

# String and M-Theory Corrections to $G_2$ Holonomy

Physics and Mathematics of  $G_2$  Compactifications

MCTP, Ann Arbor, 4'th May 2007

## Plan

- Higher-Order Corrections in String Theory
- Deformations of Special-Holonomy Backgrounds
- Preservation of Supersymmetry
- Higher-Order Corrections in M-Theory
- Generalised Holonomy

## Gravity from String Theory

One of the miracles of string theory is that it embodies general covariance, and gravity, albeit in an *a priori* rather non-transparent way.

It shows up even in perturbative string calculations around a flat Minkowski spacetime background. The 3-graviton scattering amplitude in string theory is consistent with the 3-point interaction implied by tree-level scattering in Einstein gravity.

The 4-graviton string scattering amplitude has a contribution that is also consistent with the Einstein-Hilbert term. However, there is an additional string term that is not explained by Einstein-Hilbert gravity. It is in fact the first indication of a higher-order correction to Einstein gravity:

$$I = \int d^{10}x \sqrt{-g} \left[ R + c \alpha'^3 (\text{Riemann})^4 + \dots \right]$$

where  $(\text{Riemann})^4$  is quartic in curvature.

# Corrections to Gravity Backgrounds in String Theory

Not only gravity, but the entire leading-order effective supergravity action receives higher-order string corrections. The detailed forms of some of these corrections, even at the 4-point level, are not known. However, if we restrict attention to the gravity (and dilaton) sector, then all the corrections in the effective action up to order  $\alpha'^3$  are known. This allows, in particular, a detailed discussion of the  $\alpha'^3$  corrections to purely gravitational backgrounds which, at leading order, satisfied the vacuum Einstein equations.

One particularly interesting question concerns the fate of leading-order gravitational backgrounds with special holonomy, since these, at leading order, are supersymmetric. Examples are

$$(\text{Minkowski})_4 \times K_6, \quad (\text{Minkowski})_3 \times K_7, \quad (\text{Minkowski})_2 \times K_8$$

where  $K_6$  is a Ricci-flat Calabi-Yau 6-manifold,  $K_7$  is a 7-manifold of  $G_2$  holonomy, and  $K_8$  is an 8-dimensional Ricci-flat Calabi-Yau manifold, a hyper-Kähler manifold or a manifold of  $\text{Spin}(7)$  holonomy.

# Tree-Level Corrections to Type IIA or IIB Strings

In the gravity/dilaton sector, the corrected effective action up to order  $\alpha'^3$  is given by

$$\mathcal{L} = \sqrt{-g} e^{-2\phi} \left( R + 4(\partial\phi)^2 - c \alpha'^3 Y \right)$$

where  $c$  is a known pure-number constant (proportional to  $\zeta(3)$ ).  $Y$  is quartic in curvature. The equations of motion are

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi &= c \alpha'^3 X_{\mu\nu} \\ \nabla^2 \phi - 2(\partial\phi)^2 &= \frac{1}{2} c \alpha'^3 (Y - g^{\mu\nu} X_{\mu\nu}) \end{aligned}$$

where

$$X_{\mu\nu} = \frac{e^{2\phi}}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^{10}x \sqrt{-g} e^{-2\phi} Y$$

The quartic curvature correction  $Y$  is quite complicated, as a ten-dimensional Riemannian expression. With care, we can use a simpler eight-dimensionally covariant light-cone expression, for the special case of (Minkowski)  $\times K$  backgrounds.

## The Quartic-Curvature Correction

The quartic curvature invariant is given, in light-cone gauge, by

$$Y \propto (t^{i_1 \dots i_8} t^{j_1 \dots j_8} - \frac{1}{4} \epsilon^{i_1 \dots i_8} \epsilon^{j_1 \dots j_8}) R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8}$$

and  $t^{i_1 \dots i_8}$  is defined by

$$t^{i_1 \dots i_8} M_{i_1 i_2} \dots M_{i_7 i_8} = 24 \text{tr} M^4 - 6 (\text{tr} M^2)^2, \quad \text{for all } M_{ij} = -M_{ji}$$

