Constructing compact 7-manifolds with holonomy G_2

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Riemannian geometry

Let M^n be a manifold of dimension n. Let $x \in M$. Then T_xM is the *tangent space* to M at x.

Let g be a Riemannian metric on M.

Let ∇ be the *Levi-Civita connection* of g.

Let $R(g)$ be the *Riemann curvature* of g.

Holonomy groups

Fix $x \in M$. The *holonomy group* $\mathrm{Hol}(g)$ of g is the set of isometries of T_xM given by *parallel transport* using ∇ about closed loops γ in M based at x. It is a subgroup of $O(n)$. Up to conjugation, it is independent of the basepoint x.

Berger's classification

Let M be simply-connected and g be irreducible and nonsymmetric. Then Hol(g) is one of $SO(m)$, $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m)Sp(1)$ for $m \geq 2$, or G_2 or $Spin(7)$. We call G_2 and $Spin(7)$ the *exceptional holonomy groups*. Dim(M) is 7 when Hol(g) is G_2 and 8 when Hol(g) is $Spin(7)$.

Understanding Berger's list

The four *inner product algebras* are

$\mathbb{R}$ — *real numbers*.

$\mathbb{C}$ — *complex numbers*.

$\mathbb{H}$ — *quaternions*.

$\mathbb{O}$ — *octonions*,

or *Cayley numbers*.

Here $\mathbb{C}$ is not ordered,

$\mathbb{H}$ is not commutative,

and $\mathbb{O}$ is not associative.

Also we have $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$

and $\mathbb{O} \cong \mathbb{R}^8$, with $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$.

Group	**Acts on**
$SO(m)$	$\mathbb{R}^m$
$O(m)$	$\mathbb{R}^m$
$SU(m)$	$\mathbb{C}^m$
$U(m)$	$\mathbb{C}^m$
$Sp(m)$	$\mathbb{H}^m$
$Sp(m)Sp(1)$	$\mathbb{H}^m$
G_2	$\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$
$Spin(7)$	$\mathbb{O} \cong \mathbb{R}^8$

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Geometry of G_2

The action of G_2 on $\mathbb{R}^7$ preserves the metric g_0 and a 3-form φ_0 on $\mathbb{R}^7$. Let g be a metric and φ a 3-form on M^7. We call (φ, g) a G_2-*structure* if $(\varphi, g) \cong (\varphi_0, g_0)$ at each $x \in M$. We call $\nabla\varphi$ the *torsion* of (φ, g).

If $\nabla\varphi = 0$ then (φ, g) is *torsion-free*. Also $\nabla\varphi = 0$ iff $d\varphi = d^*\varphi = 0$. If (φ, g) is torsion-free then $\mathrm{Hol}(g) \subseteq G_2$. Conversely, if g is a metric on M^7 then $\mathrm{Hol}(g) \subseteq G_2$ iff there is a G_2-structure (φ, g) with $\nabla\varphi = 0$. If M is compact and $\mathrm{Hol}(g) \subseteq G_2$ then $\mathrm{Hol}(g) = G_2$ iff $\pi_1(M)$ is finite.

The goal of the talk

To construct examples of compact 7-manifolds with holonomy G_2.

Why is this difficult?

In many problems in geometry the simplest examples are symmetric. But compact 7-manifolds with holonomy G_2 have no continuous symmetries. They are not algebraic.

Sketch of constructions:
First we choose a compact 7-manifold M. We write down an explicit G_2-structure (φ, g) on M with *small torsion*. Then we use analysis to deform to a nearby G_2-structure $(\tilde{\varphi}, \tilde{g})$ with *zero torsion.* If $\pi_1(M)$ is finite then $\mathrm{Hol}(\tilde{g}) = G_2$ as we want.

It is not easy to find G_2-structures with small torsion! Here are three methods (A)–(C).

(A) Joyce (1995,2000)

Step 1. Choose a finite group Γ of isometries of the 7-torus T^7, and a flat, Γ-invariant G_2-structure (φ_0, g_0) on T^7. Then T^7/Γ is compact, with a torsion-free G_2-structure (φ_0, g_0).

Step 2. However, T^7/Γ is an *orbifold*. We repair its singularities to get a compact 7-manifold M. We can resolve *complex* orbifolds using algebraic geometry.

If the singularities of T^7/Γ locally resemble $\mathcal{S}^1 \times \mathbb{C}^3/G$ for $G \subset SU(3)$, then we model M on a *crepant resolution* X of $\mathbb{C}^3/G$.

