

PHYSICS 523, QUANTUM FIELD THEORY II

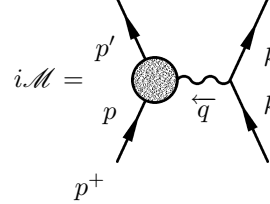
Homework 3

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JACOB LEWIS BOURJAILY

The Rosenbluth Formula

We are to prove the *Rosenbluth Formula* by considering the elastic scattering of a relativistic electron off of a proton while correcting the vertex function of the proton. The amplitude for this process is,



$$i\mathcal{M} = \bar{u}(k')(-ie\gamma_\mu)u(k)\frac{-i}{q^2}\bar{u}(p')(-ie\Gamma^\mu)u(p).$$

- a) Let us simplify the amplitude using the Gordon identity. Recall that we showed in class that the generalized vertex function Γ^μ may be written in terms of functions $F_1(q^2)$ and $F_2(q^2)$ as

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Inserting this into the amplitude and recalling the Gordon identity, we see that

$$\begin{aligned} i\mathcal{M} &= i\frac{e^2}{q^2}\bar{u}(k')\gamma_\mu u(k)\bar{u}(p')\Gamma^\mu u(p), \\ &= i\frac{e^2}{q^2}\bar{u}(k')\gamma_\mu u(k)\bar{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\right)u(p), \\ &= i\frac{e^2}{q^2}\bar{u}(k')\gamma_\mu u(k)\bar{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2 + \frac{(p'+p)^\mu}{2m}F_2 - \frac{(p'+p)^\mu}{2m}F_2\right)u(p), \\ &= i\frac{e^2}{q^2}\bar{u}(k')\gamma_\mu u(k)\bar{u}(p')\left(\gamma^\mu(F_1 + F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p), \\ &\therefore \Gamma^\mu = \gamma^\mu(F_1 + F_2) - \frac{(p'+p)^\mu}{2m}F_2. \end{aligned}$$

- b) Let us compute the spin-averaged amplitude squared directly. We see that

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{e^4}{4q^4}\sum_{\text{spin}}\bar{u}(k')\gamma_\mu u(k)\bar{u}(p')\left(\gamma^\mu(F_1 + F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p)\bar{u}(p)\left(\gamma^\nu(F_1 + F_2) - \frac{(p'+p)^\nu}{2m}F_2\right)u(p')\bar{u}(k)\gamma_\nu u(k'), \\ &= \frac{e^4}{4q^4}\text{Tr}[(\not{k}' + m_e)\gamma_\mu(\not{k} + m_e)\gamma_\nu] \times \\ &\quad \left\{ (F_1 + F_2)^2\text{Tr}[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu] - F_2(F_1 + F_2)\frac{(p'+p)^\nu}{2m}\text{Tr}[(\not{p}' + m)\gamma^\mu(\not{p} + m)] \right. \\ &\quad \left. - F_2(F_1 + F_2)\frac{(p'+p)^\mu}{2m}\text{Tr}[(\not{p}' + m)(\not{p} + m)\gamma^\nu] + F_2^2\frac{(p'+p)^\mu(p'+p)^\nu}{4m^2}\text{Tr}[(\not{p}' + m)(\not{p} + m)] \right\}, \\ &= \frac{4e^4}{q^4}(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k' \cdot k - m_e^2)) \times \\ &\quad \left\{ (F_1 + F_2)^2(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p' \cdot p - m^2)) - F_2(F_1 + F_2)(p'+p)^\mu(p'+p)^\nu \right. \\ &\quad \left. + \frac{F_2^2}{4m^2}(p'+p)^\mu(p'+p)^\nu(p' \cdot p + m^2) \right\}, \\ &= \frac{4e^4}{q^4}(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k' \cdot k - m_e^2)) \times \\ &\quad \left\{ (F_1 + F_2)^2(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p' \cdot p - m^2)) + (p'+p)^\mu(p'+p)^\nu \left(\frac{p' \cdot p + m^2}{4m^2}F_2^2 - F_2(F_1 + F_2) \right) \right\}, \\ \therefore \overline{|\mathcal{M}|^2} &= \frac{8e^4}{q^4}\left[(F_1 + F_2)^2(k' \cdot p'k \cdot p + k' \cdot pk \cdot p - k' \cdot km^2 - p' \cdot pm_e^2 + 2m^2m_e^2) \right. \\ &\quad \left. + \left(\frac{p' \cdot p + m^2}{4m^2}F_2^2 - F_2(F_1 + F_2) \right) \left(k' \cdot (p'+p)k \cdot (p'+p) - \frac{1}{2}(k' \cdot k - m_e^2)(p'+p)^2 \right) \right]. \end{aligned}$$