It was shown by Gross and Witten that  $Y$  could be written as a Berezin integral over  $SO(8)$  Majorana spinors  $\psi = (\psi_+, \psi_-)$ :

$$Y \propto \int d^{16} \psi \exp \left[ (\bar{\psi}_+ \Gamma^{ij} \psi_+) (\bar{\psi}_- \Gamma^{kl} \psi_-) R_{ijkl} \right]$$

Since the integrability condition for a covariantly-constant spinor  $\eta$  in the transverse 8-space is  $[\nabla_i, \nabla_j] \eta = \frac{1}{4} R_{ijkl} \Gamma^{kl} \eta = 0$ , it follows that a leading-order supersymmetric background will have a spinor zero-mode for at least one of the right-handed or left-handed spinors in the Berezin integral, and hence  $Y = 0$ .

## Corrections to (Minkowski) $_4 \times K_6$

Corrections to Ricci-flat Calabi-Yau manifolds were analysed long ago. It was shown (Freeman & Pope, 1986) that the variation of  $Y$ , calculated from the Berezin integral, gives

$$X_{ij} = \nabla_{\hat{i}} \nabla_{\hat{j}} S, \quad \text{where for any } V_i, \quad V_{\hat{i}} \equiv J_i^j V_j$$

$J$  is the Kähler form of the original CY background metric, and

$$S = R_{ijkl} R^{klmn} R_{mn}{}^{ij} - 2R_{ijkl} R^{km\ell n} R_m{}^i{}_n{}^j$$

is the 6-dimensional Euler density. (This agrees with sigma-model beta function calculations by Grisaru et al.)

The corrected equations of motion then imply:

$$R_{ij} = c \alpha'^3 (\nabla_i \nabla_j + \nabla_{\hat{i}} \nabla_{\hat{j}}) S, \quad \phi = -\frac{1}{2} c \alpha'^3 S$$

(Quantities on RHS calculated using the leading-order background; corrections are valid to order  $\alpha'^3$ .) In complex coordinates this corrected Einstein equation is  $R_{\alpha\bar{\beta}} = c \alpha'^3 \partial_\alpha \partial_{\bar{\beta}} S$ . The first Chern class still vanishes, but  $SU(3) \rightarrow U(3)$  holonomy. What happens to supersymmetry?

## Supersymmetry in Corrected (Minkowski) $_4 \times K_6$

The leading-order supersymmetry transformation rules also receive  $\alpha'^3$  corrections; their detailed form has never been fully derived (see Peeters, Vanhove, Westerberg for some partial results). There is a general expectation that supersymmetry should survive the corrections. This was studied by Candelas, Freeman, Pope, Sohnius & Stelle (1986) for the 6-dimensional Calabi-Yau case: Can we at least conjecture an  $\alpha'^3$  correction that will make this happen?

The modification of  $\delta\psi_\mu = \nabla_\mu\epsilon$  to  $\delta\psi_\mu = D_\mu\epsilon$ , where

$$D_i = \nabla_i + \frac{i}{2} c \alpha'^3 (\nabla_{\hat{i}} S)$$

has as integrability condition precisely the corrected Einstein equation in the CY background. In the corrected background, we shall have Killing spinors satisfying the corrected condition  $D_i \eta = 0$ ; hence supersymmetry.

We can propose  $\delta\psi_\mu = D_\mu\epsilon$  as the corrected SUSY transformation in the CY background, but since it involves the explicit use of the Kähler form (hidden in the hat), we must make sure that it is also expressible as a fully Riemannian expression, which specialises to  $D_i$  in CY backgrounds.



## Riemannian Form of Supersymmetry Correction

A Killing spinor in the leading-order CY background satisfies  $\Gamma_j \eta = -i \Gamma_{\hat{j}} \eta$ . This allows us to write a Riemannian expression that reduces to  $D_i = \nabla_i + \frac{i}{2} c \alpha'^3 (\nabla_{\hat{i}} S)$  in a six-dimensional CY background (CFPSS):

$$D_i = \nabla_i + \frac{3}{4} c \alpha'^3 \nabla_s R^r{}_{ikl} R^s{}_{tmn} R^t{}_{rpq} \Gamma^{klmnpq}$$

An alternative form, obtained by dualising in the transverse 8-space, is

$$D_i = \nabla_i - 6c \alpha'^3 \nabla_s R_{ipkl} R^{stln} R_t{}^p{}_{nq} \Gamma_{qk}$$

These, extended to the full index range, provide candidate ten-dimensional Riemannian expressions for the  $\alpha'^3$  correction to the gravitino transformation rule in string theory, that would satisfy the desideratum of implying that the supersymmetry of leading-order  $(\text{Minkowski})_4 \times K_6$  backgrounds is preserved in the face of string corrections at order  $\alpha'^3$ .

