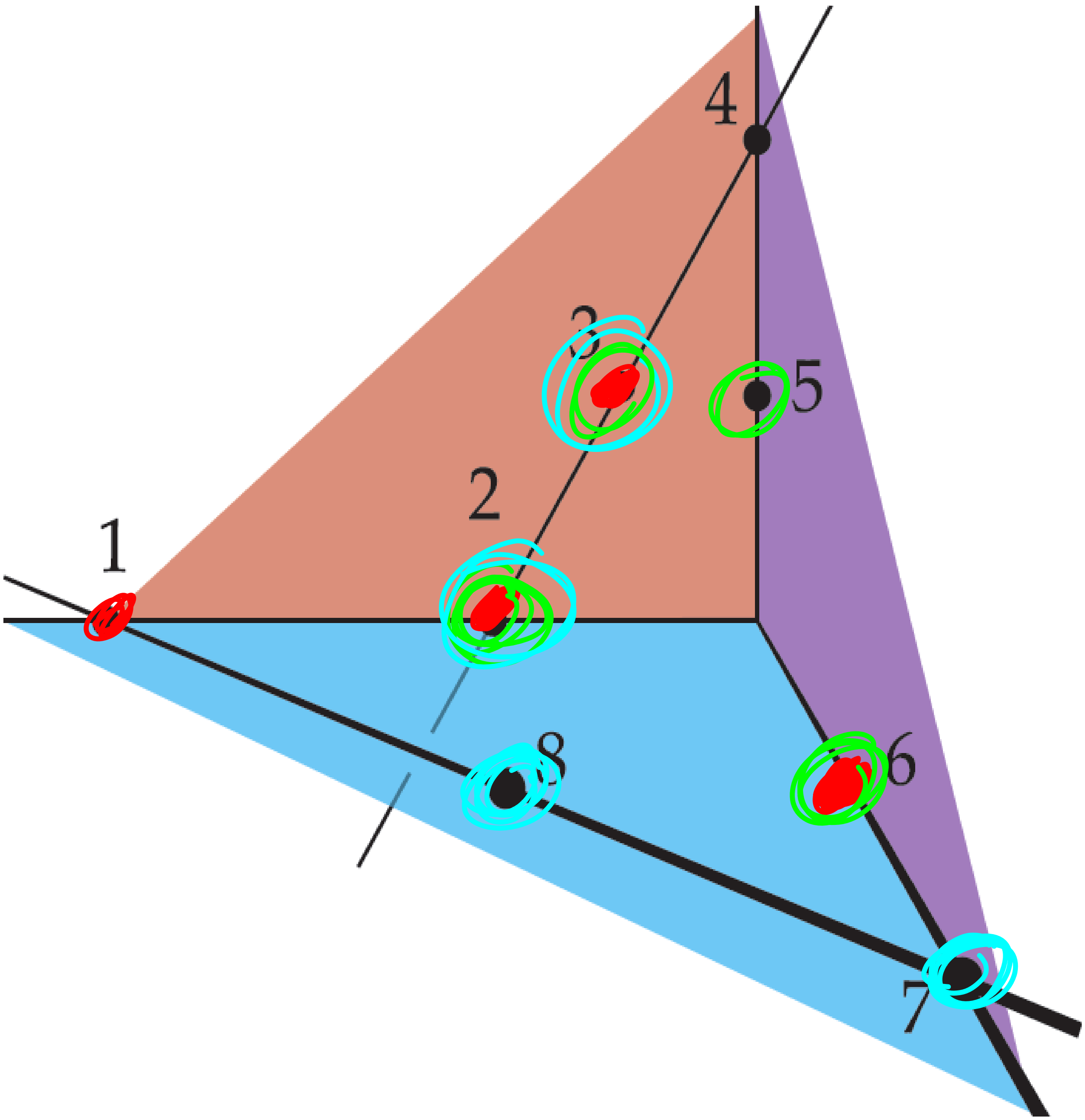
$$C_{iJ} = \sum_{\text{paths}} \prod \text{edge variables} \, !$$



1→5, 2→4, 3→8, 4→7, 5→3+8, 6→2+8, 7→1+8, 8→6+8



"Grassmann Necklace"


4
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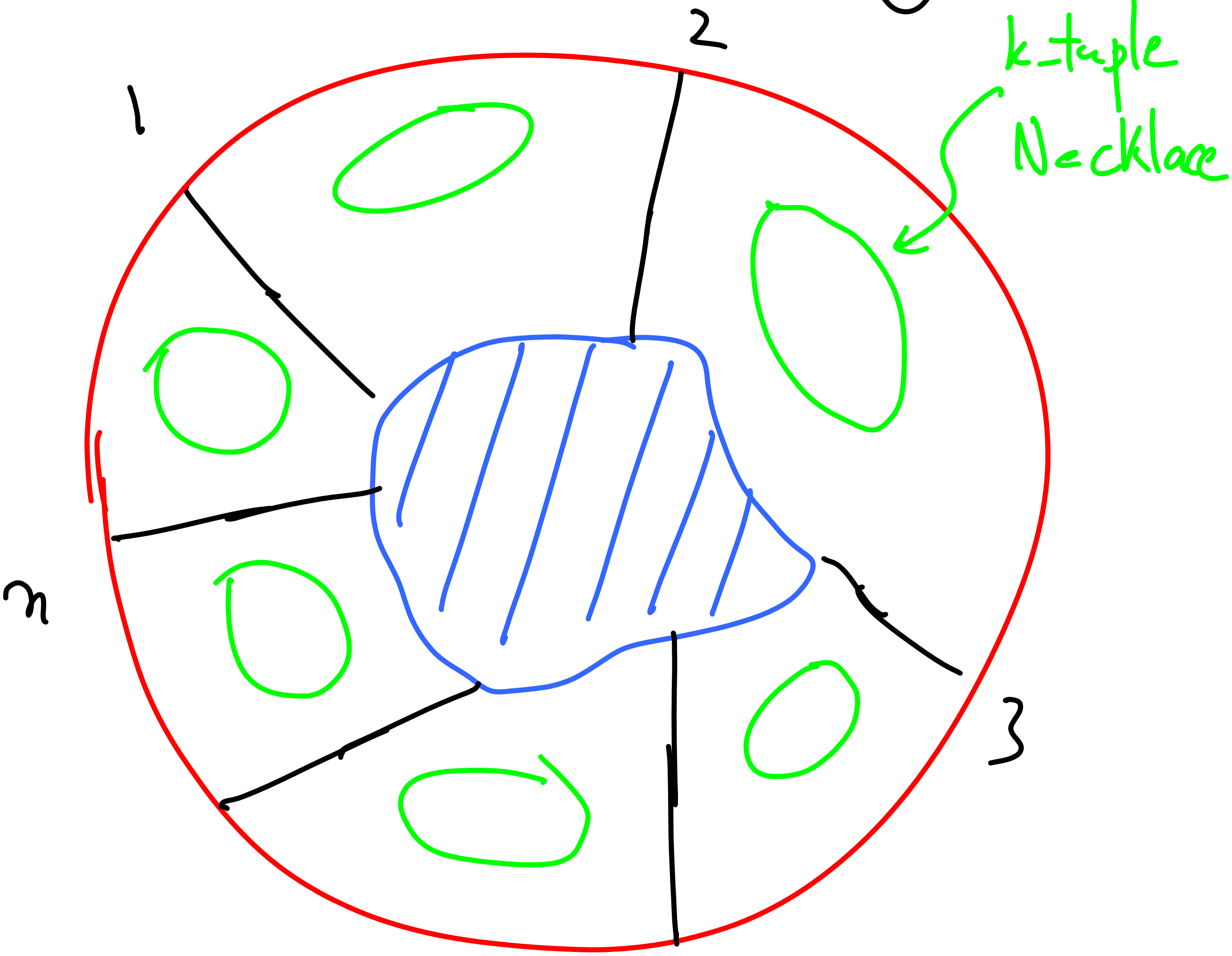
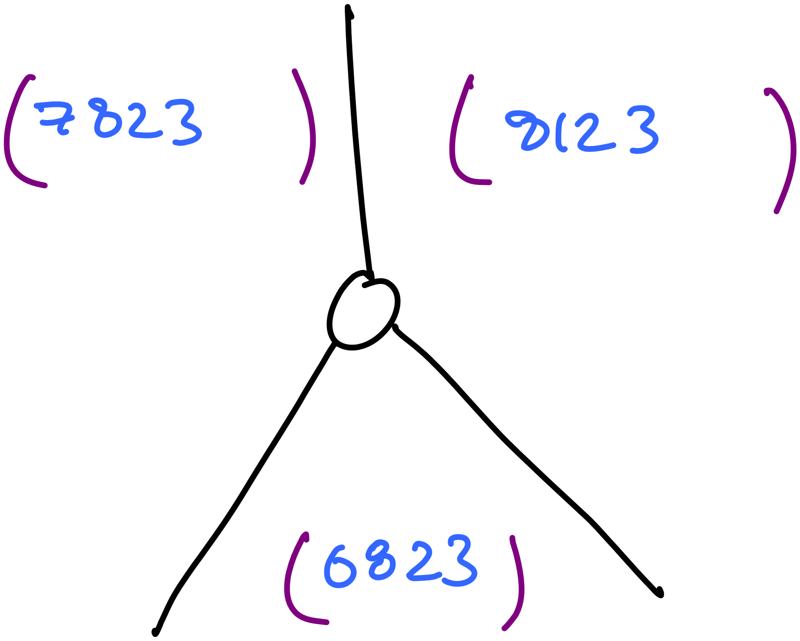
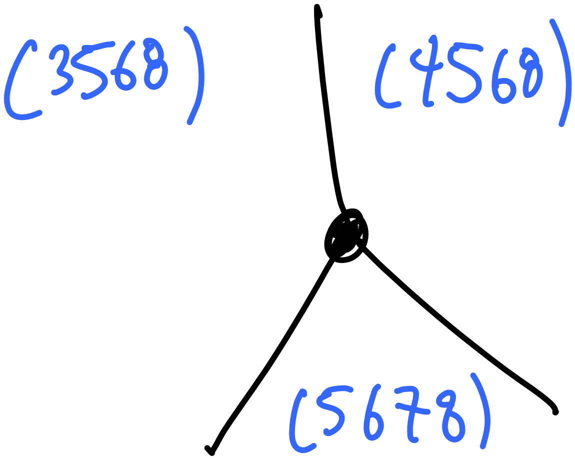
"Combinatorial Holography"

