2. We are given the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

a) Given the generators of rotations and boosts defined by,

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i = J^{0i},$$

we are to explicitly calculate all the commutation relations. We are given trivially that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Let us begin with the K's. By direct calculation,

$$\begin{aligned}[K^i, K^j] = [J^{0i}, J^{0j}] &= i(g^{0i}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}),\\ &= -iJ^{ij};\\ &= -2i\epsilon^{ijk}L^k.\end{aligned}$$

Likewise, we can directly compute the commutator between the L and K's. This also will follow by direct calculation.

$$\begin{aligned}[L^i, K^j] &= \frac{1}{2}\epsilon^{lk}[J^{ilk}, J^{0j}],\\ &= \frac{1}{2}\epsilon^{ilk}i(g^{l0}J^{ij} - g^{i0}J^{lj} - g^{lj}J^{i0} + g^{ij}J^{l0}),\\ &= i\epsilon^{ijk}J^{0k};\\ &= i\epsilon^{ijk}K^k.\end{aligned}$$

We were also to show that the operators

$$J^i_+ = \frac{1}{2}(L^i + iK^i) \qquad J^i_- = \frac{1}{2}(L^i - iK^i),$$

could be seen to satisfy the commutation relations of angular momentum. First let us compute,

$$\begin{aligned}[J_+, J_-] &= \frac{1}{4}\left[(L^i + iK^i), (L^j - iK^i)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] - i[L^i, K^j] + [K^i, K^j]\right),\\ &= 0.\end{aligned}$$

In the last line it was clear that I used the commutator $[L^i, K^j]$ derived above. The next two calculations are very similar and there is a lot of 'justification' algebra in each step. There is essentially no way for me to include all of the details of every step, but each can be verified (e.g. $i[K^i, L^j] = -i[L^j, K^i] = (-i)i\epsilon^{jik}K^k = -\epsilon^{ijk}K^k ...etc$). They are as follows:

$$\begin{aligned}[J^i_+, J^j_+] &= \frac{1}{4}\left[(L^i + iK^i), (L^j + iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] + i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k + iK^k) = i\epsilon^{ijk}J^k_+.\end{aligned}$$

Likewise,

$$\begin{aligned}[J^i_-, J^j_-] &= \frac{1}{4}\left[(L^i - iK^i), (L^j - iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] - i[K^i, L^j] - i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k + \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k - iK^k) = i\epsilon^{ijk}J^k_-.\end{aligned}$$

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b) Let us consider first the $(0, \frac{1}{2})$ representation. For this representation we will need to satisfy

$$J_+^i = \frac{1}{2}(L^i + iK^i) = 0 \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = \frac{\sigma^i}{2}.$$

This is obtained by taking $L^i = \frac{\sigma^i}{2}$ and $K^i = \frac{i\sigma^i}{2}$. The transformation law then of the $(0, \frac{1}{2})$ representation is

$$\begin{aligned}\Phi_{(0,\frac{1}{2})} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(0,\frac{1}{2})},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(0,\frac{1}{2})},\\ &= e^{-\frac{i\theta^i\sigma^i}{2}+\frac{\beta^j K^j}{2}}\Phi_{(0,\frac{1}{2})}.\end{aligned}$$

The calculation for the $(\frac{1}{2}, 0)$ representation is very similar. Taking $L^i = \frac{\sigma^i}{2}$ and $K^i = -\frac{\sigma^i}{2}$, we get

$$J_+^i = \frac{1}{2}(L^i + iK^i) = \frac{\sigma^i}{2} \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = 0.$$

Then the transformation law of the representation is

$$\begin{aligned}\Phi_{(\frac{1}{2},0)} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(\frac{1}{2},0)},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(\frac{1}{2},0)},\\ &= e^{-\frac{i\theta^i\sigma^i}{2}-\frac{\beta^j K^j}{2}}\Phi_{(\frac{1}{2},0)}.\end{aligned}$$

Comparing these transformation laws with Peskin and Schroeder's (3.37), we see that

$$\psi_L = \Phi_{(\frac{1}{2},0)} \qquad \psi_R = \Phi_{(0,\frac{1}{2})}.$$

3. **a)** We are given that T_a is a representation of some Lie group. This means that

$$[T_a, T_b] = if^{abc}T_c$$

by definition. Allow me to take the complex conjugate of both sides. Note that $[T_a, T_b] = [(-T_a), (-T_b)]$ in general and recall that f^{abc} are real.

$$\begin{aligned}[T_a, T_b]^* &= (if^{abc}T_c)^*,\\ [T_a^*, T_b^*] &= -if^{abc}T_c^*,\\ \therefore [(-T_a^*), (-T_b^*)] &= if^{abc}(-T_c^*).\end{aligned}$$

So by the definition of a representation, it is clear that $(-T_a^*)$ is also a representation of the algebra.

b) As before, we are given that T_a is a representation of some Lie group. We will take the Hermitian adjoint of both sides.

$$\begin{aligned}[T_a, T_b]^\dagger &= (if^{abc}T_c)^\dagger,\\ (T_aT_b)^\dagger - (T_bT_a)^\dagger &= -if^{abc}T_c^\dagger,\\ T_b^\dagger T_a^\dagger - T_a^\dagger T_b^\dagger &= -if^{abc}T_c^\dagger,\\ [T_b^\dagger, T_a^\dagger] &= -if^{abc}T_c^\dagger,\\ \therefore [T_a^\dagger, T_b^\dagger] &= if^{abc}T_c^\dagger.\end{aligned}$$

So by the definition of a representation, it is clear that $T_a^\dagger$ is a representation of the algebra.

c) We define the spinor representation of $SU(2)$ by $T_a = \frac{\sigma^a}{2}$ so that

$$T_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2 \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad T_3 \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will consider the matrix $S = i\sigma^2$. Clearly S is unitary because $(i\sigma^2)(i\sigma^2)^\dagger = 1$. Now, one could proceed by direct calculation to demonstrate that

$$ST_1S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -T_1^* \qquad ST_2S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -T_2^* \qquad ST_3S^\dagger = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -T_3^*.$$

This clearly demonstrates that the representation $-T_a^*$ is equivalent to that of T_a.