What about leading-order  $(\text{Minkowski})_3 \times K_7$  or  $(\text{Minkowski})_2 \times K_8$  backgrounds? Will these remain supersymmetric? What is  $D_i$  for these?

## Corrections to $G_2$ Holonomy (Minkowski) $_3 \times K_7$

We can view these as (Minkowski) $_2 \times K_8$ , where  $K_8 = R \times K_7$ . With  $K_7$  having  $G_2$  holonomy, we shall have one covariantly-constant  $SO(8)$  spinor zero-mode of each chirality. The Berezin integral for  $Y$  again vanishes in the background, and its variation can be nicely expressed in terms of special structures on the  $G_2$  manifold (Lü, Pope, Stelle, Townsend):

$$X_{ij} = c_{ikm} c_{jln} \nabla^k \nabla^l Z^{mn}$$

where  $c_{ijk} = i \bar{\eta} \Gamma_{ijk} \eta$  is the associative 3-form and

$$Z^{mn} \equiv \frac{1}{32} \epsilon^{mi_1 \dots i_6} \epsilon^{nj_1 \dots j_6} R_{i_1 i_2 j_1 j_2} \dots R_{i_5 i_6 j_5 j_6}$$

From the corrected string equations, we find that on  $K_7$  we now have

$$R_{ij} = c\alpha'^3 (\nabla_i \nabla_j S + c_{ikm} c_{jln} \nabla^k \nabla^l Z^{mn}), \quad \phi = -\frac{1}{2} c\alpha'^3 S$$

where  $S = g_{ij} Z^{ij}$  is the 6-dimensional Euler integrand again. Since  $G_2$  manifolds are Ricci-flat, the correction here has destroyed the special holonomy completely. But, in a generalised sense, maybe it hasn't...

## Supersymmetry in Corrected (Minkowski)<sub>3</sub> × K<sub>7</sub>

Can we again modify the supersymmetry transformation rule in such a way that the corrected  $G_2$  background will again remain supersymmetric? We can again ask for a modification of the gravitino transformation rule, to  $\delta\psi_\mu = D_\mu \epsilon$ , where  $D_i = \nabla_i + c\alpha'^3 Q_i$ , and require that the integrability condition  $[D_i, D_j]\epsilon = 0$  give the corrected  $G_2$  Einstein equation to order  $\alpha'^3$ . We find

$$D_i = \nabla_i - \frac{i}{2} c \alpha'^3 c_{ijk} (\nabla^j Z^{kl}) \Gamma_\ell$$

This, and the corrected  $G_2$  Einstein equation, both reduce to the previous CY results if we take  $K_7 = \mathbb{R} \times K_6$ . Thus these  $G_2$  holonomy results encompass the previous CY results.

The corrected gravitino transformation was “cooked up” to retain supersymmetry in the corrected  $G_2$  background. We must check that it at least admits a covariant Riemannian generalisation, that does not make use of special tensors peculiar to  $G_2$  backgrounds. This is more restrictive than the previous CY case. Remarkably, the previous 6-Gamma Riemannian expression still works.

## Corrections to (Minkowski) $_4 \times K_7$ in M-Theory

So far, we have considered  $\alpha'^3$  corrections at tree-level in type II string theory. The IIA string is an  $S^1$  compactification of M-theory. All the tree-level  $\alpha'^3$  corrections vanish in the limit of uncompactified M-theory. There are (Riemann) $^4$  corrections in M-theory, which correspond to one-loop  $\alpha'^3$  corrections in the IIA string.

