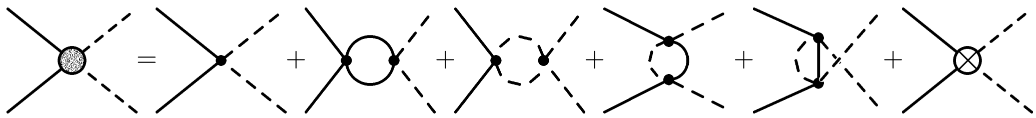
Similar to our computation above, to find β_ρ we must compute the renormalization counter-term δ_ρ. To the one-loop order, we can find δ_ρ by computing,



We notice that the symmetry factor of 2, included in our evaluation of the function $V(k)$, should not be included for the penultimate and antepenultimate diagrams because distinct fields run in the loop. Therefore, the loop integral for each of those two diagrams will contribute $2V(k)$ to to the total amplitude. Noting this subtlety, we find that

$$i\mathscr{M} = -i(\rho/3) + (-i\lambda)(-i\rho/3)\left[V(t)+V(t)\right] + (-i\rho/3)^2\left[2V(u)+2V(s)\right] - i\delta_\rho/3.$$

Recall that we have already computed the divergence of the function $V(k)$ and noted that it was independent of k. Therefore,

$$\begin{aligned} i\delta_\rho/3 &= (-i\lambda)(-i\rho/3)\left[V(t)+V(t)\right] + (-i\rho/3)^2\left[2V(u)+2V(s)\right], \\ &= -\lambda\rho/3\frac{-i}{16\pi^2}\log\frac{\Lambda^2}{M^2} - (\rho/3)^2\frac{-i}{8\pi^2}\log\frac{\Lambda^2}{M^2}, \\ \therefore \delta_\rho &= \frac{1}{16\pi^2}\left[\lambda\rho + 2\rho^2/3\right]\log\frac{\Lambda^2}{M^2}. \end{aligned}$$

Because there are no divergent self-energy diagrams in this theory to one-loop order, we have that the β-function for ρ is given precisely by twice the coefficient of the log divergence in δ_ρ.

$$\boxed{\therefore \beta_\rho = \frac{1}{8\pi^2}\left[\lambda\rho + 2\rho^2/3\right].} \quad (1.b.2)$$

Let us now consider the β-function associated with the ration λ/ρ. Using the chain rule for differentiation and the definition of the general β-function, we see that

$$\begin{aligned} \beta_{\lambda/\rho} = \frac{1}{\rho^2}\left[\beta_\lambda\rho - \beta_\rho\lambda\right] &= \frac{1}{\rho^2}\left[\frac{3\lambda^2\rho}{16\pi^2} + \frac{\rho^3}{48\pi^2} - \frac{\lambda^2\rho}{8\pi^2} - \frac{\rho^2\lambda}{12\pi^2}\right], \\ &= \frac{(\lambda/\rho)^2\rho}{16\pi^2} + \frac{\rho}{48\pi^2} - \frac{(\lambda/\rho)}{12\pi^2}, \\ &= \frac{\rho}{48\pi^2}\left[3(\lambda/\rho)^2 - 4(\lambda/\rho) + 1\right], \end{aligned}$$

$$\boxed{\therefore \beta_{\lambda/\rho} = \frac{\rho}{48\pi^2}\left(3\lambda/\rho - 1\right)\left(\lambda/\rho - 1\right).} \quad (1.c.1)$$

We see immediately that the two roots of $\beta_{\lambda/r}$ occur when $\lambda/\rho = 1, \frac{1}{3}$ and because the second derivative of $\beta_{\lambda/r}$ is $6 > 0$, we know that $\beta_{\lambda/\rho} < 0$ for $\lambda/\rho \in (\frac{1}{3}, 1)$ and $\beta_{\lambda/r} > 0$ for $\lambda/\rho > 1$. Therefore, for all $\lambda/\rho > \frac{1}{3}$, λ/ρ will flow to $\lambda/\rho = 1$. See Figure 1 below.

Therefore at large distances the couplings will flow to $\lambda = \rho$. This introduces a continuous $O(2)$ symmetry into the theory. To see this, let us define $\varphi \equiv \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$. In this notation, the Lagrangian simply reads

$$\boxed{\mathscr{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4!}\varphi^4.} \quad (1.e.1)$$

This Lagrangian is clearly invariant to $O(2)$ transformations which correspond to changing the phase of φ.

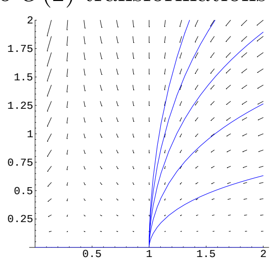


FIGURE 1. Renormalization Group Flow as a function of scale. Arrows show $p \to 0$ flow.