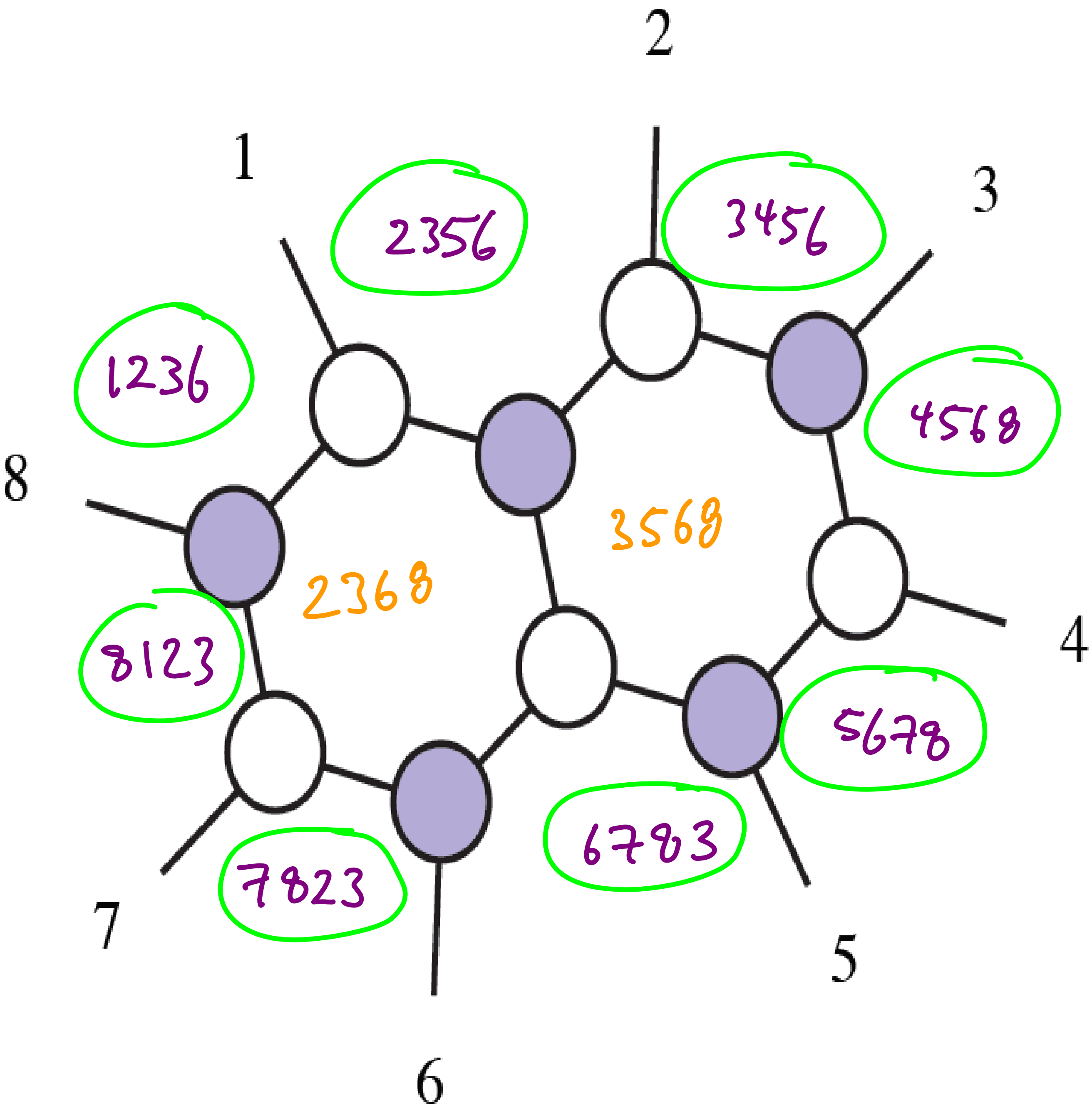




(k-1) in common,
1 different
cyclically orded



Union is (k+1),
1 missing
cyclically
ordered

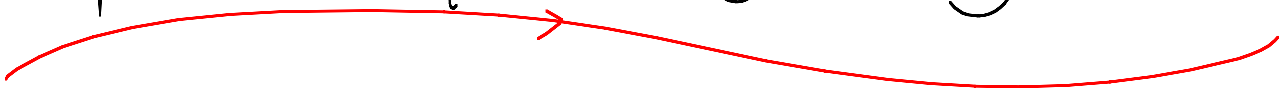

2
1
2356
3456
3
1236
4568
8
3568
2368
4
8123
5678
6783
7823
7
5
6

$$\begin{tikzcd}\text{Conf} \arrow[r, leftrightarrow] \arrow[r, bend right, red] & \text{Dual Conf}\end{tikzcd}$$

Trivial mapping between twistor/mom space residues + momentum-twistor residues:

drop the height of Deligne hook diagrams by 2!

Natural Measure on Positive Part



- Suppose we are looking for a volume form which is non-singular on the interior of the positive part, with sing. only on the boundary.

If it exists – must be

$$\frac{d^{k\times n} C}{\mathrm{vol}\,GL(k)\,(12\cdots k)\cdots(n1\cdots k-1)} \quad !$$

Remarkably, in any set of **edge** co-ordinates, we find

$$\frac{d^{k\times n} C}{\mathrm{vol}\, GL(k)\, (12..k)..(n1..k-1)} = \frac{d\,p_1}{p_1} \cdots \frac{d\,p_{k(n-k)}}{p_{k(n-k)}}$$

↳ Logarithmic singularities only on boundary of positive part!

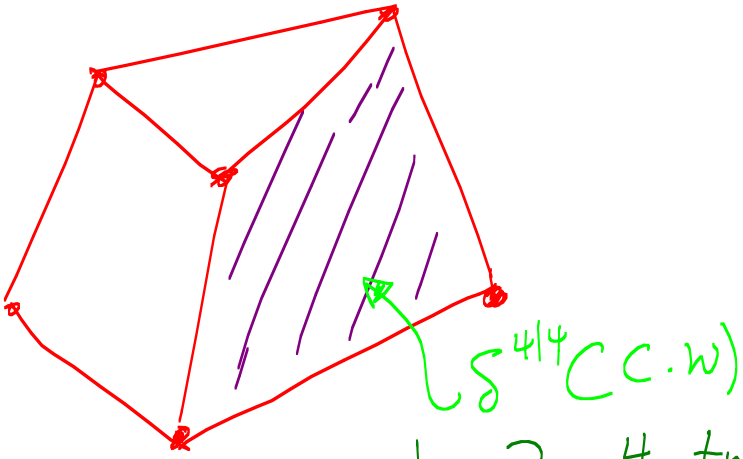
$$\frac{d^{k\times n}C}{\mathrm{vol}\,GL(k)\,(12\cdots k)\cdots(n1\cdots k-1)}$$

Manifestly GL(k) inv./
co-ordinate independent.
Geometry manifest.
Not obvious that it
purely logarithmic sing.
Not obvious what residues are

$$\frac{dp_1}{p_1}\cdots\frac{dp_{k(n-k)}}{p_{k(n-k)}}$$

Co-ordinates! Each
cover small part of
positive part.
Makes log. sing. manifest,
+ Residues easy!