At one loop, the  $\alpha'^3$  corrections in the type IIA and type IIB string differ, because of different R-R sectors circulating in the loop. In type IIB,  $SL(2, Z)$  duality implies it is the same as at tree-level:

$$Y \propto (t^{i_1 \dots i_8} t^{j_1 \dots j_8} - \frac{1}{4} \epsilon^{i_1 \dots i_8} \epsilon^{j_1 \dots j_8}) R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8} = Y_0 - E_8$$

(with no  $e^{-2\phi}$  factor, since it is 1-loop). In type IIA, we have instead

$$\tilde{Y} \propto (t^{i_1 \dots i_8} t^{j_1 \dots j_8} + \frac{1}{4} \epsilon^{i_1 \dots i_8} \epsilon^{j_1 \dots j_8}) R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8} = Y_0 + E_8$$

and in addition there is a Chern-Simons term  $B_2 \wedge t R^4$ . These lift to terms of the form  $\beta(\tilde{Y}_0 + \hat{E}_8)$  and  $\beta \hat{A}_3 \wedge t \hat{R}^4$  in M-theory. ( $\beta \sim \alpha'^3$ .)

## Eleven-Dimensional Lagrangian

There are also one-loop  $\alpha'^3$  corrections to the form-field Lagrangian terms, whose detailed structure is unknown. This prevents one from considering corrections to backgrounds in string or M-theory with fluxes where such terms would contribute at this order. But we don't need to know about such terms in order to discuss corrections to M-theory backgrounds that are purely gravitational at leading-order.

For  $(\text{Minkowski})_4 \times K_7$  backgrounds, where the curvature is restricted to seven dimensions, neither the  $\hat{E}_8$  term nor the  $\hat{A}_3 \wedge t \hat{R}^4$  term contribute at this order. It would suffice to consider the eleven-dimensional Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}} \left( \hat{R} - \frac{\beta}{1152} \hat{Y}_0 \right)$$

However, it is more elegant to exploit the freedom to add terms that vanish by the leading-order field equations. Adding these terms does not change the physics in any way; it corresponds merely to making field redefinitions. But, as is often the case, there exist more convenient, and less convenient, choices of field variable.

# M-Theory Equations for (Minkowski)<sub>4</sub> × K<sub>7</sub>

Performing the field redefinition

$$\hat{g}_{MN} \longrightarrow \left(1 + \frac{\beta}{5184} \hat{R} \hat{S}\right) \hat{g}_{MN}$$

where  $\hat{S}$  is the 6-dimensional Euler integrand, leads to

$$\mathcal{L} = \sqrt{-\hat{g}} \left( \hat{R} - \frac{\beta}{1152} (\hat{Y}_0 - \hat{R} \hat{S}) \right)$$

up to order  $\beta$ . Variation yields the equations of motion

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} = -\frac{\beta}{1152} \square S$$

$$\hat{R}_{ij} - \frac{1}{2} \hat{R} \hat{g}_{ij} = \frac{\beta}{1152} (X_{ij} + \nabla_i \nabla_j S - g_{ij} \square S)$$

Thus  $d\hat{s}_{11}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + ds_7^2$ , where the metric on  $K_7$  again satisfies

$$R_{ij} = \frac{\beta}{1152} (\nabla_i \nabla_j S + c_{ikm} c_{jln} \nabla^k \nabla^l Z^{mn})$$

The field redefinition has compensated for the absence of the dilaton, allowing us to get the same deformed  $K_7$  field equation as in string theory.

## Corrections to $(\text{Minkowski})_3 \times K_8$ in M-Theory

With curvature in 8 dimensions, the  $\beta \hat{E}_8$  and  $\beta \hat{A}_3 \wedge t \hat{R}^4$  terms contribute at order  $\beta$  to corrections to gravitational backgrounds. The relevant M-theory Lagrangian is

$$\mathcal{L} = \hat{R} \hat{*} \mathbf{1} - \frac{1}{2} \hat{*} \hat{F}_4 \wedge \hat{F}_4 - \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 + \mathcal{L}_1 \text{ with}$$

$$\mathcal{L}_1 = -\frac{\beta}{1152} (\hat{Y}_0 + \hat{E}_8 - \hat{R} \hat{S}) \hat{*} \mathbf{1} + \beta (2\pi)^4 \hat{A}_3 \wedge \hat{X}_8$$

