Physics 523, Quantum Field Theory II

Homework 7

Due Wednesday, 3rd March 2004

Jacob Lewis Bourjaily

Superficial Divergences

Let us consider φ^3 scalar field theory in $d = 4$ dimension. The Lagrangian for this theory is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3.$$

a) Let us determine the superficial divergence D for this theory in terms of the number of vertices V and the number of external lines N. From this we are to show that the theory is super-renormalizable.

In generality, the superficial divergence of a φ^n theory in d dimensions can be given by $D = dL - 2P$, where L is the number of loops and P is the number of propagators because each loop contributes a d-dimensional integration and each propagator contributes a power of 2 in the denominator. Furthermore, we see that $nV = N + 2P$ because each external line connects to one vertex and each propagator connects two and each vertex involves n lines. This implies that $P = \frac{1}{2}(nV - N)$.

Therefore, still in complete generality, the superficial divergence of a φ^n theory in d-dimensions may be written

$$\begin{aligned} D = dL - 2P &= \frac{d}{2}nV - \frac{d}{2}N - dV + d - nV + N, \\ &= d + \left(n\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N. \end{aligned}$$

Therefore, in a 4-dimensional φ^3-theory the superficial divergence is given by

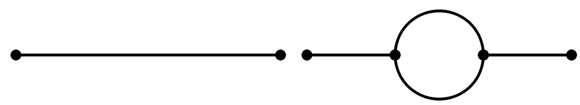
$$\boxed{D = 4 - V - N.} \tag{1.a.1}$$

ὅπερ ἔδει δεῖξαι

We see that because $D \propto -V$ the theory is *super-renormalizable*.

b) We are to show the superficially divergent diagrams for this theory that are associated with the exact two-point function.

Using equation (1.a) above, we see that the three superficially divergent diagrams in this φ^3-theory associated with the exact two-point function are:



c) Let us compute the mass dimension of the coupling constant g.

Because $\mathscr{L}$ must have dimension (mass)^4 each term should have dimension (mass)^4. Because of the $m^2\varphi^2$ term, this implies that the field φ has dimension (mass)^1. Therefore the coupling g must have dimension (mass)^1.

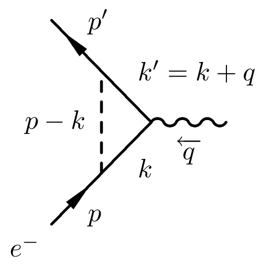
Renormalization and the Yukawa Coupling

We are to consider the theory of elementary fermions that couple to both QED and a Yukawa field ϕ governed by the interaction Hamiltonian

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}}\phi\overline{\psi}\psi + \int d^3x e A_\mu \overline{\psi}\gamma^\mu\psi.$$

a) Let us verify that $\delta Z_1 = \delta Z_2$ to the one-loop order.

We computed in homework 4 the amplitude for the $\overline{\psi}\gamma\psi$ vertex with a virtual scalar ϕ,

$i\mathscr{M} =$ 

$$= \int \frac{d^4k}{(2\pi)^4}\overline{u}(p')\frac{-i\lambda}{\sqrt{2}}\frac{i}{((p-k)^2 - m_\phi^2 + i\epsilon)}\frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)}(-ie\gamma^\mu)\frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)}\frac{-i\lambda}{\sqrt{2}}u(p),$$

In the limit where $q \to 0$, we see that this implies

$$\overline{u}(p)\delta\Gamma^\mu u(p) = i\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{\overline{u}(p)\left[(\not{k}+m)\,\gamma^\mu\,(\not{k}+m)\right]u(p)}{((p-k)^2-m_\phi^2+i\epsilon)(k^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}.$$

Using Feynman parametrization to simplify the denominator, we will use the variables

$$\ell \equiv k - zp \qquad \text{and} \qquad \Delta \equiv (1-z)^2m^2 + zm_\phi^2.$$

The numerator of the integrand is then reduced to

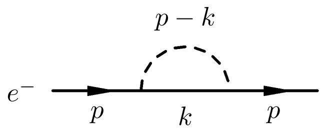
$$\begin{aligned}
\mathscr{N} &= \overline{u}(p)\left[(\not{\ell}+z\not{p}+m)\,\gamma^\mu\,(\not{\ell}+z\not{p}+m)\right]u(p),\\
&= \overline{u}(p)\left[\not{\ell}\gamma^\mu\not{\ell}+z^2\not{p}\gamma^\mu\not{p}+mz\not{p}\gamma^\mu+mz\gamma^\mu\not{p}+m^2\gamma^\mu\right]u(p),\\
&= \overline{u}(p)\left[\frac{1}{d}\ell^2(2\gamma^\mu-d\gamma^\mu)+z^2m^2\gamma^\mu+m^2z\gamma^\mu+m^2z\gamma^\mu+m^2\gamma^\mu\right]u(p),\\
&= \overline{u}(p)\left[\gamma^\mu\left(\frac{2-d}{d}\ell^2+m^2(1+z)^2\right)\right]u(p).
\end{aligned}$$