Grassmannian residues are associated with cells of the Positive Grassmannian



$D =$ dim $2n-4$ twistor/mom space

dim $4K$ mom-twistor space

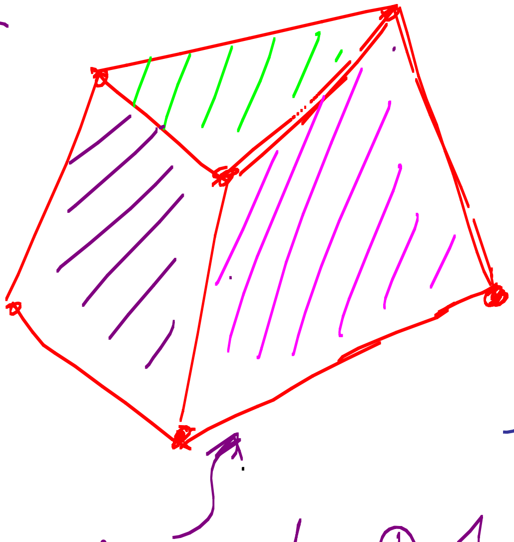
Explicit formula for any residue:

e.g. in mom. twistor space

$$\mathrm{Res} = \int \frac{dp_1}{p_1} \cdots \frac{dp_{4K}}{p_{4K}} \underbrace{\prod_{\alpha=1}^{K} \delta^{4|4}\left[C_{\alpha a}(p) Z_a \right]}_{\substack{\text{solve for } p_1, \ldots, p_{4K};\\ \text{no integral}}}$$

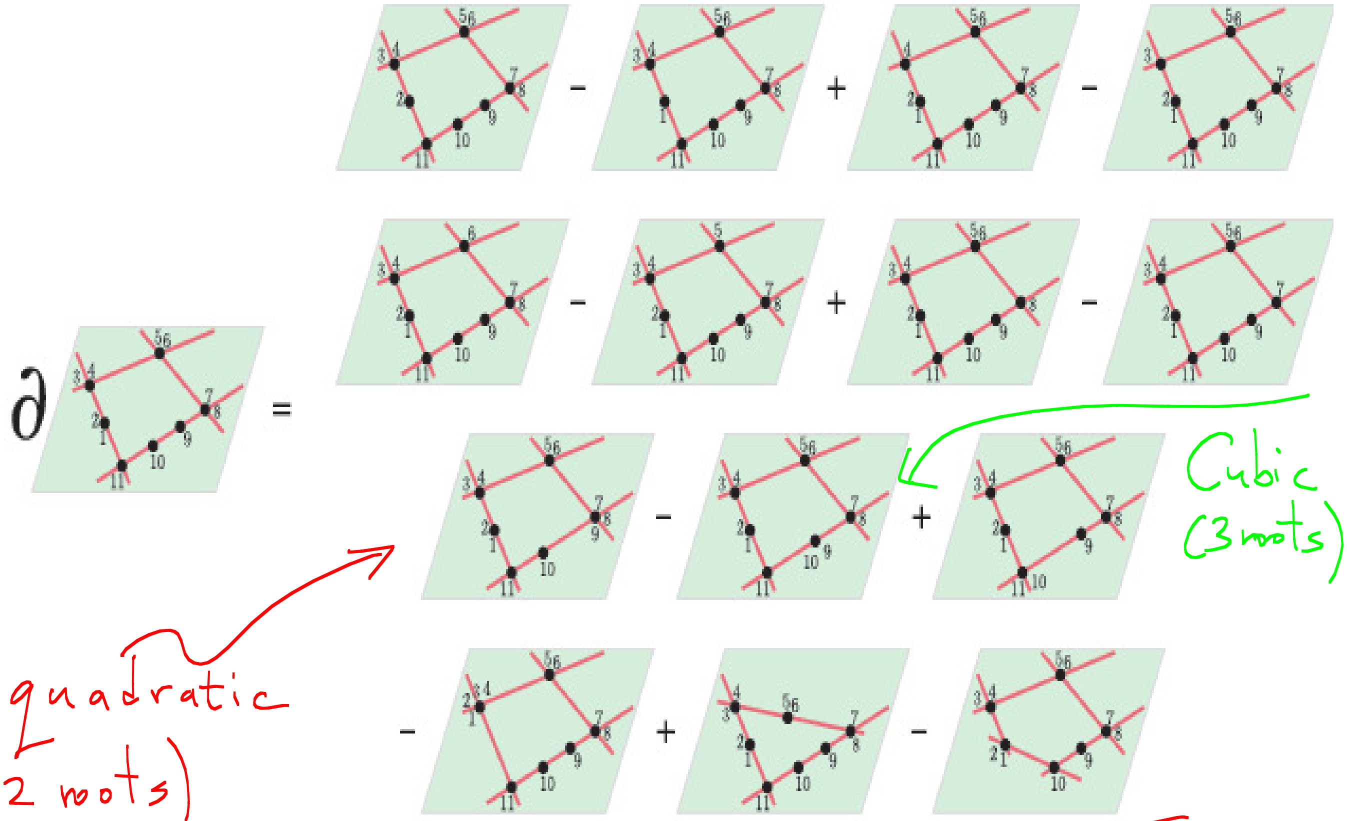
Known **explicitly**

Relations between residues:

$\partial \Big[$  $\Big] = 0$

Cells of dim D+1

Ex: 14-term identity involving rationals, $\sqrt{\ }$'s, $\sqrt[3]{\ }$'s :