- c) Let us consider the kinematics of this reaction in the initial rest frame of the proton. In this frame we see that $p = (m, \vec{0})$, $k = (E, E\hat{z})$, $k' = (E', \vec{k}')$, $p' = (E - E' + m, -\vec{k})$ with $|\vec{k}'| = E'$. We have defined the momentum transfer q such that $p' - p = q = k - k'$.

Noting that $p \cdot p' = m^2 + Em - E'm$, let us compute p'^2 .

$$\begin{aligned} p'^2 &= (p+q)^2 = p^2 + 2p \cdot q + q^2 = m^2 + 2p \cdot (p' - p) + q^2 = -m^2 + 2p' \cdot p + q^2 = m^2 + 2Em - 2E'm + q^2 = m^2, \\ &\implies q^2 = 2E'm - 2Em, \\ &\therefore E' = E + \frac{q^2}{2m}. \end{aligned}$$

If we write $k' = (E', 0, E' \sin \theta, \cos \theta)$ so that $q = (E - E', 0, -E' \sin \theta, E - E' \cos \theta)$ we see

$$q^2 = E'^2 - 2EE' + E^2 - E'^2 - E'^2 \sin^2 \theta - E^2 + 2EE' \cos \theta - E'^2 \cos^2 \theta = 2EE'(\cos \theta - 1) = -4EE' \sin^2 \frac{\theta}{2}.$$

Using our identity derived above that $E' = E + \frac{q^2}{2m}$, we may conclude that

$$\begin{aligned} q^2 &= -4E^2 \sin^2 \frac{\theta}{2} - \frac{q^2}{2m} 4E \sin^2 \frac{\theta}{2}, \\ \therefore q^2 &= -\frac{4E^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}. \end{aligned}$$

Let us now compute all of the required inner products to compute the desired amplitude squared. Noting that $p^2 = p'^2 = m^2$, $k^2 = k'^2 = 0$, and $p \cdot k = Em$ we may derive all of our necessary identities and inner products indirectly (it's more fun that way). We notice that

$$\begin{aligned} p'^2 &= m^2 = p^2 + 2p \cdot q + q^2 = -m^2 + 2p' \cdot p + q^2, \\ \therefore p' \cdot p &= m^2 - \frac{q^2}{2}. \end{aligned}$$

Similarly,

$$p'^2 = m^2 = p^2 + 2p \cdot q + q^2 = m^2 + 2p \cdot k - 2p \cdot k' + q^2,$$

but we know that $p \cdot k = Em$,

$$\therefore p \cdot k' = Em + \frac{q^2}{2}.$$

Likewise,

$$\begin{aligned} k'^2 &= 0 = k^2 - 2k \cdot q + q^2 = 2k \cdot k' + q^2 = 0, \\ \therefore k' \cdot k &= -\frac{q^2}{2}. \end{aligned}$$

And

$$k'^2 = 0 = k^2 - 2k \cdot q + q^2 = -2k \cdot p' + 2k \cdot p + q^2,$$

where we know that $k \cdot p = Em$ and

$$\therefore k \cdot p' = Em + \frac{q^2}{2}.$$

Similarly,

$$\begin{aligned} k^2 &= 0 = k'^2 + 2q \cdot k' + q^2 = 2p' \cdot k' + q^2, \\ \therefore p' \cdot k' &= Em. \end{aligned}$$

Tabulating our results, we have shown that

$$\begin{array}{lll} k' \cdot k = -\frac{q^2}{2} & p' \cdot p = m^2 - \frac{q^2}{2} & k' \cdot p = Em + \frac{q^2}{2} \\ p' \cdot k = Em + \frac{q^2}{2} & k' \cdot p' = Em & p \cdot k = Em. \end{array}$$