Combining this with our work above, we see that this implies

$$\begin{aligned}
\delta Z_1 = -\delta F_1(q=0) &= -i\frac{\lambda^2}{2}\int_0^1 dz(1-z)2\int\frac{d^d\ell}{(2\pi)^d}\left[\frac{\left(\frac{2-d}{d}\right)\ell^2}{[\ell^2-\Delta+i\epsilon]^3}+\frac{m^2(1+z)^2}{[\ell^2-\Delta+i\epsilon]^3}\right],\\
&= -i\frac{\lambda^2}{2}\int_0^1 dz(1-z)\left[\frac{2-d}{d}\frac{d}{2}\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}-\frac{i}{(4\pi)^2}\frac{m^2(1+z)^2}{\Delta}\right],\\
&\simeq \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[\frac{2-d}{2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right],\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[\frac{\epsilon-2}{2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right],
\end{aligned}$$

$$\boxed{\therefore\,\delta Z_1 = \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[1-\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right]} \tag{2.a.1}$$

Let us now compute the one-loop contribution of ϕ to the electron two-point function,



$$\Longrightarrow \Sigma_{\phi_2} = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)}{((p-k)^2-m_\phi^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

We will define the following variables for Feynman parametrization of the denominator:

$$\ell \equiv k - zp, \qquad \text{and} \qquad \Delta \equiv -z(1-z)\not{p}^2 + zm_\phi^2 + (1-z)m^2.$$

We see therefore that

$$\begin{aligned}
\Sigma_{\phi_2} &= i\frac{\lambda^2}{2}\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{z\not{p}+m}{[\ell^2-\Delta+i\epsilon]^2},\\
&= i\frac{\lambda^2}{2}\int_0^1 dz(z\not{p}+m)\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}},\\
&\simeq -\frac{\lambda^2}{32\pi^2}\int_0^1 dz(z\not{p}+m)\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right).
\end{aligned}$$

Therefore,

$$\delta Z_2 = \left.\frac{\partial\Sigma_{\phi_2}}{\partial\not{p}}\right|_{\not{p}=m} = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\left[z\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)+(zm+m)\frac{2mz(1-z)}{\Delta}\right],$$

$$\boxed{\therefore\,\delta Z_2 = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\left[z\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)+\frac{2m^2z(1+z)(1-z)}{\Delta}\right].} \tag{2.a.2}$$

Let us now compute the difference $\delta Z_2 - \delta Z_1$. We see that

$$\delta Z_2 - \delta Z_1 = \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-2z)\log\left(\frac{1}{\Delta}\right) + (1-2z)\left(\frac{2}{\epsilon} - \gamma_E + \log(4\pi)\right) - (1-z) - \frac{m^2(1-z)(1+z)}{\Delta}\left(2z-(1+z)\right)\right],$$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-2z)\log\left(\frac{1}{\Delta}\right) - (1-z) + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-z) - \frac{m^2(1-z)(1-z^2)}{\Delta} - (1-z) + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[-\frac{m^2(1-z)^2(1+z)}{\Delta} + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$

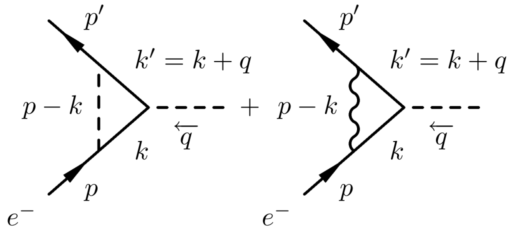
$$\boxed{\therefore \delta Z_2 - \delta Z_1 = 0.} \tag{2.a.3}$$

ὅπερ ἔδει δεῖξαι

We can expect that $Z_1 = Z_2$ quite generally in this theory because our proof of the Ward-Takahashi identity relied, fundamentally, on the local $U(1)$ gauge invariance of the A_μ term in the Lagrangian which is not altered by the addition of the scalar ϕ.

b) Let us now consider the renormalization of the $\overline{\psi}\phi\psi$ vertex in this theory.