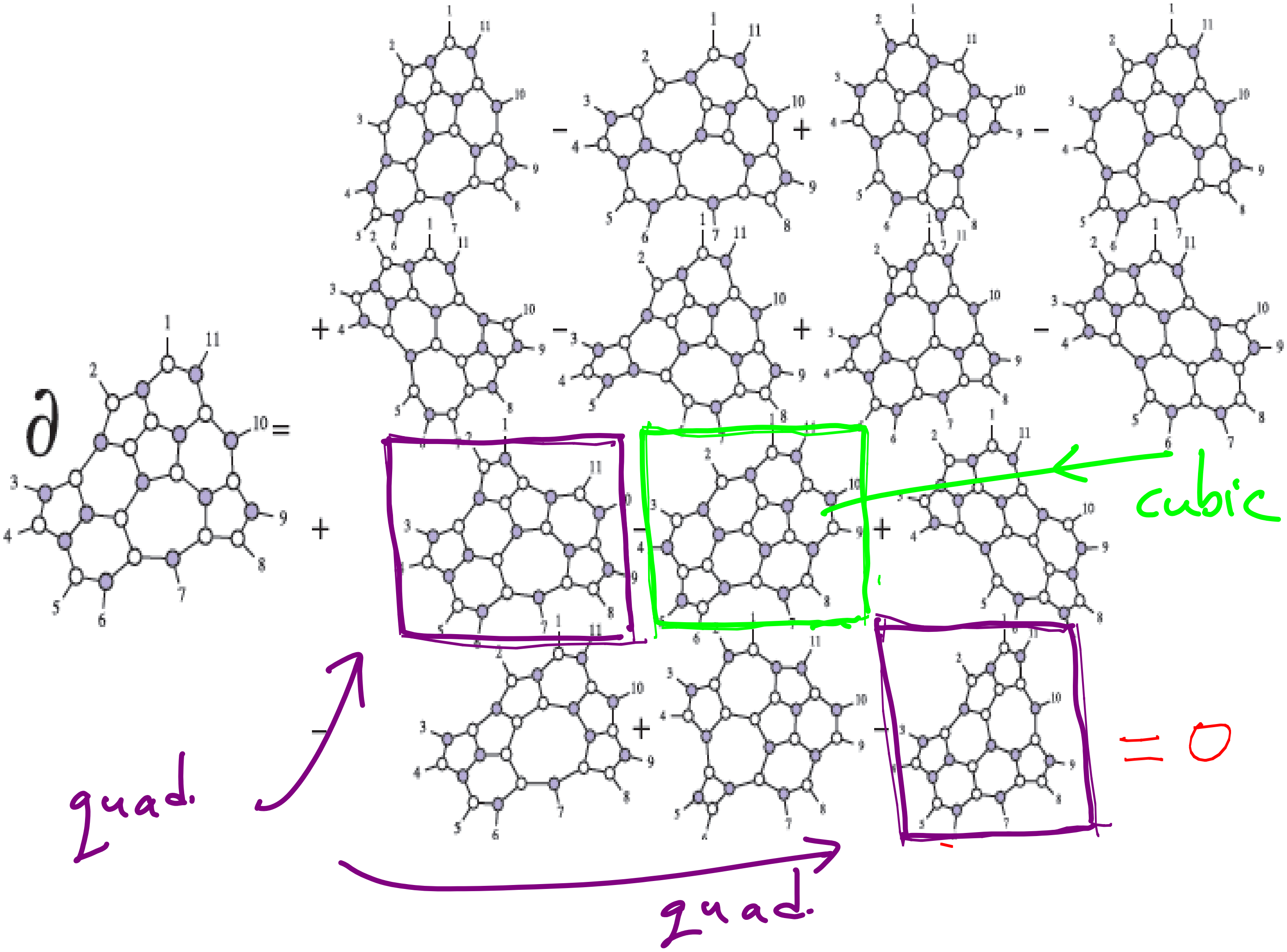


Cubic (3 roots)

quadratic (2 roots)

quadratic (2 roots)

$= 0$



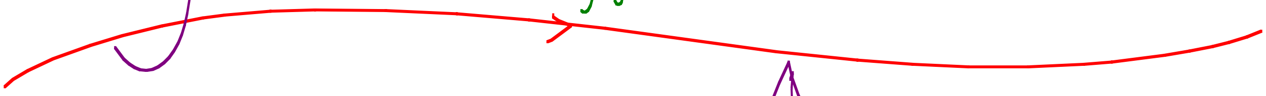
cubic

quad.

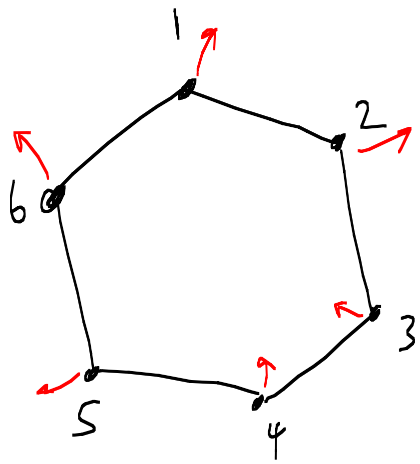
quad.

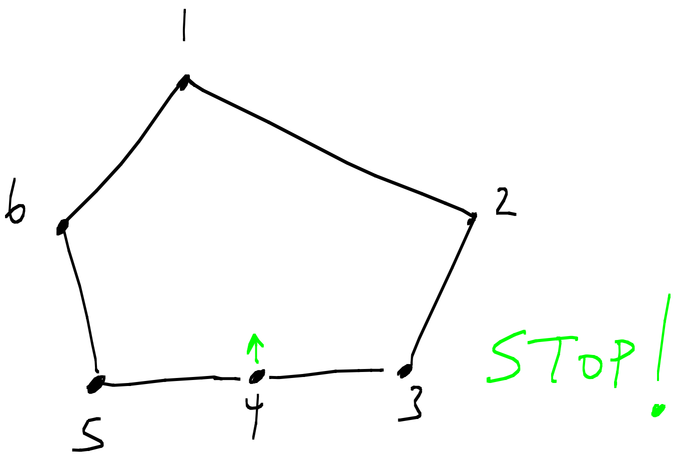
= 0

Yangian = Diffs on Positive Part



Obvious symmetry of Geometry





Must have (at least!),

$$\delta\left(i\ i{+}1\ ..\ i{+}k{-}1\right) \to 0 \quad \text{as} \quad \left(i\ i{+}1\ ..\ i{+}k{-}1\right) \to 0$$

- First non-trivial variations quadratic in C's:

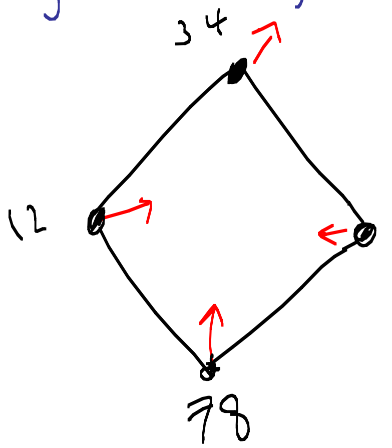
$$\delta \vec{C}_a = (\vec{C}_a \cdot \vec{\xi})\, \Omega^a_{\ b}\, \vec{C}_b$$

k-vector parameters

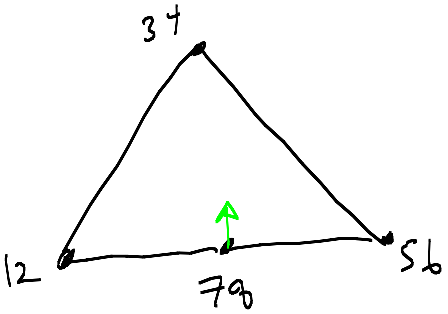
- Demanding $\delta(12..k) \to 0$ when $(12..k) \to 0$ fixes

$$\delta \vec{C}_a = (\vec{C}_a \cdot \vec{\xi}) \sum_{a<b} W_b \vec{C}_b \ ; \quad W_c \vec{C}_c = 0.$$

Remarkably, this is enough to guarantee that these diffs act nicely on all faces of the positive part!



,



STOPS!

i.e. in this config, $\delta(571) \to 0$ as $(571) \to 0$.

Of course this is exactly how the level-one Yangian generator $\sum_{a<b} \left[\mathcal{W}^A \frac{\partial}{\partial \mathcal{W}^{C'}} \mathcal{W}^C \frac{\partial}{\partial \mathcal{W}^B} \right]$

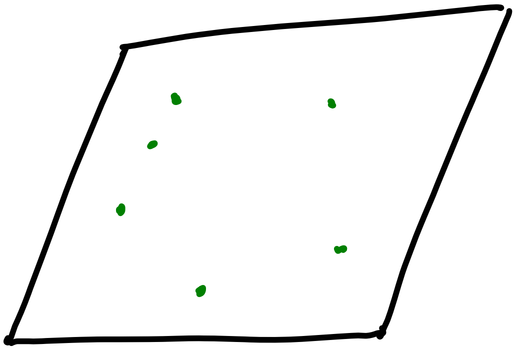
acts on $\int d^{k\times n}C \; f(C) \; \delta^{4|4}(C\cdot\mathcal{W})$

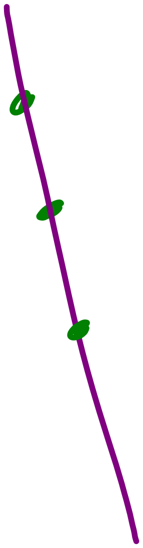
"Grassmannian rep. of Yangian".

