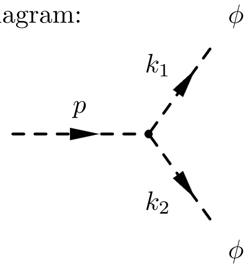
1. The Decay of a Scalar Particle

From the Lagrangian given by,

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi^2,$$

we are to determine the lifetime of a Φ particle to decay into two ϕ's to the lowest order in μ assuming that $M > 2m$.

We first notice that the interaction Hamiltonian is $\int d^3x \mu\Phi\phi\phi$. From this, we can directly calculate the amplitude associated with our desired diagram:

$i\mathcal{M} = \Phi$ 

$= -2i\mu,$

The factor of 2 comes from Bose statistics associated with the two identical final ϕ particles. So,

$$|\mathcal{M}|^2 = 4\mu^2.$$

We have shown before that we can directly compute the decay width of a particle from the amplitude by using the relation,

$$\Gamma = \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{k}|}{E_{cm}}|\mathcal{M}|^2.$$

In the center of mass frame, the rest frame of the Φ, $E_{cm} = M$, $p = (M, \vec{0}), k_1 = (M/2, \vec{k})$, and $k_2 = (M/2, -\vec{k})$. From simple kinematics it is clear that $|\vec{k}| = \left(\frac{M^2}{4} - m^2\right)^{1/2} = \frac{M}{2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}$. This leads to

$$\Gamma = \frac{4\mu^2 M^2}{64\pi^2 M^2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}\int d\Omega.$$

When we integrate over the solid angle Ω, we should only cover 2π because the ϕ's are identical. After integrating and simplifying terms we find that

$$\Gamma = \frac{\mu^2}{8\pi M}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}. \tag{1.1}$$

$$\boxed{\therefore \tau = \frac{8\pi M}{\mu^2}\left(1 - 4\frac{m^2}{M^2}\right)^{-1/2}.} \tag{1.2}$$

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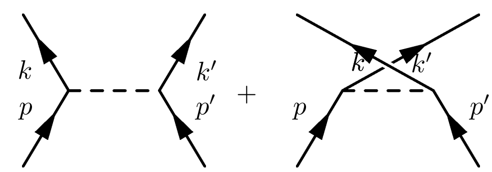
2. Massless Fermion Scattering in Yukawa Theory

a) We are to write the complete amplitude for scattering two massless fermions in Yukawa theory. From previous homework and class notes this is,

$i\mathcal{M} =$ 

$$= (-ig^2)\left(\bar{u}(k)u(p)\frac{1}{(k-p)^2 - m_\phi^2}\bar{u}(k')u(p') - \bar{u}(k)u(p')\frac{1}{(p-k')^2 - m_\phi^2}\bar{u}(k')u(p)\right). \tag{2.1}$$