These imply that

$$k \cdot (p' + p) = 2Em + \frac{q^2}{2}, \quad k' \cdot (p' + p) = 2Em + \frac{q^2}{2}, \quad \text{and} \quad (p + p')^2 = 4m^2 - q^2.$$

- d) We are to use the kinematic information derived in part (c) above to rewrite the spin-averaged amplitude squared into a more convenient form. Recall that

$$|\overline{\mathcal{M}}|^2 = \frac{8e^4}{q^4} \left[\overbrace{(F_1 + F_2)^2 (k' \cdot p' k \cdot p + k' \cdot p k \cdot p - k' \cdot k m^2 - p' \cdot p m_e^2 + 2m^2 m_e^2)}^{\text{i}} + \underbrace{\left(\frac{p' \cdot p + m^2}{4m^2} F_2^2 - F_2(F_1 + F_2) \right)}_{\text{ii}} \underbrace{\left(k' \cdot (p' + p) k \cdot (p' + p) - \frac{1}{2} (k' \cdot k - m_e^2) (p' + p)^2 \right)}_{\text{iii}} \right].$$

We note that in the approximation where $k^2 \sim 0$, we should set $m_e \rightarrow 0$. Let us compute each part separately first before combining the results.

- i. $(k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - (k' \cdot k)m^2 = (Em)^2 + (Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2.$
- ii.
$$\begin{aligned} \frac{p' \cdot p + m^2}{4m^2} F_2^2 - F_1 F_2 - F_2^2 &= \frac{1}{2} F_2^2 - \frac{q^2}{8m^2} F_2^2 - F_1 F_2 - F_2^2, \\ &= -\frac{1}{2} \left[(F_2^2 + 2F_1 F_2 + F_1^2 - F_1^2 + \frac{q^2}{4m^2} F_2^2) \right], \\ &= -\frac{1}{2} \left[((F_1 + F_2)^2 - \left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right)) \right]. \end{aligned}$$
- iii. $(k' \cdot (p' + p))(k \cdot (p' + p)) - \frac{1}{2} (k' \cdot k) (p' + p)^2 = 4(Em)^2 + 2Emq^2 + \frac{q^4}{4} + q^2 m^2 - \frac{q^4}{4}.$

Combining these results, we see that the coefficient for the $(F_1 + F_2)^2$ term will be

$$2(Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2 - 2(Em)^2 - Emq^2 - \frac{q^2}{2}m^2 = \frac{q^4}{4},$$

which can be written,

$$\frac{q^4}{4} = \frac{q^2}{2} \frac{q^2}{2} = -\frac{2E^2 m^2}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}} \frac{q^2}{2m^2}.$$

Similarly, we will combine the results above to compute the coefficient for the $(F_1^2 - \frac{q^2}{4m^2} F_2^2)$ term.

$$\begin{aligned} 2(Em)^2 + Emq^2 + \frac{q^2}{2}m^2 &= 2E^2 m^2 - \frac{4E^3 m \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}} - \frac{2E^2 m^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}, \\ &= \frac{2E^2 m^2 + 4E^3 m \sin^2 \frac{\theta}{2} - 4E^3 m \sin^2 \frac{\theta}{2} - 2m^2 E^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}, \\ &= 2E^2 m^2 \frac{1 - \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}} = \frac{2E^2 m^2 \cos^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}. \end{aligned}$$

Therefore, combining all of these results, the total spin-average amplitude squared becomes

$$|\overline{\mathcal{M}}|^2 = \frac{16e^4 E^2 m^2}{q^4 \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}\right)} \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right].$$

$\dot{\theta}\pi\epsilon\rho \dot{\epsilon}\delta\epsilon\iota \delta\epsilon\iota\xi\alpha\iota$

- e) Let us compute the differential cross section, $\frac{d\sigma}{d\cos\theta}\Big|_{\text{lab}}$. To do this, we will compute the cross section in most general terms. From elementary considerations, we calculated that

$$\begin{aligned} d\sigma &= \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f), \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \frac{d^3p' d^3k'}{(2\pi)^2} \frac{1}{4E'E'_p} \delta^{(4)}(p + k - p' - k'). \end{aligned}$$

We see that this is so because $E_{\mathcal{A}} = m$, $E_{\mathcal{B}} = E$, $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$ and there are two final states.