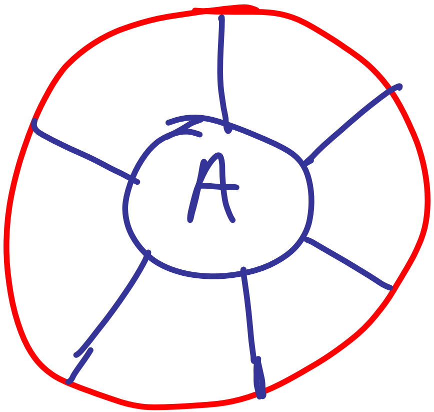
All-loop Integrand

in

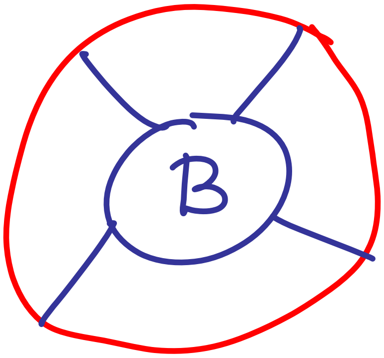
The Grassmannian



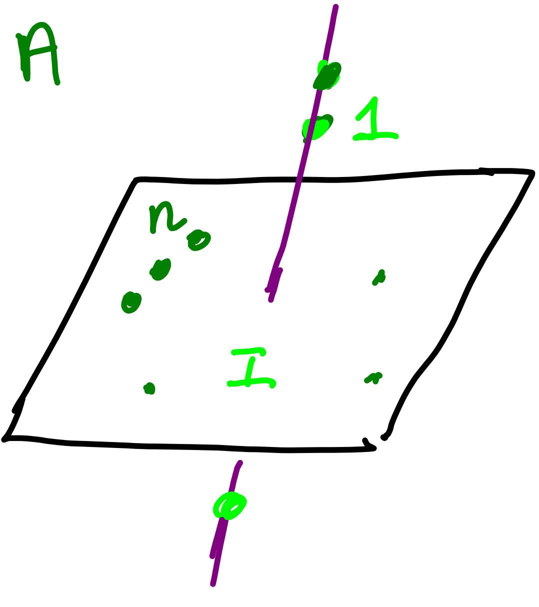


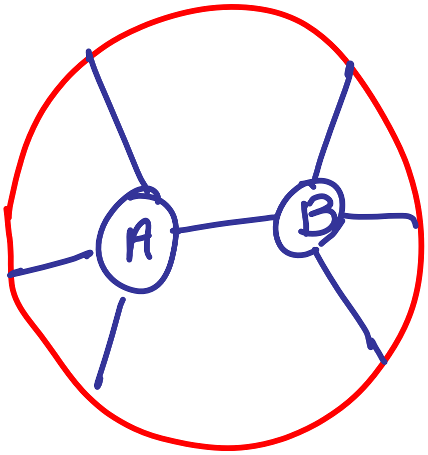


D_A



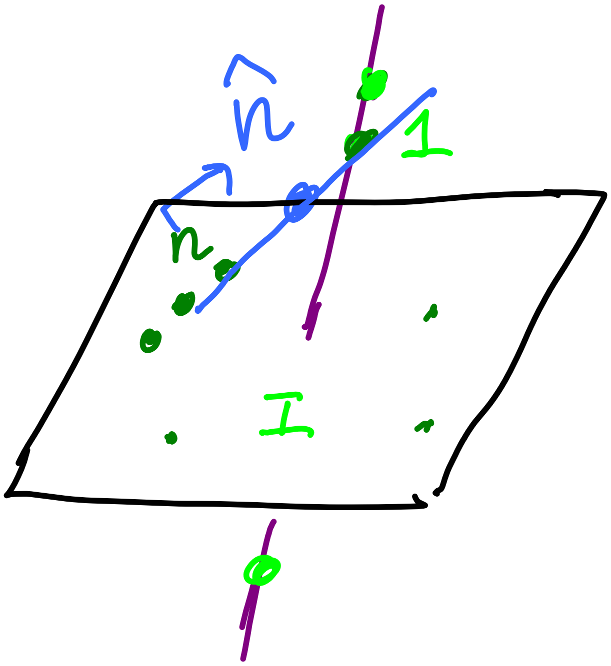
D_B

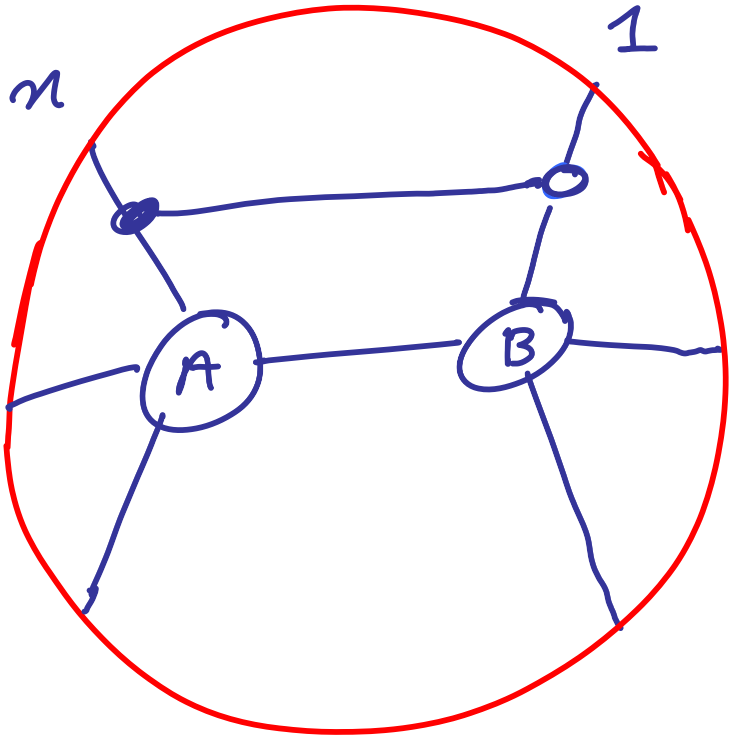




$D_A + D_B - 1$

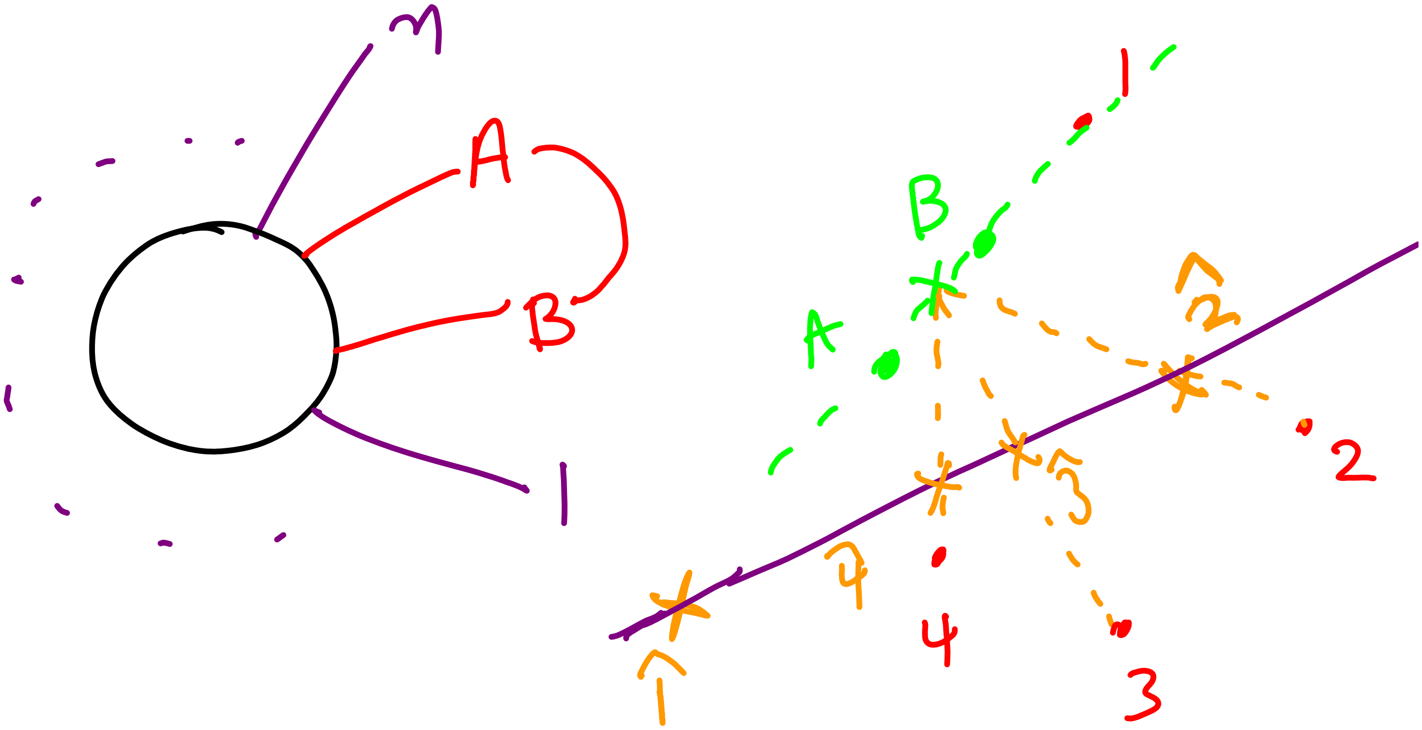
Factorization





$$D_A + D_B$$

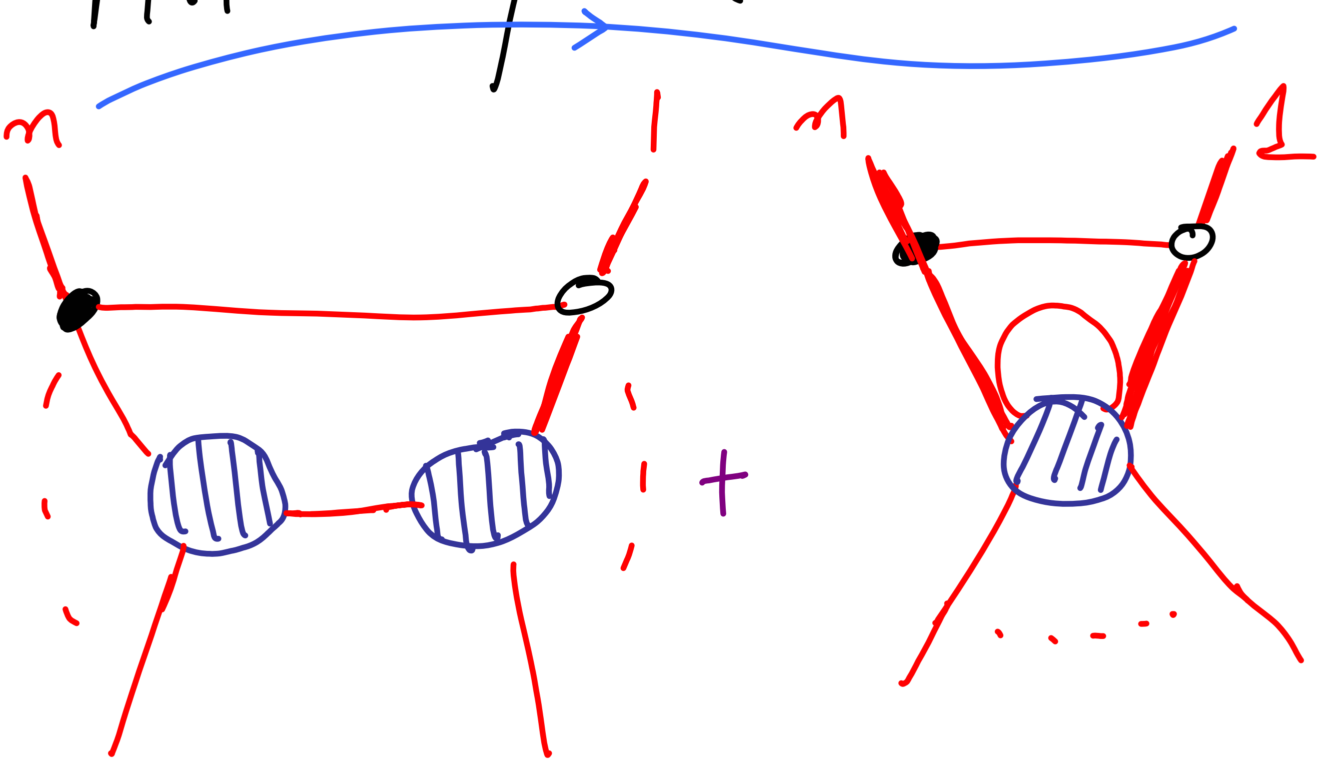
BCFW



Maps point in $G(k+1, n+2)$ cell

→ point in $G(K, n)$ cell

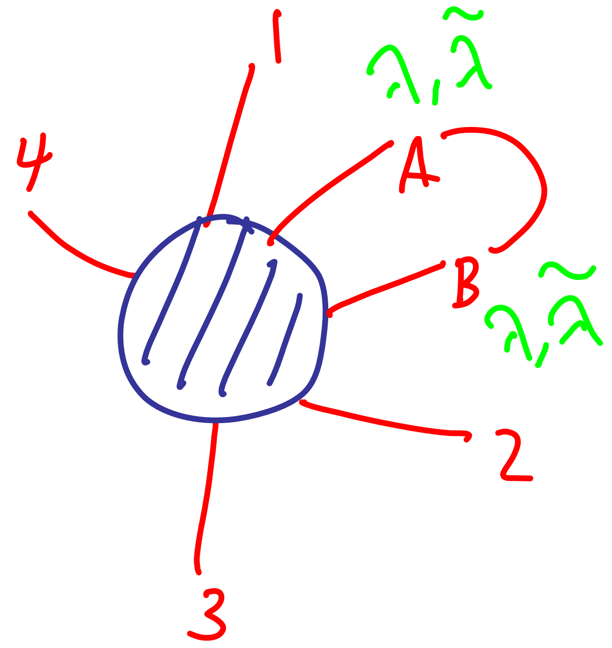
All-Loop Recursion

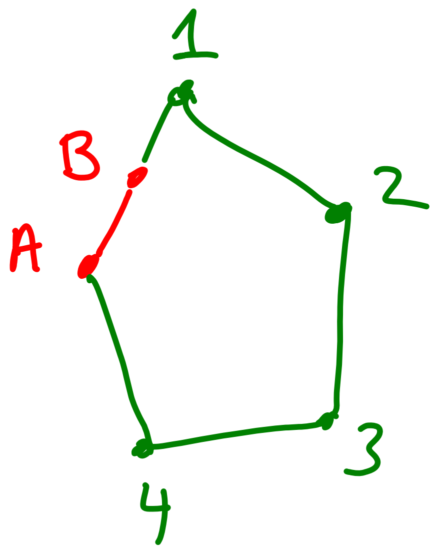


Universal Pos. Part. measure:

$$\left(\prod_i \frac{dP_i}{P_i}\right)$$ for each cell.

All sing. manifest!





$$\int \frac{d\rho_1}{\rho_1}\cdots\frac{d\rho_8}{\rho_8}\Big/GL(1)\;\frac{d\tau}{\tau}\;\delta^{4|4}\left[C(\rho)^{3,6\to 2,4}\cdot\mathcal{W}\right]$$

$$\ell=\lambda\tilde\lambda+\tau\lambda_1\tilde\lambda_2\;,\quad d^4\ell=\frac{d^2\lambda\, d^2\tilde\lambda}{GL(1)}\;\frac{d\tau}{\tau}$$

$$\rightarrow \int\frac{d^4\ell}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2\,(\ell-p_4)^2}\times M_{\text{tree}}$$

All spacetime amp.
singularities @ L-loops

↕

Hitting boundaries of "top"
Pos. Grassm. with $G(k+L, n+2L)$

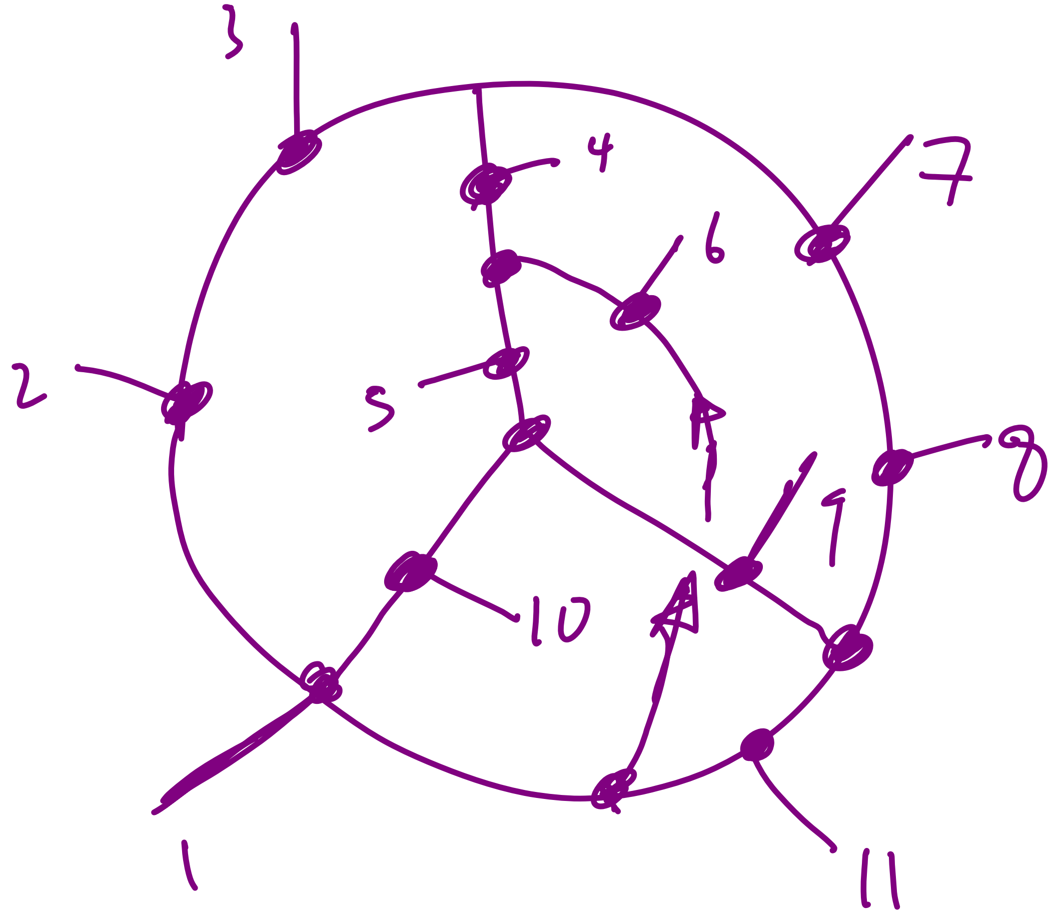
→ [Purely Combinatorial]