(including the redefinition-dependent term for convenience), where

$$\hat{X}_8 = \frac{1}{192(2\pi)^4} [\text{tr} \hat{\Theta}^4 - \frac{1}{4} (\text{tr}(\hat{\Theta})^2)^2]$$

This implies the field equations

$$\begin{aligned} \hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} &= -\frac{\beta}{1152} (\square S + E_8) g_{\mu\nu} \\ \hat{R}_{ij} - \frac{1}{2} \hat{R} \hat{g}_{ij} &= \frac{\beta}{1152} (X_{ij} + \nabla_i \nabla_j S - g_{ij} \square S) \\ d\hat{*}\hat{F}_4 - \frac{1}{2} \hat{F}_4 \wedge \hat{F}_4 &= (2\pi)^4 \beta X_8 \end{aligned}$$

where as usual  $X_{ij}$  is the variation of  $\sqrt{-g}(Y_0 - E_8)$  in  $K_8$ . The Ricci-flatness of  $K_8$  is corrected to  $R_{ij} = (\beta/1152)(X_{ij} + \nabla_i \nabla_j S)$ , and we take

$$\begin{aligned}
d\hat{s}_{11}^2 &= e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + e^{-A} ds_8^2 \\
\hat{F}_4 &= d^3x \wedge df + G_4
\end{aligned}$$

If we assume for now that  $K_8$  is non-compact we can take  $G_4$ , which lives in  $K_8$ , to be zero, and the field equations then imply

$$\Box A = \frac{\beta}{1728} E_8, \quad \Box f = \beta (2\pi)^4 *X_8$$

Note that the non-zero warp factor is forced by the  $\beta(\hat{Y}_0 + \hat{E}_8 + \dots)$  correction, while the non-zero  $\hat{F}_4$  is forced by the anomaly term  $\beta \hat{A}_3 \wedge \hat{X}_8$ .

The anomaly term is  $X_8 = 1/(192(2\pi)^4) (P_1^2 - 4P_2)$ , where  $P_i$  is  $i$ 'th Pontrjagin class. It was shown (Isham & Pope 1988) that if an 8-manifold admits a nowhere-vanishing spinor (as does a special-holonomy manifold with its covariantly-constant spinor) then there is a topological relation  $P_1^2 - 4P_2 = 8\chi$ , and hence we have  $E_8 = 576(2\pi)^4 *X_8$ . This leads to

$$f = 3A$$

This is the same relation one finds in a standard M2-brane solution.



## Supersymmetry of Corrected (Minkowski)<sub>3</sub> × K<sub>8</sub>

Taking the leading-order  $K_8$  to have  $\text{Spin}(7)$  holonomy, we find its Ricci-flatness is corrected to

$$R_{ij} = \frac{\beta}{1152} \left( \frac{1}{2} c^{mn}{}_{k(i} c^{pq}{}_{l)j} \nabla^k \nabla^l Z_{mnpq} + \nabla^k \nabla_l Z_{mnk(i} c^{mnl}{}_{j)} + \nabla_i \nabla_j S \right)$$

where  $c_{ijkl}$  is the calibrating 4-form of the  $\text{Spin}(7)$  background and

$$Z^{mnpq} = \frac{1}{64} \epsilon^{mni_1 \dots i_6} \epsilon^{pqj_1 \dots j_6} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} R_{i_5 i_6 j_5 j_6}$$

We can again look for a corrected covariant derivative, whose integrability condition yields this corrected Einstein equation. We find

$$D_i \equiv \nabla_i + \frac{\beta}{1152} Q_i = \nabla_i + \frac{\beta}{4608} c_{ijkl} \nabla^j Z^{klmn} \Gamma_{mn}$$

This reduces to our previous  $G_2$  result if  $K_8 = R \times K_7$ . It also “Riemannianises” to the same 6-Gamma expression as before! So we can find a “corrected covariantly-constant” spinor in the corrected  $K_8$  background.