Let us now integrate over $d\sigma$ to find its dependence on $\cos\theta$. During the derivation, we will make use of the fact that $E + m = E'_p + E'$ by energy conservation enforced by the dirac δ function.

We will also call upon our results above to use the identities $E' = E + \frac{q^2}{2m}$ and $q^2 = \frac{4E^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}$. Notice the insertion of the Jacobian for the change of variables to integrate over the energy portion of the δ function in line 4. We will now proceed directly by first integrating over the p' part of the integral.

$$\begin{aligned} \sigma &= \int d\sigma = \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{d^3p' d^3k'}{(2\pi)^2} \frac{1}{4E'E'_p} \delta^{(4)}(p + k - p' - k'), \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{d^3k'}{(2\pi)^2} \frac{1}{4E'E'_p} \delta^{(1)}\left(E' - E - m + \sqrt{m^2 + E^2 + E'^2 - 2EE' \cos\theta}\right), \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{E'^2 dE d\Omega}{(2\pi)^2} \frac{1}{4E'E'_p} \delta^{(1)}\left(E' - E - m + \sqrt{m^2 + E^2 + E'^2 - 2EE' \cos\theta}\right), \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{d\Omega}{(2\pi)^2} \frac{E'}{4E'_p} \left(1 + \frac{E' - E \cos\theta}{E'_p}\right)^{-1}, \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{d\Omega}{(2\pi)^2} \frac{E'}{4E'_p} \left(\frac{E'_p}{E'_p + E' - E \cos\theta}\right), \\ &= \frac{1}{4mE} |\overline{\mathcal{M}}|^2 \int \frac{d\cos\theta}{(2\pi)} \frac{E'}{4E'_p} \left(\frac{E'_p}{E'_p + E' - E \cos\theta}\right), \\ &= \frac{1}{32\pi m E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E'}{m + E(1 - \cos\theta)}, \\ &= \frac{1}{32\pi m^2 E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E'}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}, \\ &= \frac{1}{32\pi m^2 E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E + \frac{q^2}{2m}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}, \\ &= \frac{1}{32\pi m^2 E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E - \frac{2E^2 \sin^2 \frac{\theta}{2}}{m(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})}}{1 + \frac{2E}{m} \sin^2 \frac{\theta}{2}}, \\ &= \frac{1}{32\pi m^2 E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E + \frac{2E^2}{m} \sin^2 \frac{\theta}{2} - \frac{2E^2}{m} \sin^2 \frac{\theta}{2}}{(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})^2}, \\ &= \frac{1}{32\pi m^2 E} |\overline{\mathcal{M}}|^2 \int d\cos\theta \frac{E}{(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})^2}, \\ &= \frac{1}{32\pi m^2 (1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})^2} |\overline{\mathcal{M}}|^2 \int d\cos\theta, \\ &= \frac{1}{32\pi m^2 (1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})^2} |\overline{\mathcal{M}}|^2 \cos\theta, \\ &\quad \therefore \frac{d\sigma}{d\cos\theta}\Big|_{\text{lab}} = \frac{1}{32\pi m^2 (1 + \frac{2E}{m} \sin^2 \frac{\theta}{2})^2} |\overline{\mathcal{M}}|^2. \end{aligned}$$

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f) We will now derive the Rosenbluth formula. From our work above, we see that

$$\begin{aligned} \left. \frac{d\sigma}{d\cos\theta} \right|_{\text{lab}} &= \frac{16e^4 E^2 m^2 \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]}{32\pi m^2 \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)^2 q^4 \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)}, \\ &= \frac{e^4 E^2 \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]}{2\pi \frac{16E^4 \sin^4 \frac{\theta}{2}}{\left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)^2} \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)^2 \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)}, \\ &= \frac{e^4 \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]}{32\pi E^2 \sin^4 \frac{\theta}{2} \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)}, \end{aligned}$$

$$\therefore \left. \frac{d\sigma}{d\cos\theta} \right|_{\text{lab}} = \frac{\pi\alpha^2 \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]}{2E^2 \sin^4 \frac{\theta}{2} \left(1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right)}.$$

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