Physics 513, Quantum Field Theory

Homework 3

Due Tuesday, 23rd September 2003

Jacob Lewis Bourjaily

1. a) We are given complex scalar Lagrangian,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

It is clear that the canonical momenta of the field are

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^*;$$
$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^*)} = \partial_0 \phi.$$

The canonical commutation relations are then

$$[\phi(x), \partial_0 \phi^*(y)] = [\phi^*(x), \partial_0 \phi(y)] = i\delta^{(3)}(x-y),$$

with all other combinations commuting. As in Homework 2, the Hamiltonian can be directly computed,

$$\begin{aligned} H = \int d^3x \mathcal{H} &= \int d^3x \, (\pi \partial_0 \phi - \mathcal{L}), \\ &= \int d^3x \left(\pi^* \pi - 1/2\pi^*\pi + 1/2\nabla\phi^*\nabla\phi + 1/2m^2\phi^*\phi\right), \\ &= \frac{1}{2}\int d^3x \left(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi\right). \end{aligned}$$

We can use this expression for the Hamiltonian to find the Heisenberg equation of motion. We have

$$\begin{aligned} i\partial_0\phi(x) &= \left[\phi(x), \frac{1}{2}\int d^3y \, \left(\pi^*(y)\pi(y) + \nabla\phi^*(y)\nabla\phi(y) + m^2\phi^*(y)\phi(y)\right)\right], \\ &= \frac{1}{2}\int d^3y \, [\phi(x), \pi(y)]\pi^*(y), \\ &= \frac{i}{2}\int d^3y \, \delta^{(3)}(x-y)\pi^*(y), \\ &= \frac{i}{2}\pi^*(x). \end{aligned}$$

Analogously, $i\partial_0\phi^*(x) = \frac{i}{2}\pi(x)$. Notice that this derivation used the fact that ϕ commutes with everything in $\mathcal{H}$ except for π. Before we compute the commutator of $\pi^*(x)$ with the Hamiltonian, we should re-write $\mathcal{H}$ as PS did so that our conclusion will be more lucid. We have from above that

$$H = \frac{1}{2}\int d^3x \left(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi\right).$$

We can evaluate the middle term in H using Green's Theorem (essentially integration by parts). We will assume that the surface term vanishes at infinity because the fields must. This allows us to write the Hamiltonian as,

$$H = \frac{1}{2}\int d^3x \left(\pi^*\pi + \phi^*(-\nabla^2 + m^2)\phi\right).$$

Commuting this with $\pi^*(x)$, we conclude that

$$\begin{aligned} i\partial_0\pi^*(x) &= \frac{1}{2}\int d^3y\,[\pi^*(x),\phi^*(y)](-\nabla^2+m^2)\phi(y),\\ &= -\frac{i}{2}\int d^3y\,(-\nabla^2+m^2)\phi(y)\delta^{(3)}(x-y),\\ &= -\frac{i}{2}\phi(x). \end{aligned}$$

Combining the two results, it is clear that

$$\begin{aligned} \partial_0^2\phi(x) &= (\nabla)^2-m^2)\phi(x),\\ \implies (\partial_\mu\partial^\mu+m^2)\phi &= 0. \end{aligned}$$

This is just the Klein-Gordon equation. The result is the same for the complex conjugate field.

b) Because the field is no longer purely real, we cannot assume that the coefficient of $e^{i\mathbf{p}\cdot\mathbf{x}}$ in the ladder-operator Fourier expansion is the adjoint of the coefficient of $e^{-i\mathbf{p}\cdot\mathbf{x}}$. Therefore we will use the operator b. The expansion of the fields are then

$$\begin{aligned} \phi(x^\mu) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip_\mu x^\mu}+b^\dagger_{\mathbf{p}}e^{ip_\mu x^\mu}\right);\\ \phi^*(x^\mu) &= \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\left(a^\dagger_{\mathbf{q}}e^{iq_\mu x^\mu}+b_{\mathbf{q}}e^{-iq_\mu x^\mu}\right). \end{aligned}$$

It is easy to show that these allow us to define π and π^* in terms of a and b operators as well. These become,

$$\begin{aligned} \pi(x^\mu) = \partial_0\phi^*(x^\mu) &= \int\frac{d^3q}{(2\pi)^3}i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\left(a^\dagger_{\mathbf{q}}e^{iq_\mu x^\mu}-b_{\mathbf{q}}e^{-iq_\mu x^\mu}\right);\\ \pi^*(x^\mu) = \partial_0\phi(x^\mu) &= \int\frac{d^3p}{(2\pi)^3}i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(-a_{\mathbf{p}}e^{-ip_\mu x^\mu}+b^\dagger_{\mathbf{p}}e^{ip_\mu x^\mu}\right). \end{aligned}$$