Beyond the Planar Limit

Planar magic:

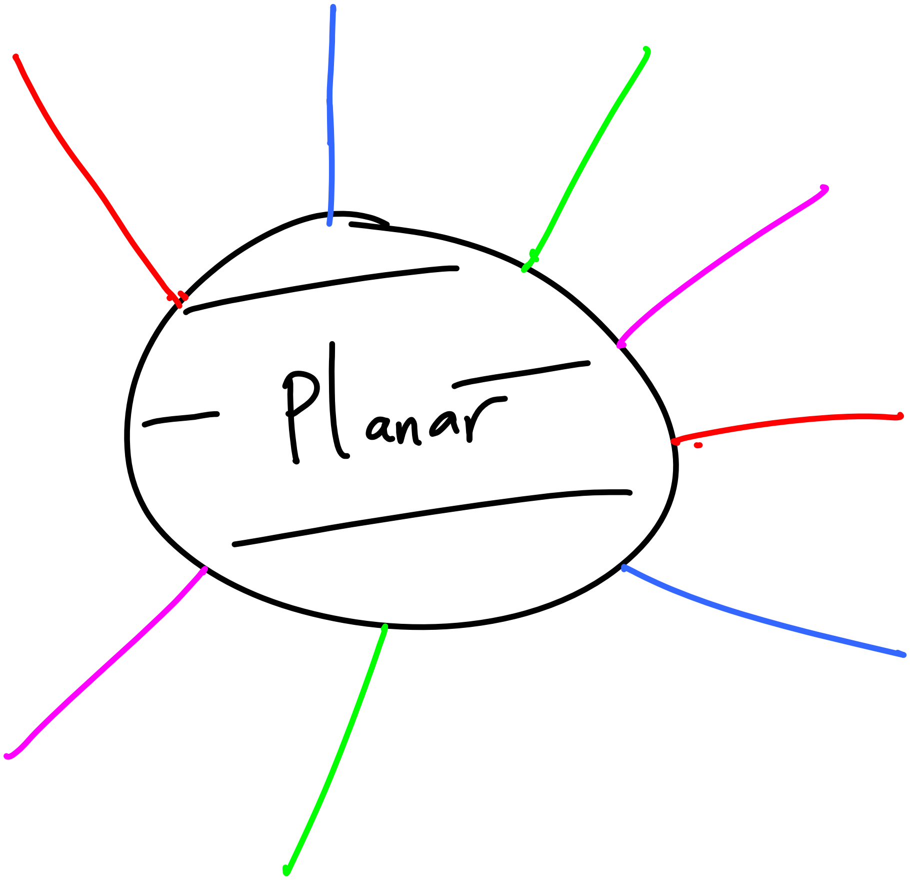
Finite # of L.S. @ <u>all</u> loop order.

$\{\longleftrightarrow$ Yangian/Grassmannian structure$\}$


1
2
3
4
5
6
7
8
9
10
11

?

Planarify First



Simple Combinatorial Thm

of distinct points in a cell of dimension d, is $\leq (d-n+2K)$

⇓ + some simple geometry

⋮

All-loop MHV result

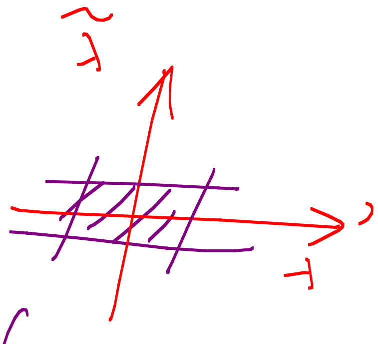
MHV leading singularities are a lin. combination of permutations of Parke-Taylor factors!

$$M_n^{MHV} = \sum (\text{Color}) \times \sum_{\substack{\text{perm.}\\ \sigma}} \frac{1}{\langle \sigma_1 \sigma_2 \rangle \cdots \langle \sigma_n \sigma_1 \rangle} \times \text{Transc.}$$

Conclusions

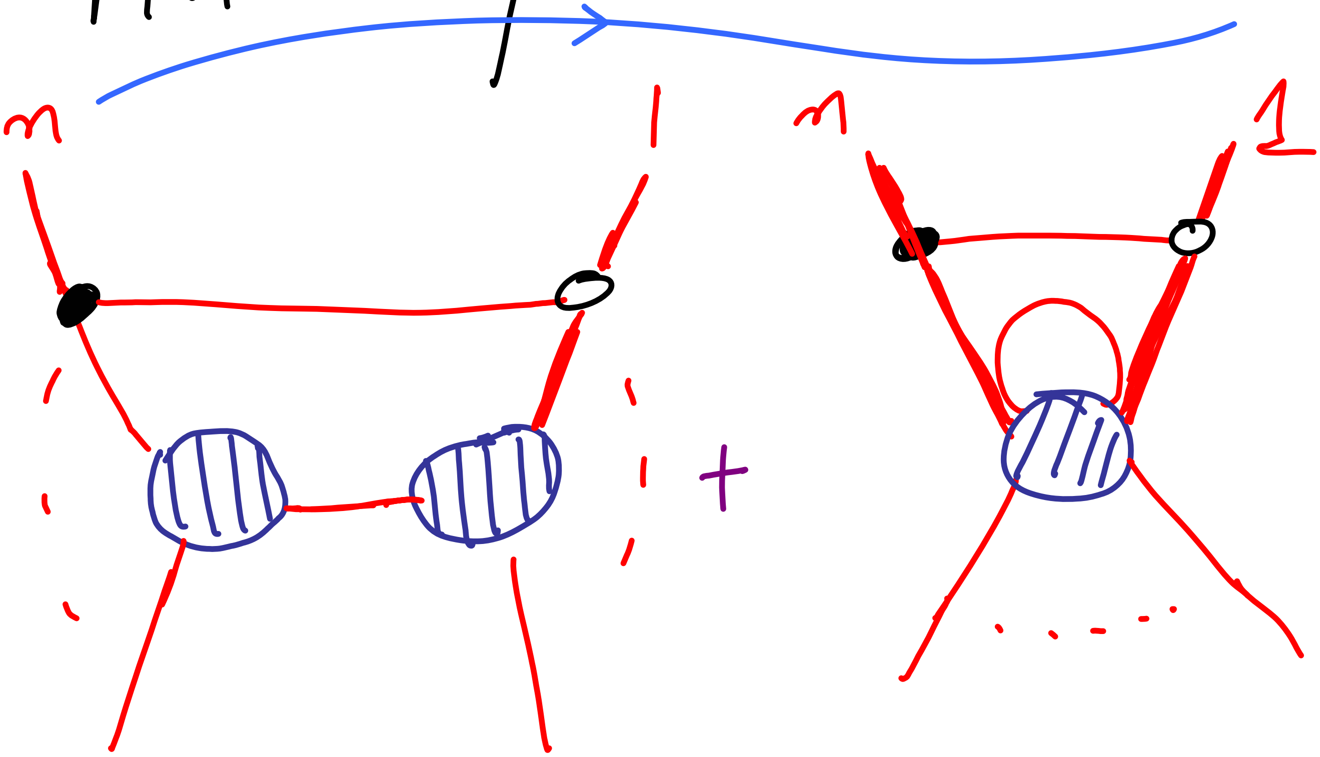
. We have seen a remarkable relationship between the Positive Grassmannian – the simplest generalization of simplices – and planar $\mathcal{N}=4$ SYM Scattering Amplitudes.

- There is a unique form with singularities only on the faces of the Polytope.
- Integrating this form against residues are localized to faces of the polytope of appropriate dimension.