Step 3. M is made by gluing patches $\mathcal{S}^1 \times X$ into T^7/Γ. Now X carries ALE metrics of holonomy $SU(3)$. As $SU(3) \subset G_2$, these give torsion-free G_2-structures on $\mathcal{S}^1 \times X$.
We join them to (φ_0, g_0) on T^7/Γ to get a family $\{(\varphi_t, g_t) : t \in (0, \epsilon)\}$ of G_2-structures on M.

Step 4. This (φ_t, g_t) has $\nabla\varphi_t = O(t^4)$. So $\nabla\varphi_t$ is small for small t. But $R(g_t) = O(t^{-2})$ and the injectivity radius $\delta(g_t) = O(t)$, since g_t becomes singular as $t \to 0$.

For small t we deform (φ_t, g_t) to $(\tilde\varphi_t, \tilde g_t)$ with $\nabla\tilde\varphi_t = 0$, using analysis. Then Hol$(\tilde g_t) = G_2$ if $\pi_1(M)$ is finite.

Other constructions

(B) Kovalev (2003)

Use Calabi–Yau analysis to construct Asymptotically Cylindrical Calabi–Yau 3-folds X_1, X_2 with one end asymptotic to $K3 \times \mathcal{S}^1 \times (0, \infty)$. Then $X_1 \times \mathcal{S}^1$ and $X_2 \times \mathcal{S}^1$ are G_2-manifolds asymptotic to $K3 \times T^2 \times (0, \infty)$.

Glue $X_1 \times \mathcal{S}^1$ and $X_2 \times \mathcal{S}^1$ together near infinity to get a compact G_2-manifold M with small torsion, then deform to zero torsion as before. The gluing swaps the two $\mathcal{S}^1$ factors. The two $K3$ surfaces must be related by a hyperkähler rotation.

(C) Joyce–Karigiannis (2007-2027?)

Let X be a Calabi–Yau 3-fold and $\sigma : X \to X$ an antiholomorphic isometric involution. Let L be the fixed point set of σ, a *special Lagrangian 3-fold* in X. Let $\mathcal{S}^1$ be $x^2 + y^2 = 1$ in $\mathbb{R}^2$ and τ act on $\mathcal{S}^1$ by $(x, y) \mapsto (x, -y)$, fixing $(\pm 1, 0)$.

Then $X \times \mathcal{S}^1$ is a G_2-manifold invariant under (σ, τ). So $(X \times \mathcal{S}^1)/\langle(\sigma, \tau)\rangle$ is a G_2-orbifold, with singular set $L \times \{(\pm 1, 0)\}$. The singularities locally look like $\mathbb{R}^4/\{\pm 1\} \times \mathbb{R}^3$.

To resolve $\mathbb{R}^4/\{\pm 1\}$, use an *Eguchi–Hanson space* Y, with holonomy SU(2). The family of E–H spaces is $\mathbb{R}^3 \setminus \{0\}$.

To resolve the singularities of $(X \times \mathcal{S}^1)/\langle(\sigma,\tau)\rangle$ to get compact M with holonomy G_2, we glue in a *family* of Eguchi–Hanson spaces Y_x parametrized by x in $L \times \{(\pm 1, 0)\}$. To choose the family we need a closed, coclosed 1-form α on $L \times \{(\pm 1, 0)\}$ which is nonzero at every point. Not yet proved – v. hard.

Including singularities
To make compact, *singular* G_2-manifolds we could modify the constructions above as follows.
(A) Leave some T^7/Γ singularities unresolved. Gives an orbifold M with holonomy G_2. Singularities always non-isolated, dim 1 or 3. (Easy. Immediate from known work.)

(B) Start with noncompact Calabi–Yau 3-folds X_1 or X_2 with k isolated conical singularities, e.g. conifold. Then M^7 is singular along k copies of $\mathcal{S}^1$. The local model is $\mathcal{S}^1 \times C$, for C a Calabi–Yau 3-fold cone.

(Difficult. Not done.)

(C) Use a closed, coclosed 1-form α on $L\times\{(\pm 1,0)\}$ which has k isolated generic zeroes $x_1,\ldots,x_k$. Then expect the construction to yield M with k isolated singular points, each topologically a cone on $\mathbb{CP}^3$. Probably modelled on Bryant–Salamon $\mathbb{CP}^3$ cone with holonomy G_2. (Very difficult. Not done.)