These allow us to directly demonstrate that

$$\begin{aligned} [\phi(x^\mu),\pi(y^\mu)] &= \int\frac{d^3pd^3q}{(2\pi)^6}\frac{-i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\left(\left[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}\right]e^{-i(p_\mu x^\mu-q_\mu x^\mu)}-\left[b^\dagger_{\mathbf{p}},b_{\mathbf{q}}\right]e^{i(p_\mu x^\mu-q_\mu x^\mu)}\right),\\ &= i\delta^{(3)}(x-y), \end{aligned}$$

while noting that

$$\left[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}\right]=\left[b_{\mathbf{p}},b^\dagger_{\mathbf{q}}\right]=(2\pi)^3\delta^{(3)}(p-q),$$

and all other terms commute. This implies that there are in fact two entirely different sets of particles with the same mass: those created by $b^\dagger$ and those created by $a^\dagger$.

c) I computed the conserved Noether charge in Homework 2 as

$$j^\mu = i\left(\phi\partial^\mu\phi^*-\phi^*\partial^\mu\phi\right).$$

We integrate this over all space to see the conserved current in the 0 component. When expressing *phi* and *pi* in terms of ladder operators, we can evaluate this directly.

$$\begin{aligned} Q &= \frac{i}{2}\int d^x\left(\phi^*(x)\pi^*(x)-\pi(x)\phi(x)\right),\\ &= \frac{i}{2}\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ix^\mu(q_\mu-p_\mu)}-a_{\mathbf{p}}b_{\mathbf{q}}e^{-ix^\mu(p_\mu+q_\mu)}+b^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{ix^\mu(p_\mu+q_\mu)}-b^\dagger_{\mathbf{p}}b_{\mathbf{q}}e^{ix^\mu(q_\mu-p_\mu)}\right)-\text{c.c.},\\ &= \frac{i}{2}\int\frac{d^3pd^3q}{(2\pi)^3}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{q}}\delta^{(3)}(p-q)-a_{\mathbf{p}}b_{\mathbf{q}}\delta^{(3)}(p+q)+b^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}\delta^{(3)}(p+q)-b^\dagger_{\mathbf{p}}b_{\mathbf{q}}\delta^{(3)}(p-q)\right)-\text{c.c.},\\ &= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}}-a_{\mathbf{p}}b_{-\mathbf{p}}+b^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}-b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right)-\text{c.c.},\\ &= i\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}}-b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right). \end{aligned}$$

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The calculation on the previous page clearly shows that particles that were created by $b^\dagger$ contribute oppositely to those created by $a^\dagger$ to the total charge. We concluded in Homework 2 that this charge was electric charge.

2. **a)** We are asked to compute the general, K-type Bessel function solution of the Wightman propagator,

$$D_W(x) \equiv \langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ipx}.$$

Because x is a space-like vector, there exists a reference frame such that $x^0 = 0$. This implies that $x^2 = -\mathbf{x}^2$. And this implies that $px = -\mathbf{p}\cdot\mathbf{x} = -|p||x|\cos(\theta) = -|p|\sqrt{-x^2}\cos(\theta)$. We can then write $D_W(x)$ in polar coordinates as

$$\begin{aligned}
D_W(x) &= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} e^{i|p|\sqrt{-x^2}\cos(\theta)} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&= \frac{1}{(2\pi)^2}\int_0^{\pi} d\theta\, e^{i|p|\sqrt{-x^2}\cos(\theta)} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&= \frac{1}{(2\pi)^2}\int_{-1}^{1} d\xi\, e^{i|p|\sqrt{-x^2}\xi} \int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}},\\
&\text{(where } \xi = \cos(\theta))\\
&= \frac{1}{4\pi^2}\int_0^\infty p^2 dp\, \frac{1}{2\sqrt{p^2+m^2}}\frac{1}{i|p|\sqrt{-x^2}}\left(e^{i|p|\sqrt{-x^2}} - e^{-i|p|\sqrt{-x^2}}\right),\\
&= \frac{1}{4\pi^2\sqrt{-x^2}}\int_0^\infty dp\, \frac{p\sin(|p|\sqrt{-x^2})}{\sqrt{p^2+m^2}}.
\end{aligned}$$

Gradsteyn and Ryzhik's equation (3.754.2) states that for a K Bessel function,

$$\int_0^\infty dx\, \frac{\cos(ax)}{\sqrt{\beta^2+x^2}} = K_0(a\beta)).$$

By differentiating both sides with respect to a, it is shown that

$$-\int_0^\infty dx\, \frac{a\sin(ax)}{\sqrt{\beta^2+x^2}} = -\beta K_0'(a\beta) = \beta K_1(a\beta).$$

We can use this identity to write a more concise equation for $D_W(x)$. We may conclude

$$D_W(x) = \frac{m}{4\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2}).$$

b) We may compute directly,

$$\begin{aligned}
iD(x) &= \langle 0|[\phi(x),\phi(0)]|0\rangle,\\
&= \langle 0|\phi(x),\phi(0)|0\rangle - \langle 0|\phi(0),\phi(x)|0\rangle,\\
&= D_W(x) - D_W(-x),\\
\implies D(x) &= i(D_W(-x) - D_W(x)).
\end{aligned}$$

Similarly,

$$D_1(x) = \langle 0|\{\phi(x),\phi(0)\}|0\rangle = D_W(x) + D_W(-x).$$

It is clear that both function 'die off' very rapidly at large distances. I was not able to conclude that they were truly vanishing, but they are certainly nearly-so at even moderately small distances.