Physics 513, Quantum Field Theory

Homework 9

Due Tuesday, 18th November 2003

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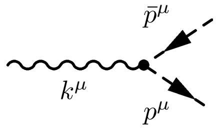
The Decay of Vector into Two Scalars

We are to compute the decay rate of unpolarized vector particles of mass M into two scalars of mass m. We should calculate the decay rate in the rest frame.

Defining $\tilde{p}^\mu = (\bar{p} - p)^\mu$, the amplitude for the decay diagram is given by



$$= i\mathcal{M} = \epsilon_\mu i f \tilde{p}^\mu.$$

It is quite straightforward to calculate the spin-averaged square of the amplitude,

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{3}\sum_{\text{spin}} \epsilon_\mu i f \tilde{p}^\mu \epsilon^*_\nu (-i) f \tilde{p}^\nu,\\ &= \frac{f^2}{3}\left(\frac{k_\mu k_\nu}{M^2} - g_{\mu\nu}\right)\tilde{p}^\mu \tilde{p}^\nu,\\ &= \frac{f^2}{3}\left(\frac{(k_\mu \tilde{p}^\mu)^2}{M^2} - \tilde{p}^2\right).\end{aligned}$$

Now, because we are computing this in the rest frame where $k_\mu = (M, 0)$ and $\tilde{p}^\mu = (0, -2|\vec{p}|)$, $k_\mu \tilde{p}^\mu = 0$. Similarly, we know that $\tilde{p}^2 = 4|\vec{p}|^2$. Therefore,

$$\overline{|\mathcal{M}|^2} = \frac{4f^2|\vec{p}|^2}{3}.$$

Note that $|\vec{p}| = E^2 - m^2 = \left(\frac{M^2}{4} - m^2\right)^{1/2}$. Using this and the equation for the decay rate found in Peskin and Schroeder,

$$\begin{aligned}\Gamma &= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}|\mathcal{M}|^2,\\ &= \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{p}|}{M}\frac{4f^2|\vec{p}|^2}{3},\\ &= \frac{f^2}{24\pi^2 M^2}\int d\Omega |\vec{p}|^3,\end{aligned}$$

$$\therefore \Gamma = \frac{f^2\left(\frac{M^2}{4} - m^2\right)^{3/2}}{6\pi M^2}.$$

Mott's Formula

We are to generalize problem 2 of Homework 8 in the relativistic case. We computed then the general amplitude to be

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2}\bar{u}^{s'}(p_f)\gamma^0 u^s(p).$$

To compute the spin averaged amplitude, it will be helpful to recall our earlier kinematic result that $(p_f - p)^4 = 16|\vec{p}|^4 \sin^4\theta/2$. Let us now compute the amplitude squared in the spin-averaged case.

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{2}\frac{Z^2 e^4}{(p_f - p)^4}\sum_{\text{spin}} \bar{u}^s(p)\gamma^0 u^{s'}(p_f)\bar{u}^{s'}(p_f)\gamma^0 u^s(p),\\ &= \frac{Z^2 e^4}{32|\vec{p}|^4 \sin^4\theta/2}\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right).\end{aligned}$$

It will be helpful to break up the trace into its four additive pieces.

$$\text{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p} + m)\right) = \text{Tr}\left(\gamma^0\not{p}_f\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0 m\gamma^0\not{p}\right) + \text{Tr}\left(\gamma^0\not{p}_f\gamma^0 m\right) + \text{Tr}\left(\gamma^0 m\gamma^0 m\right).$$

It should be clear that the two middle terms are both zero because there is an odd number of γ's. The last term is nearly trivial, $\text{Tr}\left(\gamma^0 m\gamma^0 m\right) = 4m^2$. Let us now work on the first term.

$$\begin{aligned}
\text{Tr}\left(\gamma^0 \not{p}_f \gamma^0 \not{p}\right) &= p_{f_\mu} p_\nu \text{Tr}\left(\gamma^0\gamma^\mu\gamma^0\gamma^\nu\right), \\
&= 4p_{f_\mu} p_\nu \left(g^{0\mu}g^{0\nu} - g^{00}g^{\mu\nu} + g^{0\nu}g^{\mu 0}\right), \\
&= 4\left(2E^2 - p_{f_\mu}p^\mu\right), \\
&= 4\left(2E^2 - E^2 + \vec{p_f}\vec{p}\right), \\
&= 4\left(E^2 + |\vec{p}|^2\cos\theta\right).
\end{aligned}$$

Using these results, we have that

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[E^2 + |\vec{p}|^2\cos\theta + m^2\right], \\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - |\vec{p}|^2(1-\cos\theta)\right], \\
&= \frac{Z^2e^4}{8|\vec{p}|^4\sin^4\theta/2}\left[2E^2 - 2|\vec{p}|^2\sin^2\theta/2\right], \\
&= \frac{Z^2e^4E^2}{4|\vec{p}|^4\sin^4\theta/2}\left[1 - \left(\frac{|\vec{p}|}{E}\right)^2\sin^2\theta/2\right], \\
&= \frac{Z^2e^4}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].
\end{aligned}$$

In the last two lines we have used the fact that $\vec{p}/E = \beta$. Now, we showed in Homework 8 that

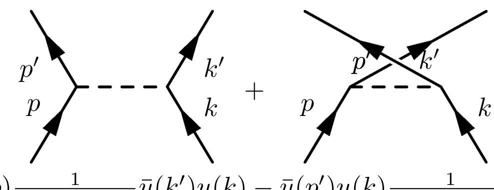
$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}.$$

Using the fine structure constant to simplify notation, where $\alpha^2 = \frac{e^4}{16\pi^2}$, it is clear that

$$\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4\beta^2|\vec{p}|^2\sin^4\theta/2}\left[1 - \beta^2\sin^2\theta/2\right].$$

Helicity Amplitudes in Yukawa Theory

We are to consider the amplitude given by,

$i\mathcal{M} =$ 

$$= (-ig^2)\left(\bar{u}(p')u(p)\frac{1}{(p'-p)^2-m_\phi^2}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{1}{(p'-k)^2-m_\phi^2}\bar{u}(k')u(p)\right).$$

a) We are to derive the selection rules for helicity for this theory.

We can best understand the selection rules by requiring that one of the spinors is in a projection. To bring the projection operator to the neighboring spinor (in either diagram and starting from any outside term) requires that the projection anticommutes through a γ^0. Therefore, the interaction *must* flip the spins. Exempli Gratia, $\bar{u}\frac{1+\gamma^5}{2}u_R = u^\dagger\gamma^0\frac{1+\gamma^5}{2}u_R = \bar{u}_L u_R$.

b) Given these selection rules, what are the non-vanishing amplitudes? These are the only possible terms that involve both incoming states flipping their spin in the outgoing states. So, the nonzero amplitudes are $\mathcal{M}_{LL;RR}, \mathcal{M}_{RR;LL}, \mathcal{M}_{LR;RL}, \mathcal{M}_{RL;LR}, \mathcal{M}_{RL;RL}, \mathcal{M}_{LR;LR}$.

c) We are to use problem 5 of Homework 5 to compute the explicit form of the two-spinors. We should use this to find the eigenvectors $u_\lambda(p)$ at very high energies. This is a relatively straight forward calculation. We derived quite some time ago that in the high energy limit for general