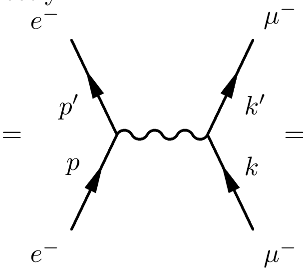
Mott's Formula (II)

In homework 9, we derived Mott's formula (the relativistic Rutherford formula). We are now to derive it by considering the spin-averaged amplitude squared of the scattering of an electron with a muon in the limit that the mass of the muon is much larger than the energy of the electron.

a) We are to compute the spin-averaged amplitude squared for $e^-\mu^-$ scattering for general m_e and m_μ.

Let us compute this directly.

$i\mathcal{M} =$



$= \frac{ie^2}{q^2}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k).$

We can compute the spin-averaged square of the amplitude directly. This becomes

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4}{4q^4}\operatorname{Tr}\left[(\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\operatorname{Tr}\left[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\right], \\
&= \frac{4e^4}{q^4}\left[p'^\mu p^\nu + p'^\nu p^\mu + g\mu\nu(m_e^2 - p\cdot p')\right] \times \left[k'_\mu k_\nu + k'_\nu k_\mu + g_{\mu\nu}(m_\mu^2 - k\cdot k')\right], \\
\therefore \overline{|\mathcal{M}|^2} &= \frac{8e^4}{q^4}\left[(p\cdot k') + (p'\cdot k)_{(}p\cdot k)(p'\cdot k') - m_\mu^2(p\cdot p') - m_e^2(k\cdot k') + 2m_\mu^2 m_e^2\right].
\end{aligned}$$

b) Taking the limit where m_μ is large, we can consider the case that the center of mass frame of the collision is the muon's rest frame. Therefore, we have that $k \approx k' = (m_\mu, \vec{0})$. E represents the energy of the electron. In this case, we can drastically simplify our kinematics.

$$p\cdot k = Em_\mu \qquad k\cdot k' = m_\mu^2 \qquad p\cdot p' = E^2 - \vec{p}\vec{p'} = E^2 - \vec{p}^{\,2}\cos\theta.$$

We can use this to directly write our spin-averaged squared amplitude

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{8e^4}{q^4}\left[2E^2m_\mu^2 - m_\mu^2(E^2 - \vec{p}^{\,2}\cos\theta) + m_e^2m_\mu^2\right], \\
&= m_\mu^2\frac{16e^4}{q^4}\left(E^2 - \vec{p}^{\,2}\sin^2\theta/2\right), \\
\therefore \overline{|\mathcal{M}|^2} &= \frac{m_\mu^2 e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).
\end{aligned}$$

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In the last step we reduced the formula to one which will greatly help us in part (c) below.

c) We are to derive Mott's formula by taking the limit where m_μ is very large in the center of mass frame. As we stated before, this approximation is identical to assuming that the center of mass frame is actually the rest frame of the muon so our amplitude calculated in part (b) is correct to the second order. We know that the final velocity of the muon is zero and that the center of mass energy is approximately m_μ (to the first order) in this frame so we may write,

$$\begin{aligned}
\left.\frac{d\sigma}{d\Omega}\right|_{\text{cm}} &= \frac{1}{4E_aE_b|v_a - v_b|}\frac{|\vec{p}|}{(2\pi)^2 4E_{cm}}\overline{|\mathcal{M}|^2}, \\
&= \frac{1}{4Em_\mu\beta}\frac{|\vec{p}|}{(2\pi)^2 4m_\mu}\frac{m_\mu^2e^4}{\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right), \\
&= \frac{e^4}{16\pi^2 4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right), \\
\therefore \left.\frac{d\sigma}{d\Omega}\right|_{\text{cm}} &= \frac{\alpha^2}{4\beta^2\vec{p}^{\,2}\sin^4\theta/2}\left(1 - \beta^2\sin^2\theta/2\right).
\end{aligned}$$

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