The gravitino transformation rule is then

$$\delta\hat{\psi}_M = \hat{\nabla}_M \hat{\epsilon} + \frac{\beta}{1152} \hat{Q}_M \hat{\epsilon} - \frac{1}{288} \hat{F}_{N_1 \dots N_4} \hat{\Gamma}_M^{N_1 \dots N_4} \hat{\epsilon} + \frac{1}{36} \hat{F}_{MN_1 \dots N_3} \hat{\Gamma}^{N_1 \dots N_3} \hat{\epsilon}$$

with  $\hat{Q}_\mu = 0$  and  $\hat{Q}_i = Q_i$ . Collecting all the contributions up to order  $\beta$ , the “M2-brane relation”  $f = 3A$  cancels spin-connection terms against field-strength terms (in a standard M2-brane fashion), and we find a Killing spinor

$$\hat{\epsilon} = e^{\frac{1}{2}A} \epsilon \otimes \eta$$

where  $\epsilon$  is constant in  $(\text{Minkowski})_3$  and  $\eta$  satisfies the corrected covariant constancy condition

$$\nabla_i \eta + \frac{\beta}{4608} c_{ijkl} \nabla^j Z^{klmn} \Gamma_{mn} \eta = 0$$

in the corrected internal space  $K_8$ . (Previous discussions omitted the  $\mathcal{O}(\beta)$  correction to the gravitino transformation rule, and some omitted the contribution from the  $\sqrt{-g}(Y_0 + E_8)$  correction to the Einstein equations.)

## (Minkowski)<sub>3</sub> × K<sub>8</sub> Solutions with Compact K<sub>8</sub>

With  $K_8$  non-compact, we could take  $G_4$  in  $\hat{F}_4 = d^3x \wedge df + G_4$  to be zero. Adding in a flux  $G_4$  in  $K_8$  is optional, provided that it is taken to be sufficiently small that  $\mathcal{O}(\beta)$  corrections involving quadratic and higher powers of  $\hat{F}_4$  in M-theory are unimportant (since we don't know in detail what they are). Hawking & Taylor-Robinson, and Becker & Becker, have studied the conditions for a solution, with preserved supersymmetry, having  $G_4 \neq 0$ . The conclusion is that it can be added provided  $G_4$  is self-dual, and that

$$G_{ijkl} \Gamma^{jkl} \eta = 0$$

where  $\eta$  is the Killing spinor on  $K_8$ .

If  $K_8$  is compact, with non-zero Euler number, the inclusion of  $G_4$  becomes obligatory. This is seen by integrating  $d\hat{F}_4 + \frac{1}{2}\hat{F}_4 \wedge \hat{F}_4 = \beta(2\pi)^4 \hat{X}_8$  over  $K_8$ , yielding

$$\int_{K_8} G_4 \wedge G_4 = \frac{\beta(2\pi)^4}{12} \chi$$

## Corrections to $(\text{Minkowski})_1 \times K_{10}$ in M-Theory

There is one further special-holonomy case that can be studied, where the curved background is a Ricci-flat Kähler 10-manifold, with  $SU(5)$  holonomy. This cannot form part of a vacuum in perturbative string theory, which has only nine spacelike dimensions. It can in principle occur in M-theory. It is an interesting example, because it probes aspects of M-theory that cannot be probed directly from light-cone studies in perturbative string theory. However, there is good reason to think that the  $\mathcal{O}(\beta)$  corrections in M-theory are valid in a genuinely eleven-dimensional sense.

The eleven-dimensional Einstein equations, with their  $\mathcal{O}(\beta)$  corrections, are given by

$$\begin{aligned}\hat{R}_{00} - \frac{1}{2}\hat{R}\hat{g}_{00} &= -\frac{\beta}{1152}\square S g_{00} + \frac{\beta}{576}\hat{E}_{00} \\ \hat{R}_{ij} - \frac{1}{2}\hat{R}\hat{g}_{ij} &= \frac{\beta}{1152}(X_{ij} + \nabla_i\nabla_j S - g_{ij}\square S) + \frac{\beta}{576}\hat{E}_{ij}\end{aligned}$$

after imposing the  $(\text{Minkowski})_1 \times K_{10}$  Ricci-flat Kähler background conditions in the correction terms on the right-hand sides. Here  $\hat{E}_{MN}$ , coming from the variation of the 8-dimensional Euler integrand  $\hat{Y}_2$ , is given by