The two diagrams at the one-loop level that contribute to $\overline{u}(p')\delta\Gamma u(p)$ are



These diagrams yield

$$\overline{u}(p')\delta\Gamma u(p) = \int\frac{d^dk}{(2\pi)^d}\overline{u}(p')\left[\left(-i\frac{\lambda}{\sqrt{2}}\right)\frac{i}{((p-k)^2 - m_\phi^2 + i\epsilon)}\frac{i(\not{k}+\not{q}+m)}{((k+q)^2 - m^2 + i\epsilon)}\frac{i(\not{k}+m)}{(k^2-m^2+i\epsilon)}\left(-i\frac{\lambda}{\sqrt{2}}\right)\right.$$
$$\left. + (-ie\gamma^\mu)\frac{i(\not{k}+\not{q}+m)}{((k+q)^2-m^2)}\frac{-i}{((p-k)^2-\mu^2)}\frac{i(\not{k}+m)}{(k^2-m^2)}(-ie\gamma_\mu)\right]u(p).$$

Taking the limit where $q \to 0$ and introducing the variables

$$\ell \equiv k - zp, \qquad \Delta_1 \equiv (1-z)^2m^2 + zm_\phi^2, \quad \text{and} \quad \Delta_2 \equiv (1-z)^2m^2 + z\mu^2,$$

this becomes,

$$\overline{u}(p)\delta\Gamma u(p) = \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\overline{u}(p)\left[i\lambda^2\frac{\ell^2 + (1+z)^2m^2}{(\ell^2-\Delta_1+i\epsilon)^3} - 2ie^2\frac{d\ell^2 + m^2\left(d(z^2+1) + 2z(2-d)\right)}{(\ell^2-\Delta_2+i\epsilon)^3}\right]u(p).$$

Therefore,

$$\delta Z_1' = -\delta F_1' = \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\left[-i\lambda^2\frac{\ell^2 + (1+z)^2m^2}{(\ell^2-\Delta_1+i\epsilon)^3} + 2ie^2\frac{d\ell^2 + m^2\left(d(z^2+1)+2z(2-d)\right)}{(\ell^2-\Delta_2+i\epsilon)^3}\right],$$

$$= \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\left[-i\lambda^2\frac{\ell^2}{(\ell^2-\Delta_1+i\epsilon)^3} + 2ie^2\frac{d\ell^2}{(\ell^2-\Delta_2+i\epsilon)^3}\right] + \text{ finite terms},$$

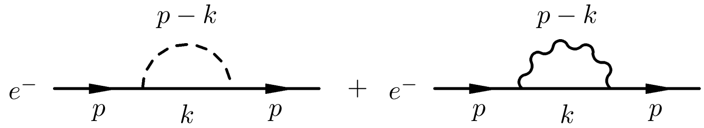
$$= \int_0^1 dz(1-z)\left[\frac{\lambda^2}{4}\frac{d}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta_1^{2-d/2}} - \frac{e^2}{2}\frac{d^2}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta_2^{2-d/2}}\right] + \text{ finite terms},$$

$$= \int_0^1 dz(1-z)\left[\frac{\lambda^2}{16\pi^2}\left(\frac{2}{\epsilon} - \log\Delta_1 - \gamma_E + \log(4\pi) - \frac{1}{2}\right) - \frac{2\alpha}{\pi}\left(\frac{2}{\epsilon} - \log\Delta_2 - \gamma_E + \log(4\pi) - 1\right)\right] + \text{ finite terms},$$

$$= \int_0^1 dz(1-z)\frac{2}{\epsilon}\left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi}\right) + \text{ finite terms},$$

$$\boxed{\therefore \delta Z_1' = \frac{1}{\epsilon}\left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi}\right) + \text{ finite terms.}} \tag{2.b.2}$$

Now let us compute $\delta Z_2'$. We see that this factor comes from the diagrams,



We see that we have already computed both of these contributions; the first diagram's contribution was computed above and the second diagram's contribution was computed in homework 6.

Therefore, we note that

$$\boxed{\delta Z_2' = \frac{1}{\epsilon}\left(-\frac{\lambda^2}{32\pi^2} - \frac{\alpha}{2\pi}\right) + \text{ finite terms.}} \tag{2.b.3}$$

Combining these results, we have that

$$\boxed{\therefore\, \delta Z_2' - \delta Z_1' = \frac{3}{\epsilon}\left(\frac{\alpha}{2\pi} - \frac{\lambda^2}{32\pi^2}\right) + \text{ finite terms} \neq 0.} \tag{2.b.4}$$

ὅπερ ἔδει δεῖξαι