- The obvious diffs of this Polytope ⟶ Yangian invariance of the residues.

All-Loop Recursion



Universal Pos. Part. measure:

$$\left(\prod_i \frac{dP_i}{P_i}\right)$$ for each cell.

All sing. manifest!

Loop amplitudes are anomalies

Ex:

$$M^{MHV}_{1\text{-loop}} = \sum_{i,j} \int d\log\frac{\langle AB\, i\, i{+}1\rangle}{\langle AB * i\rangle}\; d\log\frac{\langle AB\, i\, i{+}1\rangle}{\langle AB * i{+}1\rangle}\; d\log\frac{\langle AB\, j\, j{+}1\rangle}{\langle AB * j\rangle}\; d\log\frac{\langle AB\, j\, j{+}1\rangle}{\langle AB * j{+}1\rangle}$$

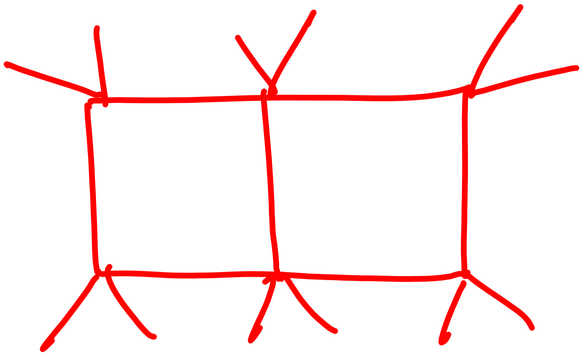
$\uparrow$

(Highly) total derivative!

Each term finite (gen. i, j), non-zero only because real contour of integration induces boundaries in integral!

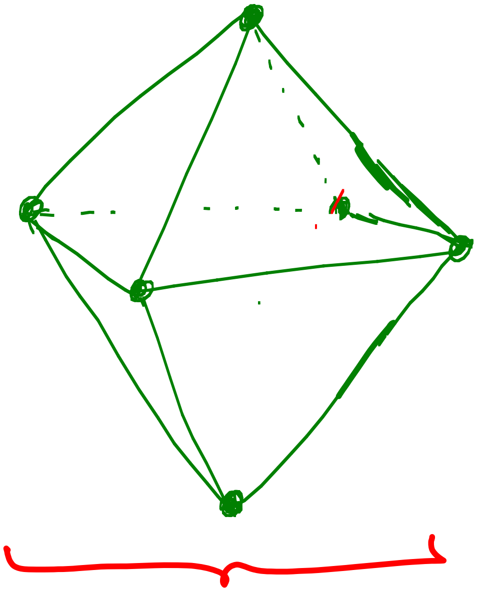
Sing. manifest! Ideally set up to compute integral ??

{ To "do integral", need far more than polylogs, (usual) symbols ....

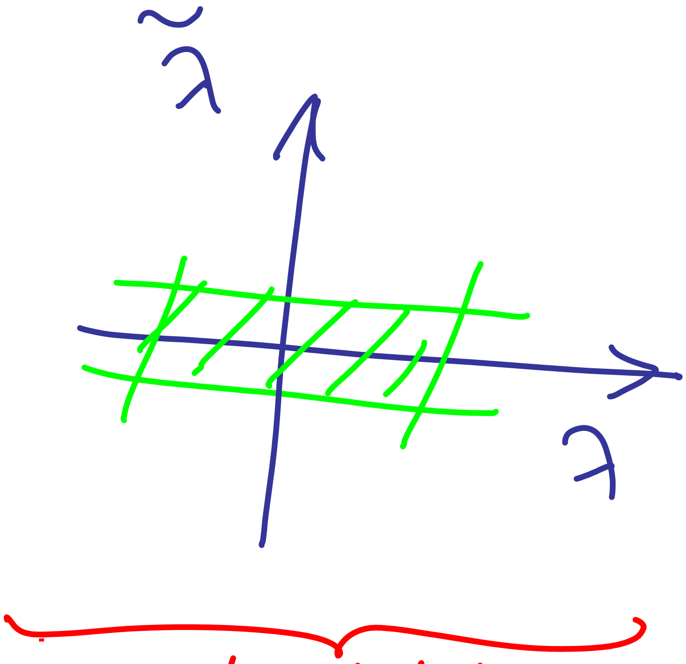


appears in N^4MHV amp,

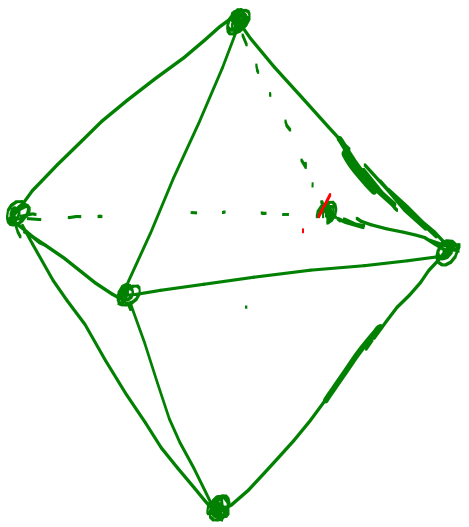
Elliptic, not polylog }

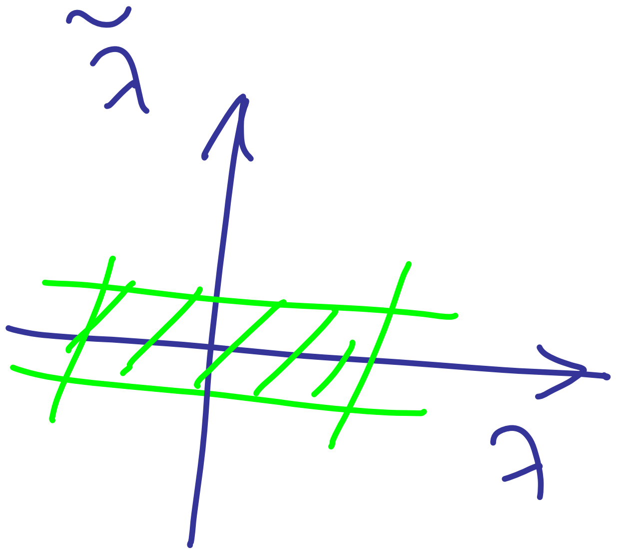


Positive Grassm.



External data





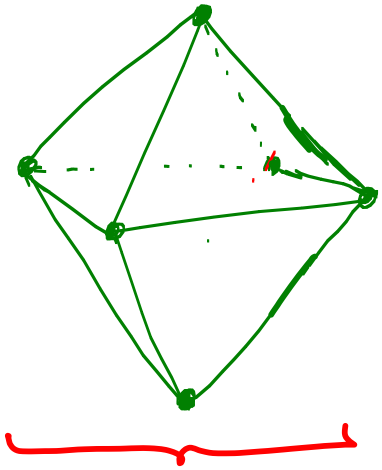
Positive Grassm.

External data

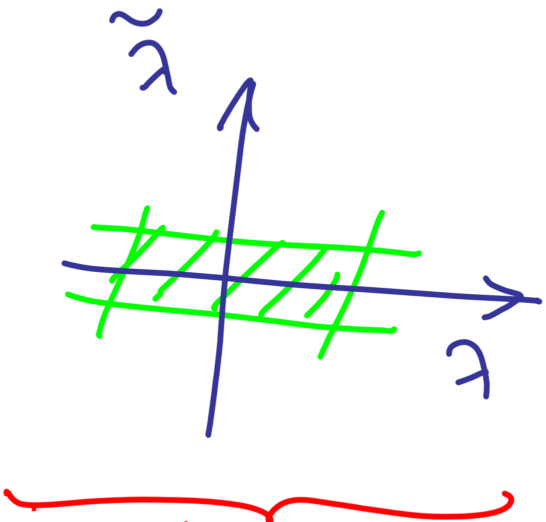
Amazing Combinatorics

Even more amazing combinatorics!

[Knutson, Lam, Speyer '09]



Pos. Grassm.



External data

→ Algebraic functions of data

[L.S; all-loop integrand]

→ Transcendental functions

[Polylogs + much more....]

A peek @ non-planar magic

$$M_n^{MHV} = \sum (\text{Color}) \times \sum_{\substack{\text{perm.} \\ \sigma}} \frac{1}{\langle \sigma_1 \sigma_2 \rangle \cdots \langle \sigma_n \sigma_1 \rangle} \times \text{Transc.}$$

[New L.S. appear @ NMHV + beyond]

Hopefully more progress to report for Amps 2012!