$$\begin{aligned}\hat{E}_{00} &= \frac{1}{2}Y_2, \\ \hat{E}_i{}^j &= E_i{}^j \equiv -\frac{9!}{2^9} \delta_{ii_1 \dots i_8}^{jj_1 \dots j_8} R^{i_1 i_2}{}_{j_1 j_2} \dots R^{i_7 i_8}{}_{j_7 j_8},\end{aligned}\quad (1)$$

in the  $(\text{Minkowski})_1 \times K_{10}$  background. Expecting a warped deformation, we make the ansatz

$$d\hat{s}_{11}^2 = -e^{2A} dt^2 + e^{-\frac{1}{4}A} ds_{10}^2, \quad \hat{F}_4 = G_3 \wedge dt + G_4$$

taking, initially, only  $G_3$  non-zero in  $K_{10}$ . The corrected equations of motion then imply

$$\begin{aligned}R_{ij} &= \frac{\beta}{1152} \left( \nabla_{\hat{i}} \nabla_{\hat{j}} S + \nabla_i \nabla_j S + 2E_{ij} + \frac{1}{4}E_8 g_{ij} \right) \\ \square A &= \frac{\beta}{1728} E_8 \\ d*G_3 &= (2\pi)^4 \beta X_8\end{aligned}$$

The Ricci tensor becomes, as usual, non-zero, and the equation for the warp factor has the 8-dimensional Euler integrand  $E_8$  as its source. The equation for the 3-form  $G_3$  on  $K_{10}$  is integrable, and has  $X_8$  as source.

## Supersymmetry and $(\text{Minkowski})_1 \times K_{10}$ Solutions

For deformed special-holonomy solutions  $(\text{Minkowski})_2 \times K_8$ , we got away with making one universal addition, at order  $\alpha'^3$  (or  $\beta$ ), to the gravitino transformation rule. This was first deduced by supposing the continued supersymmetry of deformed Calabi-Yau solutions  $(\text{Minkowski})_4 \times K_6$ . It is highly non-trivial that a Riemannian gravitino correction is possible, that implies continued supersymmetry of all the deformed special-holonomy solutions. Since complete and explicit results for the transformation rules up to order  $\alpha'^3$  are not available, this is the best we have.

We certainly expect further correction terms, but presumably they would play no rôle for  $SU(3)$ ,  $G_2$  or  $Spin(7)$  backgrounds. It would have been quite possible that extra terms were needed for  $(\text{Minkowski})_1 \times K_{10}$  backgrounds. Remarkably, however, the existing corrected gravitino transformation rule implies that supersymmetry is again preserved in the  $(\text{Minkowski})_1 \times K_{10}$  background.

# Generalised Holonomy

Generalised holonomy was introduced (Duff, Liu, Hull,...) as a way to characterise the occurrence of Killing spinors in supergravity backgrounds with fluxes, which modify the usual Riemannian spin connection in the supersymmetry transformation rule

$$\delta\psi_\mu = \nabla_\mu \epsilon \quad \longrightarrow \quad \delta\psi_\mu = \mathcal{D}_\mu \epsilon$$

The integrability condition  $[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0$  extends the usual generators  $R_{\mu\nu ab}\Gamma^{ab}$  of Riemannian holonomy to an enlarged set of generators (with more Gamma-matrix structures) of generalised holonomy.

We can also employ the idea of generalised holonomy in the string-corrected backgrounds discussed earlier. In cases where the original special-holonomy manifold has dimension  $\leq 7$ , there will be no flux contributions at all, and the generalised holonomy comes just from the string-corrected supersymmetry transformation rule

$$\mathcal{D}_\mu = \nabla_\mu + \frac{3}{4}c\alpha'^3 \nabla_\nu R^\rho{}_{\mu ab} R^\nu{}_{\sigma cd} R^\sigma{}_{\rho ef} \Gamma^{abcdef}$$

The commutators of these 2-Gamma and 6-Gamma matrices generate also 10-Gamma matrices (dual to 1-Gamma), so in M-Theory the algebra closes on  $\{\Gamma_a, \Gamma_{ab}, \Gamma_{a_1 \dots a_6}\}$ .

# Generalised Structure Group in M-Theory

Splitting  $a = (0, i)$ , we have

$$\begin{array}{ll} \text{Hermitean} & : \quad \Gamma_i, \quad \Gamma_{0i}, \quad \Gamma_{0i_1 \dots i_5}, \quad 10 + 10 + 252 = 272 \\ \text{anti-Hermitean} & : \quad \Gamma_0, \quad \Gamma_{ij}, \quad \Gamma_{i_1 \dots i_6}, \quad 1 + 45 + 210 = 256 \end{array}$$

This describes the maximally-split algebra  $Sp(32)$ , with  $256 + 16$  non-compact generators (16 Cartan) and 256 compact generators. This is the *Generalised Structure Group* for purely gravitational backgrounds in M-theory. (It is  $SL(16, \mathbb{R})$  in string theory.) In specific backgrounds, we can then find the *Generalised Holonomy Group* as the subgroup realised by the non-vanishing generators.

Inclusion of the 4-form as well enlarges the generalised structure group to  $SL(32, \mathbb{R})$ . This is unaltered by the presence of the higher-order curvature correction.

We may also consider the *Generalised Transverse Structure Group* and the *Generalised Transverse Holonomy Group*. Here, we factor out the Minkowski spacetime in a corrected  $\text{Minkowski} \times K$  background, and just focus on the curved transverse space  $K$ .

We shall only include the contribution of form-field fluxes if these are *forced* by the  $\alpha'^3$  string corrections or M-theory corrections.



## Example: Seven-Dimensional Transverse Space

Consider the example of a 7-dimensional transverse space. The deformed background is purely gravitational, and since  $\Gamma_{a_1 \dots a_6}$  is dual to  $\Gamma_a$ , we have closure on  $\{\Gamma_a, \Gamma_{ab}\}$ . These generate  $SO(8)$ . This is the *Generalised Transverse Structure Group*.

We can determine the *Generalised Transverse Holonomy Group* for an originally- $G_2$  transverse space by explicit computation of the  $\alpha'^3$  curvature contribution in  $\mathcal{D}_\mu$ . We took the example of cohomogeneity-1 metrics with  $S^3 \times S^3$  principal orbits. Calculation shows the generalised transverse holonomy is  $SO(7)$ .

So at leading (uncorrected) order, the Killing spinor is the singlet in

$$SO(7) \longrightarrow G_2 : \quad 8 \longrightarrow 7 + 1$$

After including the  $\alpha'^3$  corrections, the Killing spinor is the singlet in

$$SO(8) \longrightarrow SO(7) : \quad 8 \longrightarrow 7 + 1$$

## General Dimensions of Transverse Space

Original special holonomy groups, generalised transverse structure groups and generalised transverse holonomy groups for backgrounds  $Minkowski \times K_n$  in M-theory:

$n$	Orig. Hol.	GT Struct.	GT Hol.
6	$SU(3)$	$SO(6) \times U(1)$	$SU(3) \times U(1)$
7	$G_2$	$SO(8)$	$SO(7)$
8	$Spin(7)$	$SO(8)_+ \times SO(8)_-$	$SO(8)_+ \times Spin(7)_-$
10	$SU(5)$	$SL(16, \mathbb{C})$	H

For  $K_{10}$ , an example gives

$$H = [U(1) \times SL(5, \mathbb{C}) \times SL(5, \mathbb{C})] \times [\mathbb{C}_1^{(10,1)} \oplus \mathbb{C}_3^{(10,5)}]$$

(Subscripts are  $U(1)$  charges.)

## Conclusions

- Special-holonomy backgrounds are important in string and M-theory, since they can describe supersymmetric ground states.
- In all except the special case of hyper-Kähler backgrounds, the special holonomy is either reduced (e.g.  $SU(3) \rightarrow U(3)$ ) or completely destroyed (e.g.  $G_2 \rightarrow SO(7)$ ) by the  $\mathcal{O}(\alpha'^3)$  corrections.
- In a generalised sense, however, the structure of the special holonomy group survives; the  $\alpha'^3$ -corrected gravitino transformation rule still implies the existence of Killing spinors. Generalised holonomy.
- We still lack a full *derivation* of the order  $\alpha'^3$  corrections to the gravitino transformation rule.
- $(\text{Minkowski})_1 \times K_{10}$  backgrounds with  $SU(5)$  holonomy provide a rich arena for probing the structure of M-theory.